Year 12 *Trial Exam Paper*

2014

SPECIALIST MATHEMATICS

Written examination 2

Worked solutions

This book presents:

- worked solutions, giving you a series of points to show you how to work through the questions
- mark allocations
- tips on how to approach the questions

SECTION 1

Question 1

Answer is C.

Worked solution

The centre of the ellipse is $\left(\frac{-3}{2}\ ,\ \frac{-3}{2}\right)$ with a = 3 and b = 2, so the equation is:

$$\frac{\left(x+\frac{3}{2}\right)^2}{9}+\frac{\left(y+\frac{3}{2}\right)^2}{4}=1$$

$$\frac{\left(\frac{2x+3}{2}\right)^2}{9}+\frac{\left(\frac{2y+3}{2}\right)^2}{4}=1$$

$$\frac{(2x+3)^2}{36}+\frac{(2y+3)^2}{16}=1$$

Question 2

Answer is C.

Worked solution

Rearranging the parametric equations to:

$\sec(t)=\frac{x-1}{2}$ and $\tan(t)=\frac{y}{3}$

and using the identity $\tan^2(t) + 1 = \sec^2(t)$ gives

$$\frac{y^2}{9}+1=\frac{(x-1)^2}{4}$$

$$\frac{(x-1)^2}{4}-\frac{y^2}{9}=1$$

If graphs of $x=2\sec(t)+1$ and $y=3\tan(t)$ for $t\in\left(\frac{-\pi}{2}\ ,\ \frac{\pi}{2}\right)$ are sketched then $x\in[3,\infty)$ and $y\in R$ can be determined from these graphs.

Tip

- *The graphs of* $x=2\sec(t)+1$ and $y=3\tan(t)$ *are best sketched using a CAS calculator.*
- *On the TI-inspire, the correct input would be*

  $f(x)=2\sec(x)+1\ \left|\frac{-\pi}{2}<x<\frac{\pi}{2}\right.$ and $f(x)=3\tan(x)\ \left|\frac{-\pi}{2}<x<\frac{\pi}{2}\right.$

Question 3

Answer is C.

Worked solution

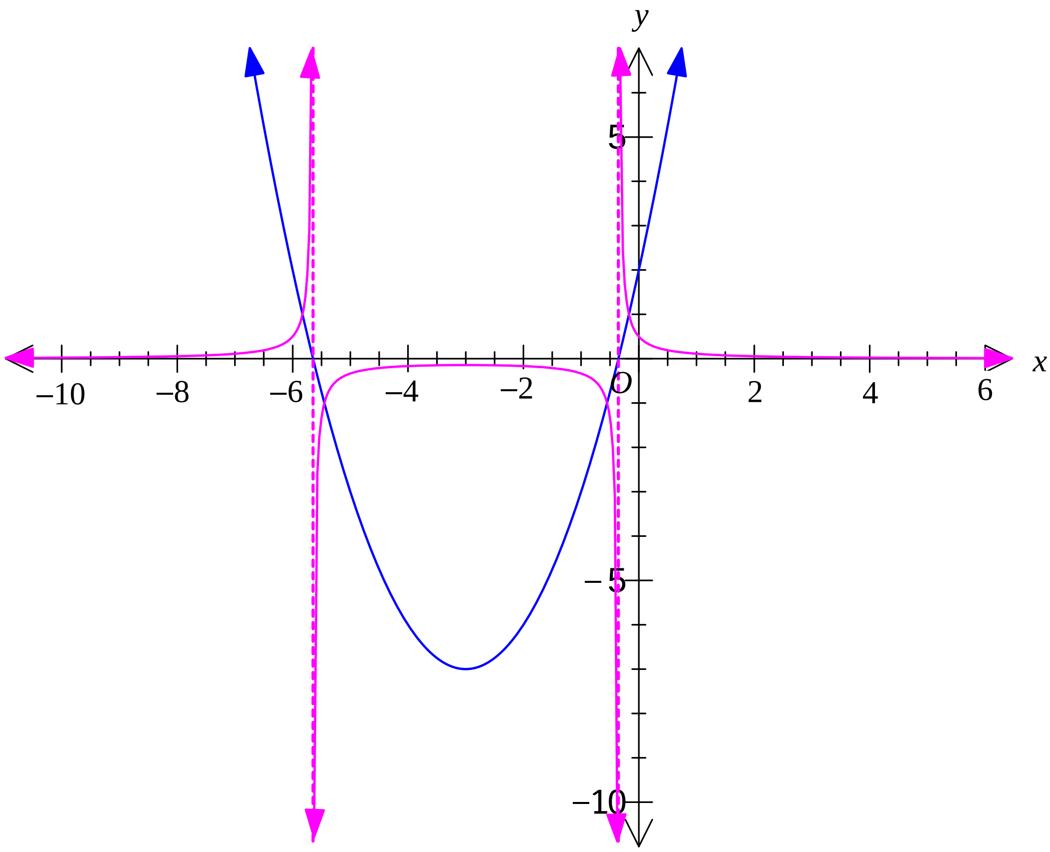
For the reciprocal graph of a quadratic function to have vertical asymptotes and a local maximum, the quadratic function $ax^2 + bx + c$ must have a minimum point ($a > 0$) and have two x-intercepts. Hence, the discriminant of the quadratic must be positive, therefore:

$$\Delta > 0$$

$$b^2 - 4ac > 0$$

$$b^2 > 4ac$$

A graphical example of such a quadratic function and its reciprocal is shown below.



Question 4

Answer is C.

Worked solution

If $\cos(x) > 0$ and $\operatorname{cosec}(x) < 0$, then x is in the fourth quadrant, so:

$$\cot(-x) = -\cot(x)$$

$$\cot(-x) = \frac{-\cos(x)}{\sin(x)}$$

$$\cot(-x) = \frac{-a}{\frac{-1}{b}}$$

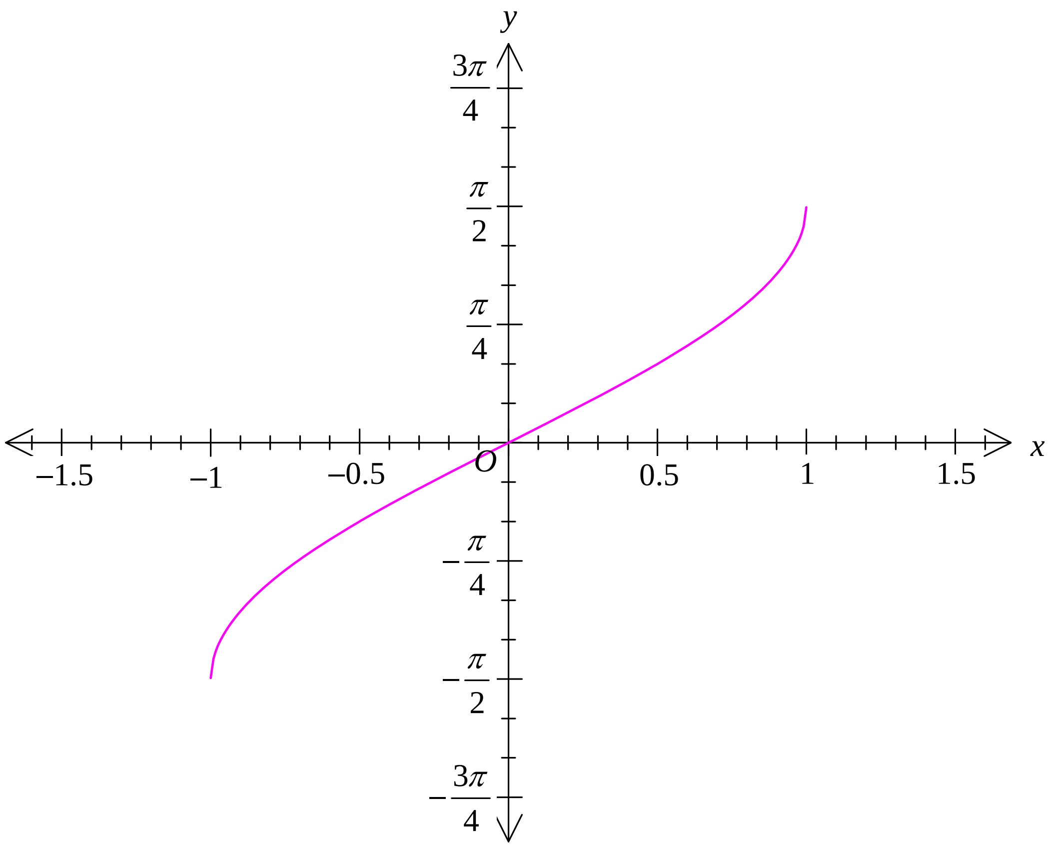
$$\cot(-x) = ab$$

Question 5

Answer is E.

Worked solution

From the graph of $y = \sin^{-1}(x)$ below, it can be seen that the graph in question has been reflected in the x-axis (or y-axis) and dilated by a factor 2 from the x-axis. Hence, $b = -2$.



It has also been translated $\frac{1}{2}$ unit in the negative direction of the x-axis and $\frac{\pi}{4}$ unit in the positive direction of the y-axis; hence, $c = \frac{1}{2}$ and $a = \frac{\pi}{4}$.

Tip

- *Although every graph represented in each alternative can be sketched on a graphic calculator, this would be far too time consuming. It is better to use a transformation approach, as outlined in the solution for this type of question.*

Question 6

Answer is D.

Worked solution

Converting $z = a + ai$, where $a < 0$, into polar form gives

$$z = |a|\sqrt{2}\ \text{cis}\frac{-3\pi}{4}$$

Then

$$z^8 = (|a|\sqrt{2})^8\ \text{cis}\left(\frac{-24\pi}{4}\right)$$

$z^8 = 16|a|^8 \text{cis}(0)$, which is located on the real axis.

So, iz^8 is z^8 rotated 90° anticlockwise and will be located on the imaginary axis.

Question 7

Answer is B.

Worked solution

$$\begin{aligned} z^3 - z^2 + iz^2 - z + 1 - i &= 0 \\ z^2(z - 1 + i) - (z - 1 + i) &= 0 \\ (z - 1 + i)(z^2 - 1) &= 0 \end{aligned}$$

So, $z = 1 - i$ and $z = \pm 1$.

$z = 1 - i$ is the only one of these solutions given in the alternatives.

Instead of factorising by grouping, alternatives **B**, **C** and **D** could have been substituted into the cubic equation. Alternatives **A** and **E** could have been disregarded as these are factors and not solutions.

Question 8

Answer is A.

Worked solution

The distance from any point on the broken line to (0, 0), which is represented by $|z|$,

is equal to the distance from any point on the broken line to the point $(1,-1)$, which is represented by $|z-1+i|$.

Hence, the equation of the broken line is $|z|=|z-1+i|$ and the equation of the shaded region is then $|z| > |z-1+i|$, as the distance from any point in the shaded region to the point (0, 0) is greater than the distance from that point to $(1,-1)$.

The relationships given can be converted to Cartesian form, although this would be time consuming. When converted to Cartesian form, alternative **A** would be:

$$
\begin{aligned}
|z| &> |z-1+i| \\
|x+yi| &> |x+yi-1+i| \\
|x+yi| &> |(x-1)+(y+1)i| \\
\sqrt{x^2+y^2} &> \sqrt{(x-1)^2+(y+1)^2} \\
\cancel{x^2}+\cancel{y^2} &> \cancel{x^2}-2x+1+\cancel{y^2}+2y+1 \\
2y &< 2x-2 \\
y &< x-1
\end{aligned}
$$

which is the area under the line $y=x-1$.

Alternatives **C** and **D** can be disregarded as these represent rays originating from (1, 0) and $(0,-1)$ respectively.

Question 9

Answer is B.

Worked solution

$z=\dfrac{-2}{i-1}$ can be rearranged to $z=\dfrac{2}{1-i}$.

Dividing gives

$$z=\frac{2}{1-i}\times\frac{1+i}{1+i}$$
$$z=\frac{2+2i}{2}$$
$$z=1+i$$

And $\overline{z}=1-i$

$$\left|\overline{z}\right|=\sqrt{1^2+\left(-1\right)^2}$$

So, $a=\sqrt{2}$

$\tan(\theta)=-1$

$\therefore\ \theta=\dfrac{-\pi}{4}$, as $\overline{z}$ is in the fourth quadrant.

Question 10

Answer is B.

Worked solution

$$\frac{dx}{dy}=\frac{1}{15}\left(13+x^2\right)$$
$$\frac{dy}{dx}=\frac{15}{13+x^2}$$
$$y=\frac{15}{\sqrt{13}}\int\frac{\sqrt{13}}{13+x^2}\ dx$$
$$y=\frac{15\sqrt{13}}{13}\int\frac{\sqrt{13}}{13+x^2}\ dx$$

Question 11

Answer is D.

Worked solution

Let $u = 1 - x$, then $\frac{du}{dx} = -1$ and $x = 1 - u$.

When $x = 5$, $u = -4$.

And when $x = 3$, $u = -2$, so

$$\int_3^5 \left(\frac{2-3x}{\sqrt{1-x}}\right) dx$$

becomes

$$-\int_{-2}^{-4} \left(\frac{2-3(1-u)}{\sqrt{u}}\right) \frac{du}{dx}\, dx$$

$$-\int_{-2}^{-4} \left(\frac{3u-1}{\sqrt{u}}\right) du$$

$$\int_{-4}^{-2} \left(3u^{\frac{1}{2}} - u^{\frac{-1}{2}}\right) du$$

Question 12

Answer is D.

Worked solution

$$\frac{dV}{dt} = 500\text{ cm}^3/\text{min}$$

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi\left(\frac{h}{4}\right)^2 h$$

$$V = \frac{1}{3}\pi\left(\frac{h^3}{16}\right)$$

$$V = \frac{\pi h^3}{48}$$

$$\frac{dV}{dh} = \frac{\pi h^2}{16}$$

Now $\dfrac{dh}{dt} = \dfrac{dV}{dt}.\dfrac{dh}{dV}$

$$\frac{dh}{dt} = 500.\frac{16}{\pi h^2}$$

When $d = 10,\ h = 20.$

So, $\dfrac{dh}{dt} = \dfrac{20}{\pi}$ cm/min

Tip

- *You need to consider that the required derivative might be in terms of a variable that is different from the one that needs to be substituted. In this problem* $\frac{dh}{dt}$ *was in terms of* h *and the substitution was* d *= 10, meaning it was necessary to find the value of* h *that corresponded to* d *= 10.*

Question 13

Answer is B.

Worked solution

Now

$$\frac{2x-1}{(x+2)^2} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2}$$

$$= \frac{A(x+2)+B}{(x+2)^2}$$

$$= \frac{Ax+2A+B}{(x+2)^2}$$

$\therefore\ A=2$

and $2A+B=-1$

$\therefore B=-5$

So, $\int \frac{2x-1}{(x+2)^2}\,dx = \int \frac{2}{(x+2)} - \frac{5}{(x+2)^2}\,dx$

Question 14

Answer is D.

Worked solution

$\overrightarrow{BA} = \underset{\sim}{a} - \underset{\sim}{b}$

$\overrightarrow{MP} = MA + AP$

$\overrightarrow{MP} = \frac{1}{2}(\underset{\sim}{a} - \underset{\sim}{b}) - \frac{1}{5}\underset{\sim}{a}$

$\overrightarrow{MP} = \frac{3}{10}\underset{\sim}{a} - \frac{1}{2}\underset{\sim}{b}$

Question 15

Answer is A.

Worked solution

If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are linearly dependent, then $\underset{\sim}{a} = n\underset{\sim}{b}$, where n is a constant.

$$2\underset{\sim}{j} - \underset{\sim}{k} = n\left(-\underset{\sim}{j} + m\underset{\sim}{k}\right)$$

$$2\underset{\sim}{j} - \underset{\sim}{k} = -n\underset{\sim}{j} + mn\underset{\sim}{k}$$

So, $n = -2$

$mn = -1$

$\therefore \; m = \dfrac{1}{2}$

Question 16

Answer is D.

Worked solution

$v = e^{3x}$

Now $a = v\dfrac{dv}{dx}$

$a = e^{3x} 3e^{3x}$

$a = 3e^{6x}$

Question 17

Answer is B.

Worked solution

$$\left(\underset{\sim}{b} . \underset{\sim}{\hat{a}}\right) \, . \, \underset{\sim}{\hat{a}} = \left[(\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k})\frac{1}{7}(6\underset{\sim}{i} - 3\underset{\sim}{j} + 2\underset{\sim}{k})\right]\frac{1}{7}(6\underset{\sim}{i} - 3\underset{\sim}{j} + 2\underset{\sim}{k})$$

$$= \frac{11}{7} . \frac{1}{7}(6\underset{\sim}{i} - 3\underset{\sim}{j} + 2\underset{\sim}{k})$$

$$= \frac{11}{49}(6\underset{\sim}{i} - 3\underset{\sim}{j} + 2\underset{\sim}{k})$$

Question 18

Answer is A.

Worked solution

$x = \cos(2t)$ and $y = \cos^2(t)$

Now $\cos(2t) = 2\cos^2(t) - 1$

$$x = 2y - 1$$

$$2y = x + 1$$

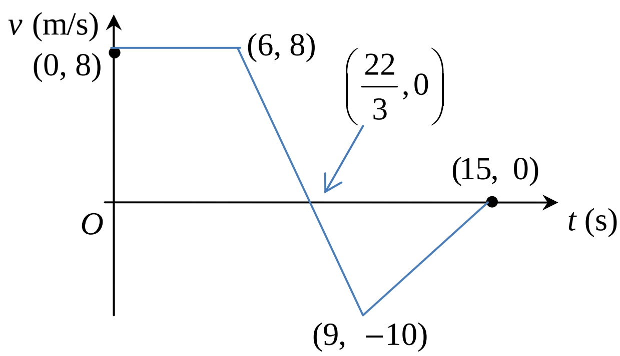
$$y = \frac{1}{2}(x+1) \text{ and when } t \geq 0, \text{ then } -1 \leq x \leq 1$$

$$y = \frac{1}{2}(x+1), \; -1 \leq x \leq 1$$

Question 19

Answer is B.

Worked solution



The intercept is $\frac{22}{3}$ (obtained from determining the straight line equation $v = -6t + 44$).

Area under trapezium representing easterly displacement $= \frac{1}{2} \times \left(6 + \frac{22}{3}\right) \times 8 = \frac{160}{3}$.

Area under triangle representing westerly displacement $= \frac{1}{2} \times \frac{23}{3} \times 10 = \frac{115}{3}$.

Resultant displacement is $\frac{160}{3} - \frac{115}{3} = 15$ m east of *O*.

Question 20

Answer is A.

Worked solution

$y_{n+1} = y_n + hf\left(x_n, y_n\right)$ and $x_{n+1} = x_n + h$

$$y_1 = y_0 + 0.5f(x_0, y_0)$$
$$y_1 = 0 + 0.5(2 - 0)$$
$$y_1 = 1$$

$y_2 = y_1 + 0.5f(x_1, y_1)$ and $x_1 = x_0 + h = 2.5$

$$y_2 = 1 + 0.5(2.5 - 1)$$
$$y_2 = 1.750$$

Question 21

Answer is B.

Worked solution

Let $3M$ = mass of larger object.
Then M = mass of smaller object.

Equations of motion on each mass are:

$3Mg - T = 3Ma \quad (1)$ larger mass

$T - Mg = Ma \quad (2)$ smaller mass

(1) + (2) gives:

$$2Mg = 4Ma$$

$$a = \frac{g}{2} \text{ m/s}^2$$

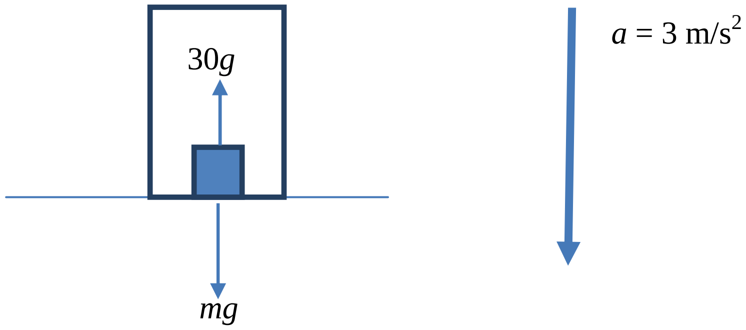
Tip

- *For connected particles, the equations of motion must be written for each particle separately, knowing that the acceleration is the same in both equations.*

Question 22

Answer is C.

Worked solution



Since the lift is accelerating downwards:

$$mg - 30g = ma$$

$$mg - 30g = 3m$$

$$mg - 3m = 30g$$

$$m(g - 3) = 30g$$

$$m = \frac{30g}{g - 3}$$

$$m \approx 43 \text{ kg}$$

END OF SECTION 1

SECTION 2

Question 1a.

Worked solution

x-intercept $(\text{let } y = 0)$:

$0 = ax + b$

$x = \dfrac{-b}{a}$

y-intercept $(\text{let } x = 0)$:

$y = b$

Intercepts are $\left(\dfrac{-b}{a}, 0\right)$ and $(0, b)$.

Mark allocation: 1 mark

- 1 mark for $\left(\dfrac{-b}{a}, 0\right)$ and $(0, b)$.

Question 1b.

Worked solution

$$V = \pi \int_0^b \left(\frac{y-b}{a}\right)^2 dy$$

$$V = \frac{\pi b^3}{3a^2}$$

Mark allocation: 2 marks

- 1 mark for using the correct formula for rotating around the *y*-axis; i.e.
  $V = \pi \int_0^b \left(\dfrac{y-b}{a}\right)^2 dy$
- 1 mark for correct answer in terms of *a* and *b*; i.e. $V = \dfrac{\pi b^3}{3a^2}$

Question 1c.

Worked solution

$$3y\sin(x)+\frac{2}{y^2}=\frac{1}{y}$$

$$\frac{d}{dx}\left[3y\sin(x)\right]+\frac{d}{dx}\left[\frac{2}{y^2}\right]=\frac{d}{dx}\left[\frac{1}{y}\right]$$

$$\left[\frac{d}{dx}(3y)\sin(x)+\frac{d}{dx}(\sin(x))3y\right]+\frac{d}{dy}\left[\frac{2}{y^2}\right]\frac{dy}{dx}=\frac{d}{dy}\left[\frac{1}{y}\right]\frac{dy}{dx}$$

$$\left[\frac{d}{dy}(3y)\frac{dy}{dx}\sin(x)+\frac{d}{dx}(\sin(x))3y\right]+\frac{d}{dy}\left[\frac{2}{y^2}\right]\frac{dy}{dx}=\frac{d}{dy}\left[\frac{1}{y}\right]\frac{dy}{dx}$$

$$3\frac{dy}{dx}\sin(x)+3y\cos(x)-\frac{4}{y^3}\frac{dy}{dx}=\frac{-1}{y^2}\frac{dy}{dx}$$

$$\frac{dy}{dx}\left[3\sin(x)-\frac{4}{y^3}+\frac{1}{y^2}\right]=-3y\cos(x)$$

$$\frac{dy}{dx}=\frac{-3y\cos(x)}{3\sin(x)-\frac{4}{y^3}+\frac{1}{y^2}}$$

$$\frac{dy}{dx}=\frac{-3y^4\cos(x)}{3y^3\sin(x)-4+y}$$

Mark allocation: 4 marks

- 1 method mark for using implicit differentiation
- 1 method mark for correctly using product rule in the implicit differentiation process
- 2 marks for correct answer $\frac{dy}{dx}=\frac{-3y\cos(x)}{3\sin(x)-\frac{4}{y^3}+\frac{1}{y^2}}$ or $\frac{dy}{dx}=\frac{-3y^4\cos(x)}{3y^3\sin(x)-4+y}$.
- Deduct 1 of these marks for a minor error

Question 1d.
Worked solution

When $x = \pi,\ 3y\sin(x) + \dfrac{2}{y^2} = \dfrac{1}{y}$ becomes:

$$\frac{2}{y^2} = \frac{1}{y}$$
$$2y = y^2$$
$$2y - y^2 = 0$$
$$y(2 - y) = 0$$
$$y = 2 \text{ only since } y \neq 0.$$

When $x = \pi$ and $y = 2$

$$\frac{dy}{dx} = \frac{6}{\frac{-1}{2} + \frac{1}{4}}$$
$$\frac{dy}{dx} = -24$$

Equation of tangent is:

$$y - 2 = -24(x - \pi)$$
$$y = -24x + 24\pi + 2$$
$$\therefore\ a = -24 \text{ and } b = 24\pi + 2$$

Mark allocation: 3 marks

- 1 mark for determining $y = 2$ when $x = \pi$
- 1 mark for calculating $\dfrac{dy}{dx} = -24$ at the point $(\pi, 2)$ and finding the tangent's equation; i.e. $y = -24x + 24\pi + 2$
- 1 mark for stating $a = -24$ and $b = 24\pi + 2$

Question 1e.

Worked solution

$V = \dfrac{\pi b^3}{3a^2}$ (from part **b**)

$V = \dfrac{\pi\left(12\pi + 1\right)^3}{216}$

Mark allocation: 1 mark

- 1 mark for $V = \dfrac{\pi\left(12\pi + 1\right)^3}{216}$.

Question 2a.

Worked solution

$$\frac{dT}{dt} = b(100 - T),\ b > 0$$

$$\frac{dt}{dT} = \frac{1}{b(100 - T)}$$

$$t = \frac{1}{b}\int \frac{1}{100 - T}\,dT$$

$$t = \frac{-1}{b}\log_e|100 - T| + c$$

$$-b(t - c) = \log_e|100 - T|$$

$$100 - T = e^{-b(t-c)}$$

$$100 - T = e^{-bt}e^{bc}$$

$$100 - T = Ae^{-bt},\text{ where } A = e^{bc}$$

$$T = 100 - Ae^{-bt}$$

When $t = 0,\ T = 3 \Rightarrow A = 97$

So, $T = 100 - 97e^{-bt}$

Mark allocation: 4 marks

- 1 mark for inverting $\frac{dT}{dt} = b(100 - T)$.
- 1 mark for correctly antidifferentiating to $t = \frac{-1}{b}\log_e|100 - T| + c$.
- 1 mark for rewriting the constant and writing it as $T = 100 - Ae^{-bt}$.
- 1 mark for evaluating A = 97.

Question 2b.

Worked solution

If $T = Be^{-kt} + 20$

then $\frac{dT}{dt} = -kBe^{-kt}$ (1)

Also $\frac{dT}{dt} = -k(T - 20)$

$\frac{dT}{dt} = -k(Be^{-kt} + 20 - 20)$ since $T = Be^{-kt} + 20$

$\frac{dT}{dt} = -kBe^{-kt}$ (2)

Verified by differentiation when equation (1) is compared with equation (2).

Mark allocation: 2 marks

- 1 mark for differentiating $T = Be^{-kt} + 20$ to get $\frac{dT}{dt} = -kBe^{-kt}$.
- 1 mark for substituting $T = Be^{-kt} + 20$ into $\frac{dT}{dt} = -k(T - 20)$ to get $\frac{dT}{dt} = -kBe^{-kt}$.
- No credit should be given if the differential equation $\frac{dT}{dt} = -k(T - 20)$ is solved by integration to obtain $T = Be^{-kt} + 20$.

Question 2c.

Worked solution

$T = Be^{-kt} + 20,\ t \ge t_1$

When $t = t_1,\ T = 60$ and when $t = 3t_1,\ T = 40$.

$60 = Be^{-kt_1} + 20 \qquad (1)$

$40 = Be^{-3kt_1} + 20 \qquad (2)$

From equation (1):

$$60 = Be^{-kt_1} + 20$$

$$e^{-kt_1} = \frac{40}{B}$$

$$-kt_1 = \log_e\left(\frac{40}{B}\right)$$

From equation (2):

$$40 = Be^{-3kt_1} + 20$$

$$e^{-3kt_1} = \frac{20}{B}$$

$$-kt_1 = \frac{1}{3}\log_e\left(\frac{20}{B}\right)$$

Hence:

$$\log_e\left(\frac{40}{B}\right) = \frac{1}{3}\log_e\left(\frac{20}{B}\right)$$

$$3\log_e\left(\frac{40}{B}\right) = \log_e\left(\frac{20}{B}\right)$$

$$\left(\frac{40}{B}\right)^3 = \frac{20}{B}$$

$$\frac{40^3}{B^3} = \frac{20}{B}$$

$$20B^2 = 40^3$$

$$B^2 = \frac{40^3}{20}$$

$$B^2 = 2 \times 40^2$$

$$B = 40\sqrt{2}$$

Mark allocation: 3 marks

- 1 mark for setting up equations $60 = Be^{-kt_1} + 20$ (1) and $40 = Be^{-3kt_1} + 20$ (2)
- 1 mark for using substitution for $-kt_1$ to obtain $\log_e\left(\frac{40}{B}\right) = \frac{1}{3}\log_e\left(\frac{20}{B}\right)$
- 1 mark for showing that $B = 40\sqrt{2}$

Question 2d.

Worked solution

$$40 = Be^{-3kt_1} + 20$$

$$-kt_1 = \frac{1}{3}\log_e\left(\frac{20}{B}\right)$$

If $B = 40\sqrt{2}$ and $k = \frac{1}{4}$, then:

$$-\frac{1}{4}t_1 = \frac{1}{3}\log_e\left(\frac{20}{40\sqrt{2}}\right)$$

$$t_1 = \frac{-4}{3}\log_e\left(\frac{20}{40\sqrt{2}}\right)$$

$$t_1 = 1.38$$

$$3t_1 = 4.16$$

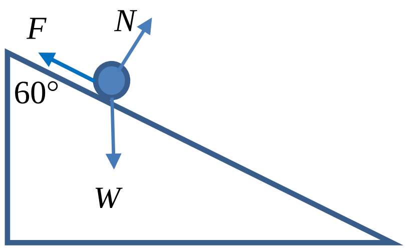
The total time was less than 5 minutes.

Mark allocation: 2 marks

- 1 mark for substituting $k = \frac{1}{4}$ and $B = 40\sqrt{2}$ into $40 = Be^{-3kt_1} + 20$.
- 1 mark for finding $3t_1 = 4.16$.

Question 3a.

Worked solution

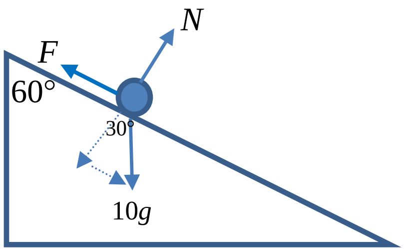


Mark allocation: 1 mark

- 1 mark for three arrows inserted in the correct direction on the diagram. Arrows can be labelled weight, normal reaction and frictional force.

Question 3b.

Worked solution



Resolving forces acting perpendicular to plane:
$10g\cos(30°) = N$, where N is the normal reaction.

Resolving forces acting parallel to plane:
$10g\sin(30°) - F = 10a$

Now
$F = \mu N$ (where F is the frictional force and μ is the coefficient of friction)

$F = \mu 10g\cos(30°)$

$F = \dfrac{5g\sqrt{3}}{2}$

So, equation of motion becomes:

$$10g\sin(30°) - \frac{5g\sqrt{3}}{2} = 10a$$

$$5g - \frac{5g\sqrt{3}}{2} = 10a$$

$$2g - g\sqrt{3} = 4a$$

$$a = \frac{g\left(2-\sqrt{3}\right)}{4}$$

Mark allocation: 3 marks

- 1 mark for determining the frictional force $F = \dfrac{5g\sqrt{3}}{2}$.
- 1 mark for the equation of motion $10g\sin(30°) - \dfrac{5g\sqrt{3}}{2} = 10a$.
- 1 mark for showing $a = \dfrac{g\left(2-\sqrt{3}\right)}{4}$

Question 3c.

Worked solution

$u = 0,\ a = \dfrac{g\left(2-\sqrt{3}\right)}{4},\ v = 5$

$s = \dfrac{v^2 - u^2}{2a}$

$s = \dfrac{25}{\dfrac{g\left(2-\sqrt{3}\right)}{2}}$

$s = \dfrac{50}{g\left(2-\sqrt{3}\right)}$ metres

Mark allocation: 2 marks

- 1 mark for using $s = \dfrac{v^2 - u^2}{2a}$
- 1 mark for obtaining the exact value of the distance $s = \dfrac{50}{g\left(2-\sqrt{3}\right)}$ metres

Question 3d.

Worked solution

Exit speed from upper plane is 5 m/s.

So velocity $= 5\cos(30°)\,\underset{\sim}{\mathrm{i}} - 5\sin(30°)\,\underset{\sim}{\mathrm{j}}$

$$\underset{\sim}{\mathrm{v}} = \left(\frac{5\sqrt{3}}{2}\underset{\sim}{\mathrm{i}} - \frac{5}{2}\underset{\sim}{\mathrm{j}}\right) \text{ m/s}$$

Mark allocation: 1 mark

- 1 mark for writing and evaluating $5\cos(30°)\,\underset{\sim}{\mathrm{i}} - 5\sin(30°)\,\underset{\sim}{\mathrm{j}}$ to $\underset{\sim}{\mathrm{v}} = \left(\dfrac{5\sqrt{3}}{2}\underset{\sim}{\mathrm{i}} - \dfrac{5}{2}\underset{\sim}{\mathrm{j}}\right)$ m/s.

Question 3e.

Worked solution

$\underset{\sim}{a}(t) = -0.4t\,\underset{\sim}{i} - (g - 0.4t)\,\underset{\sim}{j}$

$\underset{\sim}{v}(t) = -0.2t^2\,\underset{\sim}{i} - (gt - 0.2t^2)\,\underset{\sim}{j} + \underset{\sim}{c}_1$

When $t = 0$, $\underset{\sim}{v} = \left(\frac{5\sqrt{3}}{2}\underset{\sim}{i} - \frac{5}{2}\underset{\sim}{j}\right) \Rightarrow \underset{\sim}{c}_1 = \left(\frac{5\sqrt{3}}{2}\underset{\sim}{i} - \frac{5}{2}\underset{\sim}{j}\right)$

$\underset{\sim}{v}(t) = \left(\frac{5\sqrt{3}}{2} - 0.2t^2\right)\underset{\sim}{i} - \left(\frac{5}{2} + gt - 0.2t^2\right)\underset{\sim}{j}$

$\underset{\sim}{r}(t) = \left(\frac{5t\sqrt{3}}{2} - \frac{1}{15}t^3\right)\underset{\sim}{i} - \left(\frac{5t}{2} + \frac{1}{2}gt^2 - \frac{1}{15}t^3\right)\underset{\sim}{j} + \underset{\sim}{c}_2$

When $t = 0$, $\underset{\sim}{r} = 0\underset{\sim}{i} + 0\underset{\sim}{j} \Rightarrow \underset{\sim}{c}_2 = 0\underset{\sim}{i} + 0\underset{\sim}{j}$

Hence, $\underset{\sim}{r}(t) = \left(\frac{5t\sqrt{3}}{2} - \frac{1}{15}t^3\right)\underset{\sim}{i} - \left(\frac{5t}{2} + \frac{1}{2}gt^2 - \frac{1}{15}t^3\right)\underset{\sim}{j}$

The lower plane has the Cartesian equation $y = -x$, so when the ball lands on the lower plane the *y* coordinate of the position vector = –*x* coordinate of position vector.

$$\frac{5t\sqrt{3}}{2} - \cancel{\frac{1}{15}t^3} = \frac{5t}{2} + \frac{1}{2}gt^2 - \cancel{\frac{1}{15}t^3}$$

$$\frac{5t\sqrt{3}}{2} = \frac{5t}{2} + \frac{1}{2}gt^2$$

$\frac{1}{2}gt^2 + \frac{5t}{2} - \frac{5t\sqrt{3}}{2} = 0$

$t = 0$ and $gt + 5 - 5\sqrt{3} = 0$

$\therefore\ t = 0$ and $t = \dfrac{5\left(\sqrt{3} - 1\right)}{g}$

Hence, $t = \dfrac{5\left(\sqrt{3} - 1\right)}{g}$ seconds is the time when the ball is in the air.

So, *a* = 5 and *b* = 3.

SECTION 2

Mark allocation: 5 marks

- 1 mark for the first antidifferentiation and determining $\underset{\sim}{c}_1$.

$$\underset{\sim}{v}(t) = \left(\frac{5\sqrt{3}}{2} - 0.2t^2\right)\underset{\sim}{i} - \left(\frac{5}{2} + gt - 0.2t^2\right)\underset{\sim}{j}$$

- 1 mark for the second antidifferentiation and determining $\underset{\sim}{c}_2$.

$$\underset{\sim}{r}(t) = \left(\frac{5t\sqrt{3}}{2} - \frac{1}{15}t^3\right)\underset{\sim}{i} - \left(\frac{5t}{2} + \frac{1}{2}gt^2 - \frac{1}{15}t^3\right)\underset{\sim}{j}$$

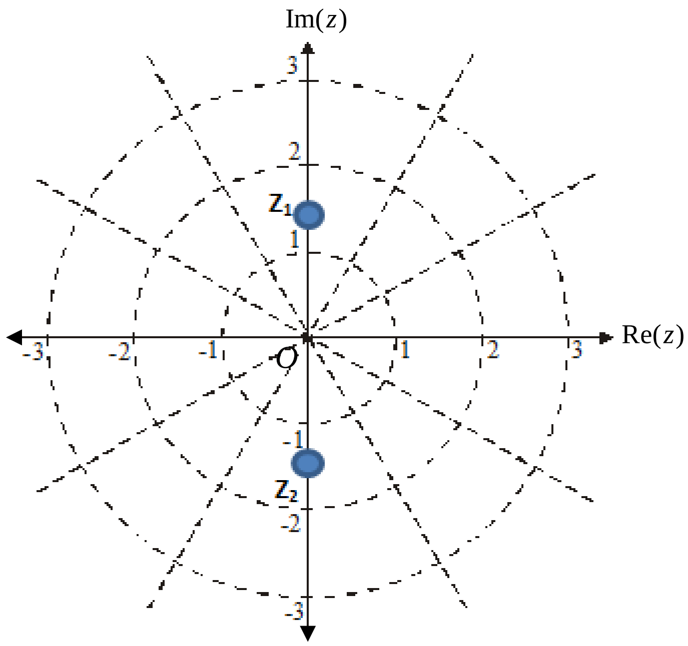
- 1 method mark for equating the components of the general position expression according to $y = -x$.

$$\frac{5t\sqrt{3}}{2} - \frac{1}{15}t^3 = \frac{5t}{2} + \frac{1}{2}gt^2 - \frac{1}{15}t^3$$

- 1 mark for factorising and solving to obtain $t = 0$ and $t = \frac{5\left(\sqrt{3}-1\right)}{g}$.
- 1 mark for evaluating $a = 5$ and $b = 3$.

Question 4a.

Worked solution



$z^2+2=0$

$z^2=-2$

$z^2=2\,\text{cis}(\pi)$ in polar form

Let $z=r\,\text{cis}\,\theta$, then

$(r\,\text{cis}\,\theta)^2=2\,\text{cis}((\pi)+2k\pi)$

$r^2\text{cis}(2\theta)=2\,\text{cis}((\pi)+2k\pi)$

$r=\sqrt{2}$ since $r>0$ and

$2\theta=\pi+2\text{k}\pi$, for $k\in Z$

$\theta=\dfrac{\pi}{2}+k\pi$, for $k\in Z$

So, $\theta=\dfrac{-\pi}{2}$ and $\dfrac{\pi}{2}$

Hence, $z_1=\sqrt{2}\,\text{cis}\left(\dfrac{\pi}{2}\right)$ and $z_2=\sqrt{2}\,\text{cis}\left(\dfrac{-\pi}{2}\right)$, when using the principal argument $-\pi<\text{Arg}\,z\le\pi$.

Mark allocation: 3 marks

- 1 method mark for using either a valid polar or Cartesian process to solve $z^2 + 2 = 0$
- 1 mark for two correct solutions
- 1 mark for plotting solutions correctly on the Argand diagram

Question 4b.

Worked solution

$|z-z_1|-|z-z_2|=1,$ where $z_1=\sqrt{2}\,i$ and $z_2=-\sqrt{2}\,i.$

Let $z=x+yi.$

$$\left|(x+yi)-\sqrt{2}\,i\right|-\left|(x+yi)+\sqrt{2}\,i\right|=1$$

$$\left|x+\left(y-\sqrt{2}\right)i\right|-\left|x+\left(y+\sqrt{2}\right)i\right|=1$$

$$\sqrt{x^2+\left(y-\sqrt{2}\right)^2}-\sqrt{x^2+\left(y+\sqrt{2}\right)^2}=1$$

$$\sqrt{x^2+\left(y-\sqrt{2}\right)^2}=1+\sqrt{x^2+\left(y+\sqrt{2}\right)^2}$$

$$x^2+\left(y-\sqrt{2}\right)^2=1+2\sqrt{x^2+\left(y+\sqrt{2}\right)^2}+x^2+\left(y+\sqrt{2}\right)^2$$

$$x^2+y^2-2\sqrt{2}y+2=1+2\sqrt{x^2+\left(y+\sqrt{2}\right)^2}+x^2+y^2+2\sqrt{2}y+2$$

$$-4\sqrt{2}y-1=2\sqrt{x^2+\left(y+\sqrt{2}\right)^2}$$

Squaring both sides gives:

$$32y^2+8\sqrt{2}y+1=4(x^2+y^2+2\sqrt{2}y+2)$$

$$32y^2+8\sqrt{2}y+1=4x^2+4y^2+8\sqrt{2}y+8)$$

$$28y^2-4x^2=7$$

Mark allocation: 3 marks

- 1 mark for substituting let $z=x+yi$ to obtain $\left|x+\left(y-\sqrt{2}\right)i\right|-\left|x+\left(y+\sqrt{2}\right)i\right|=1$
- 1 mark for squaring both sides $x^2+\left(y-\sqrt{2}\right)^2=1+2\sqrt{x^2+\left(y+\sqrt{2}\right)^2}+x^2+\left(y+\sqrt{2}\right)^2$
- 1 mark for simplifying to $28y^2-4x^2=7$

Question 4c.

Worked solution

$$28y^2 - 4x^2 = 7$$

$$4y^2 - \frac{4}{7}x^2 = 1$$

$$\frac{y^2}{\frac{1}{4}} - \frac{x^2}{\frac{7}{4}} = 1$$

So, $a = \frac{\sqrt{7}}{2}$ and $b = \frac{1}{2}$.

Vertices are $\left(0, \pm\frac{1}{2}\right)$.

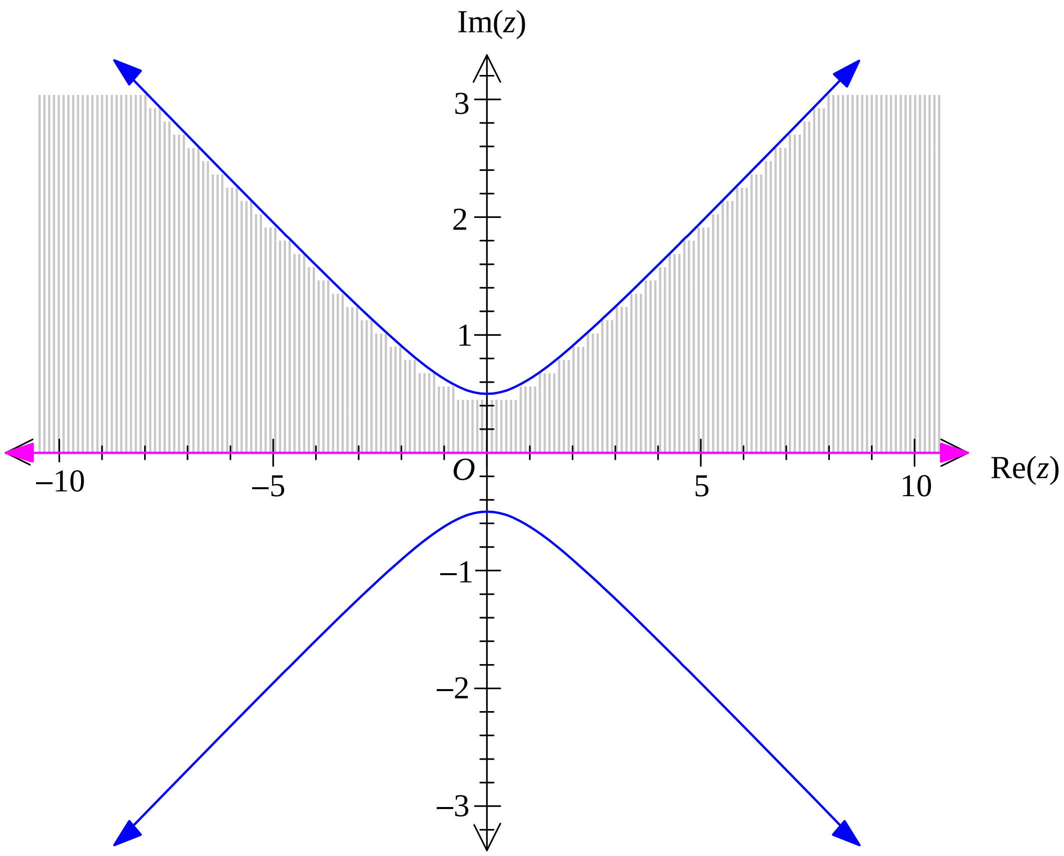
Asymptotes are $y = \pm\frac{\frac{1}{2}}{\frac{\sqrt{7}}{2}}x$ or $y = \pm\frac{\sqrt{7}}{7}x$.

Mark allocation: 3 marks

- 1 mark for writing hyperbola as $\frac{y^2}{\frac{1}{4}} - \frac{x^2}{\frac{7}{4}} = 1$ and identifying $a = \frac{\sqrt{7}}{2}$ and $b = \frac{1}{2}$.
- 1 mark for vertices $\left(0, \pm\frac{1}{2}\right)$.
- 1 mark for asymptotes $y = \pm\frac{\sqrt{7}}{7}x$.

Question 4d.

Worked solution



Mark allocation: 2 marks

- 1 mark for sketching the hyperbola $28y^2 - 4x^2 = 7$ and straight line $y = 0$.
- 1 mark for shading the area between Re(z) axis and the upper branch of hyperbola, including boundaries.

Question 4e.

Worked solution

The solutions are $z=\pm\sqrt{2}i$ and $z=a\pm ai$ (conjugate root theorem), so the factors are:

$$(z+\sqrt{2}i)(z-\sqrt{2}\,i)[(z-a)+ai)][(z-a)-ai)]=0$$
$$(z^2+2)[(z-a)^2+a^2]=0$$
$$(z^2+2)(z^2-2az+2a^2)=0$$
$$z^4-2az^3+(2a^2+2)z^2-4az+4a^2=0$$

Equating coefficients with $z^4-2az^3+2\left(3a+1\right)z^2-4az+12a=0$ gives $2a^2+2=2\left(3a+1\right)$ or $4a^2=12a$.

Solves to either a = 0 or 3.

So, $a=3$ since $a\in R\setminus\{0\}$.

Mark allocation: 3 marks

- 1 mark for finding all four solutions in terms of a.
- 1 mark for expanding factorised version to $z^4-2az^3+\left(2a^2+2\right)z^2-4az+4a^2=0$.
- 1 mark for equating coefficients to solve for a = 3.

Question 5a.

Worked solution

$$y = \frac{1}{2k}\left(e^{kx} + e^{-kx} - 2\right)$$

The point (a, b) lies on the curve, so the height of each post is $b = \frac{1}{2k}\left(e^{ak} + e^{-ak} - 2\right)$.

Mark allocation:1 mark

- 1 mark for $b = \frac{1}{2k}\left(e^{ak} + e^{-ak} - 2\right)$.

Question 5b.

Worked solution

$$y = \frac{1}{2k}\left(e^{kx} + e^{-kx} - 2\right)$$

$$\frac{dy}{dx} = \frac{1}{2k}\left(ke^{kx} - ke^{-kx}\right)$$

$$\frac{dy}{dx} = \frac{1}{2}\left(e^{kx} - e^{-kx}\right)$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{4}\left(e^{kx} - e^{-kx}\right)\left(e^{kx} - e^{-kx}\right)$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{4}\left(e^{2kx} - 1 - 1 + e^{-2kx}\right)$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{4}\left(e^{2kx} + e^{-2kx} - 2\right)$$

Mark allocation: 2 marks

- 1 mark for $\frac{dy}{dx} = \frac{1}{2}\left(e^{kx} - e^{-kx}\right)$.
- 1 mark for squaring to get $\left(\frac{dy}{dx}\right)^2 = \frac{1}{4}\left(e^{2kx} + e^{-2kx} - 2\right)$.

Question 5c.

Worked solution

$$l = 2\int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$l = 2\int_0^a \sqrt{1+\frac{1}{4}\left(e^{2kx}+e^{-2kx}-2\right)}\,dx$$

$$l = 2\int_0^a \sqrt{\frac{4+e^{2kx}+e^{-2kx}-2}{4}}\,dx$$

$$l = 2\int_0^a \sqrt{\frac{e^{2kx}+2+e^{-2kx}}{4}}\,dx$$

$$l = 2\int_0^a \sqrt{\left(\frac{e^{kx}+e^{-kx}}{2}\right)^2}\,dx$$

$$l = \int_0^a (e^{kx}+e^{-kx})\,dx$$

$$l = \left[\frac{1}{k}e^{kx}-\frac{1}{k}e^{-kx}\right]_0^a$$

$$l = \left(\frac{1}{k}e^{ak}-\frac{1}{k}e^{-ak}\right)-\left(\frac{1}{k}-\frac{1}{k}\right)$$

$$l = \left(\frac{1}{k}e^{ak}-\frac{1}{k}e^{-ak}\right)$$

Mark allocation: 4 marks

- 1 mark for substituting $\left(\frac{dy}{dx}\right)^2$ to obtain $l = 2\int_0^a \sqrt{\frac{4+e^{2kx}+e^{-2kx}-2}{4}}\,dx$.
- 1 mark for simplifying to $l = 2\int_0^a \sqrt{\left(\frac{e^{kx}+e^{-kx}}{2}\right)^2}\,dx$.
- 1 mark for finding square root of integrand $l = \int_0^a (e^{kx}+e^{-kx})\,dx$.
- 1 mark for integrating and evaluating to obtain $l = \left(\frac{1}{k}e^{ak}-\frac{1}{k}e^{-ak}\right)$.

Question 5d.

Worked solution

Substitute $k=\frac{1}{8}$ and $b=5$ into $b=\frac{1}{2k}\left(e^{ak}+e^{-ak}-2\right)$ to obtain $5=a\left(e^{\frac{a}{8}}+e^{\frac{-a}{8}}-2\right)$.

a = 8.53385… (Using *solve* function on CAS.)

Substituting this value of a into $l=\left(\frac{1}{k}e^{ak}-\frac{1}{k}e^{-ak}\right)$ gives:

l = 20.49 metres

Mark allocation: 3 marks

- 1 mark for substituting $k=\frac{1}{8}$ and $b=5$ into $b=\frac{1}{2k}\left(e^{ak}+e^{-ak}-2\right)$ to obtain a.
- 1 mark for using CAS to evaluate a = 8.53385…
- 1 mark for using $l=\left(\frac{1}{k}e^{ak}-\frac{1}{k}e^{-ak}\right)$ to evaluate l = 20.49 metres.

END OF SOLUTIONS BOOK