The Mathematical Association of Victoria

Trial Exam 2013

SPECIALIST MATHEMATICS

Written Examination 2

STUDENT NAME ____

Reading time: 15 minutes
Writing time: 2 hours

QUESTION AND ANSWER BOOK

Structure of Book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
1	22	22	22
2	5	5	58
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set squares, aids for curve sketching, one bound reference, one approved CAS calculator or CAS software and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Materials supplied
- Question and answer book of 23 pages with a detachable sheet of miscellaneous formulas at the back.
- Answer sheet for multiple-choice questions.

Instructions
- Detach the formula sheet from the back of this book during reading time.
- Write your name in the space provided above on this page.
- All written responses must be in English.

At the end of the examination
- Place the answer sheet for multiple-choice questions inside the front cover of this book.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

SECTION 1

Instructions for Section 1

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Take the **acceleration due to gravity** to have magnitude g m/s^2, where g = 9.8.

Question 1

Let g be a function with domain R, such that $g(x) = \dfrac{1}{2x^2 - px + 2}$, where p is a real constant. The set of possible values of p is

A. $p \in \{-4, 4\}$

B. $p \in R \setminus \{-4, 4\}$

C. $p \in (-4, 4)$

D. $p \in R \setminus [-4, 4]$

E. $p \in [-4, 4]$

Question 2

The path of a particle has parametric equations $x = \cos(4t)$ and $y = 6\sin^2(2t)$, where $t \geq 0$. The particle moves on part of a curve defined by a

A. parabola
B. circle
C. ellipse
D. hyperbola
E. straight line

Question 3

Let g be a function with rule $g(x) = 2\sin^{-1}\left(\dfrac{x+1}{3}\right) - \dfrac{\pi}{4}$. The maximal domain and range of g are, respectively,

A. $\left[-\dfrac{3\pi}{4}, \dfrac{3\pi}{4}\right]$ and $[-2, 2]$

B. $\left[-\dfrac{3\pi}{4}, \dfrac{5\pi}{4}\right]$ and $[-4, 2]$

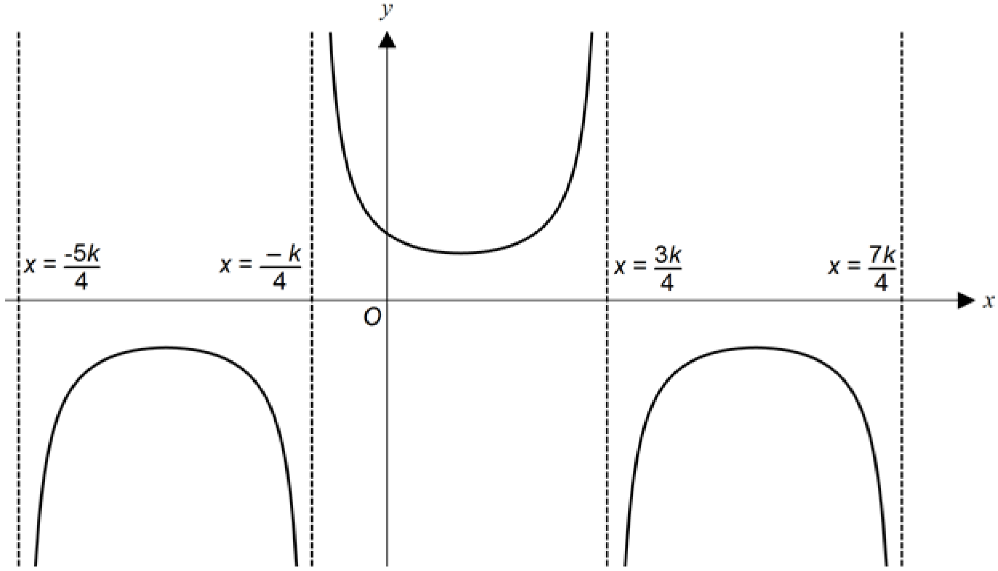
C. $[-4, 2]$ and $\left[-\dfrac{5\pi}{4}, \dfrac{3\pi}{4}\right]$

D. $[-4, 2]$ and $\left[-\dfrac{3\pi}{4}, \dfrac{5\pi}{4}\right]$

E. $[-2, 2]$ and $\left[-\dfrac{5\pi}{4}, \dfrac{5\pi}{4}\right]$

Question 4

The graph of $y = f(x)$ is shown.



If k is a real number, the rule of f could be

A. $f(x) = \sec\left(k\left(x - \frac{\pi}{4}\right)\right)$

B. $f(x) = \text{cosec}\left(k\left(x - \frac{\pi}{4}\right)\right)$

C. $f(x) = \sec\left(\frac{2\pi}{k}\left(x - \frac{k}{2}\right)\right)$

D. $f(x) = \text{cosec}\left(\frac{\pi}{k}\left(x - \frac{k}{4}\right)\right)$

E. $f(x) = \sec\left(\frac{\pi}{k}\left(x - \frac{k}{4}\right)\right)$

Question 5

Given that $(-3, 2)$ is the centre of the ellipse with equation $4x^2 + mx + 9y^2 - ny + 36 = 0$, the values of m and n are

A. $m = 9$ and $n = 4$

B. $m = 6$ and $n = 4$

C. $m = 6$ and $n = -4$

D. $m = 24$ and $n = 36$

E. $m = -24$ and $n = 36$

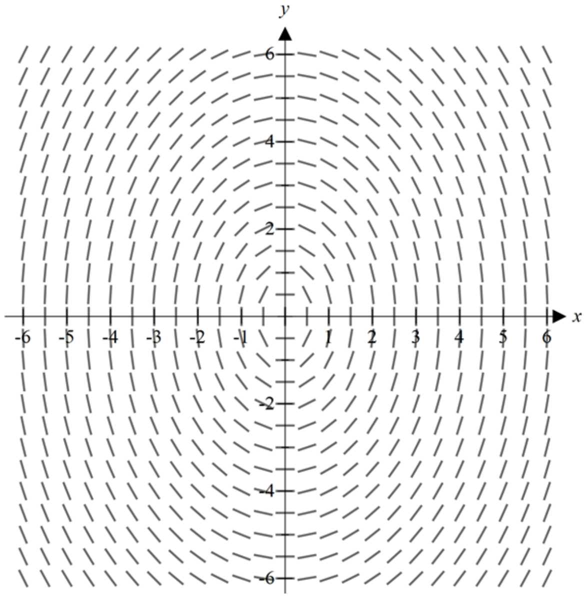
Question 6

Euler's method is used to solve the differential equation $\frac{dy}{dx} = 16^x$, with initial values $x = 1$ and $y = 0$, and a step size of $\frac{1}{4}$. The value of y when $x = \frac{3}{2}$ is

A. 4
B. 12
C. 28
D. $\dfrac{4}{\log_e(2)}$
E. $\dfrac{12}{\log_e(2)}$

Question 7

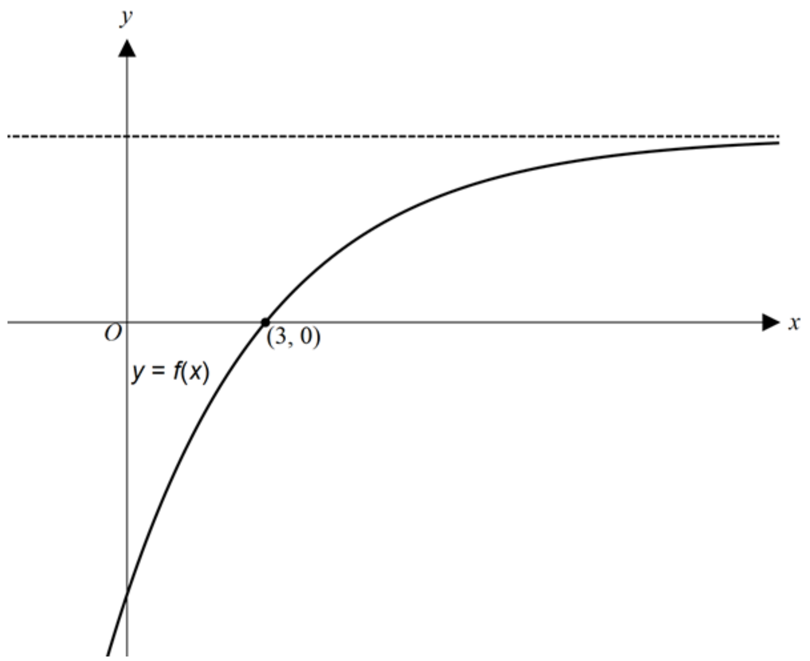
The direction field shown below is that of a particular first order differential equation.



If $m > 1$, the differential equation could be

A. $\dfrac{x^2}{m^2} + \dfrac{dy}{dx} = 1$
B. $\dfrac{x^2}{m^2} - \dfrac{dy}{dx} = 1$
C. $\dfrac{dy}{dx} = \dfrac{mx}{y}$
D. $\dfrac{dy}{dx} = -\dfrac{mx}{y}$
E. $\dfrac{dy}{dx} = -\dfrac{my}{x}$

Question 8
The graph of a smooth continuous function *f* is shown.



If $(3,0)$ is a point on the graph, which one of the following is true?

A. $f''(3) = f(3)$

B. $f''(3) > f(3)$

C. $f''(3) > f'(3)$

D. $f''(3) = f'(3)$

E. $f''(3) < f'(3)$

Question 9
If $z \in C$, which one of the following relations does **not** have a graph that is a straight line?

A. $\overline{z} - z = 0$

B. $|z+i| - |z| = 0$

C. $1 - |z+i| = 0$

D. $i^2 z + \overline{z} = 0$

E. $\text{Re}(z) + 2\,\text{Im}(z) = 0$

Question 10

Let $u = 1 - i,\ u \in C$. If $u^n = ai$, where *n* is a positive integer and *a* is a non-zero real constant, then u^{n-1} is equal to

A. $u = a(1+i)$

B. $u = a(1-i)$

C. $u = \dfrac{a}{2}(1+i)$

D. $u = \dfrac{a}{2}(1-i)$

E. $u = \dfrac{-a}{2}(1-i)$

Question 11

Let $P(z)$ be a polynomial with **integer** coefficients. If the linear factors of the polynomial include $(z+i)$, $(z-1)$ and $\left(z+1-\sqrt{2}\right)$, the degree of $P(z)$ must be **at least**

A. 2
B. 3
C. 4
D. 5
E. 6

Question 12

Given that $2\dfrac{dx}{dt} - x^2 - 4 = 0$, and that $x = 2$ at $t = 0$, then

A. $x = 2\tan\left(t + \dfrac{\pi}{4}\right)$

B. $x = 2\log_e\left(\dfrac{t^2+1}{2}\right)$

C. $x = 2\tan\left(t - \dfrac{\pi}{4}\right)$

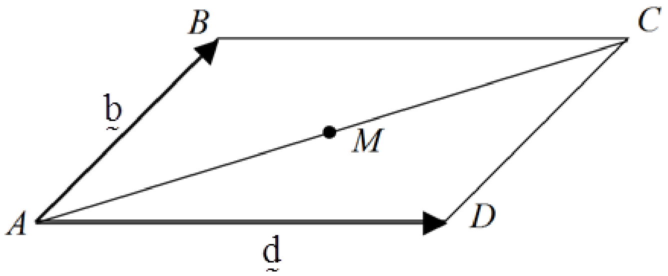
D. $x = \dfrac{1}{6}x^3 + 2x - 1$

E. $x = \dfrac{1}{2}\left(t^2 + t + 1\right)$

Question 13
If vector $\underset{\sim}{p} = 2a\underset{\sim}{i} + 3\underset{\sim}{j} - 5\underset{\sim}{k}$ is perpendicular to vector $\underset{\sim}{q} = a\underset{\sim}{i} + 4\underset{\sim}{j} - 2a\underset{\sim}{k}$, where a is a real constant, then the value of a could be

A. $a = 2$ or $a = 3$
B. $a = 4$ or $a = 3$
C. $a = -2$ or $a = -3$
D. $a = \frac{2}{3}$ or $a = 4$
E. $a = -4$ or $a = \frac{2}{3}$

Question 14
Consider the parallelogram *ABCD* below. *M* is the midpoint of *AC*.



If $\overrightarrow{AB} = \underset{\sim}{b}$ and $\overrightarrow{AD} = \underset{\sim}{d}$, then $\overrightarrow{BM}$ may be expressed as

A. $\frac{1}{2}\left(\underset{\sim}{b} + \underset{\sim}{d}\right)$
B. $\frac{1}{2}\left(\underset{\sim}{b} - \underset{\sim}{d}\right)$
C. $\frac{1}{2}\left(\underset{\sim}{d} - \underset{\sim}{b}\right)$
D. $\underset{\sim}{b} + \frac{1}{2}\left(\underset{\sim}{d} - \underset{\sim}{b}\right)$
E. $\underset{\sim}{b} - \frac{1}{2}\left(\underset{\sim}{d} - \underset{\sim}{b}\right)$

Question 15
A golf ball is hit from an origin at ground level. At time t the velocity vector of the ball is given by $\underset{\sim}{v} = 12\underset{\sim}{i} + \left(18 - 6t\right)\underset{\sim}{j},\ t \geq 0$, where $\underset{\sim}{i}$ is a unit vector in the horizontal forward direction and $\underset{\sim}{j}$ is a unit vector in the upward vertical direction.
The position vector of the golf ball at its maximum height is given by

A. $12\underset{\sim}{i} + 0\underset{\sim}{j}$
B. $36\underset{\sim}{i} + 27\underset{\sim}{j}$
C. $12\underset{\sim}{i} + 27\underset{\sim}{j}$
D. $36\underset{\sim}{i} + 54\underset{\sim}{j}$
E. $12\underset{\sim}{i} - 27\underset{\sim}{j}$

Question 16

Let P and Q be points on the graph of $xy + y^2 - x^2 - 11 = 0$, such that the gradient of the tangent to the curve at P and at Q is $\frac{1}{8}$. The coordinates of P and Q could be

A. $(2,3)$ and $(-2,-3)$

B. $(2,3)$ and $(2,-5)$

C. $(2,-5)$ and $(-2,5)$

D. $(-2,5)$ and $(-2,-3)$

E. $(2,5)$ and $(-2,3)$

Question 17

Using a suitable substitution $\int_0^{\frac{1}{2}} \left(\sqrt{\frac{\cos^{-1}(x)}{1-x^2}} \right) dx$ can be expressed completely in terms of u as

A. $\int_0^{\frac{1}{2}} \left(\sqrt{\frac{u}{1-\cos^2(u)}} \right) du$

B. $\int_0^{\frac{\pi}{6}} \left(\frac{\sqrt{u}}{\sin^2(u)} \right) du$

C. $\int_0^{\frac{1}{2}} \left(\sqrt{u} \right) du$

D. $-\int_{\frac{\pi}{2}}^{\frac{\pi}{3}} \left(\sqrt{u} \right) du$

E. $\int_{\frac{\pi}{2}}^{\frac{\pi}{3}} \left(\sqrt{u} \right) du$

Question 18

A particle moves along the positive x-axis such that $a = -\frac{4}{x^2}, x > 0$, where $a\,\text{ms}^{-2}$ is the acceleration of the particle when it is x metres from the origin. If the speed of the particle is $3\,\text{ms}^{-1}$ at $x = 1$, the particle will have a speed of $2\,\text{ms}^{-1}$ at x equal to

A. 3 m

B. 4 m

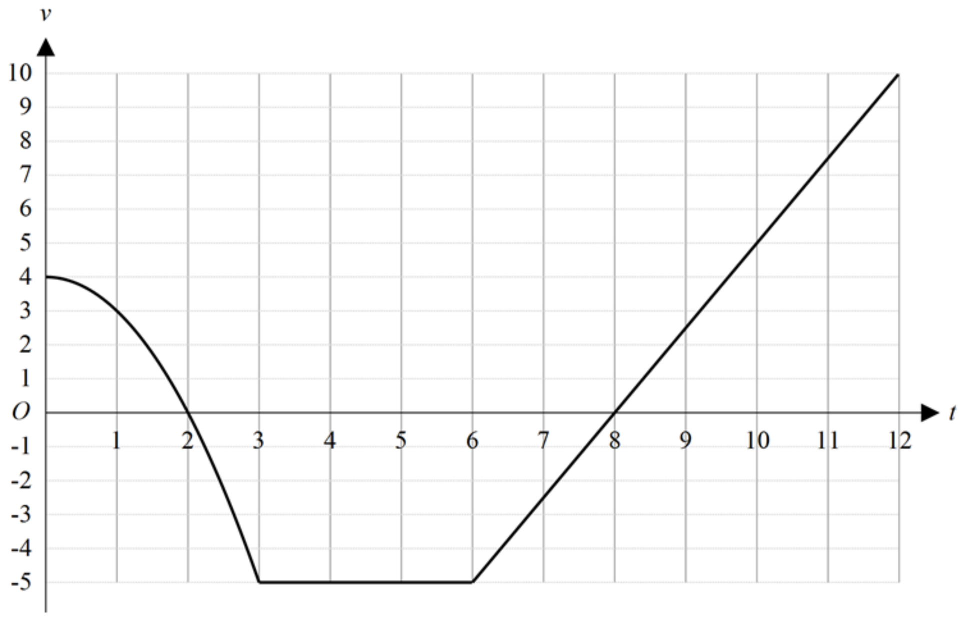
C. $\frac{8}{3}$ m

D. $\frac{3}{2}$ m

E. $\frac{8}{13}$ m

Question 19

The velocity–time graph below shows the motion of a particle travelling in a straight line, where v ms^{-1} is its velocity at time t seconds.

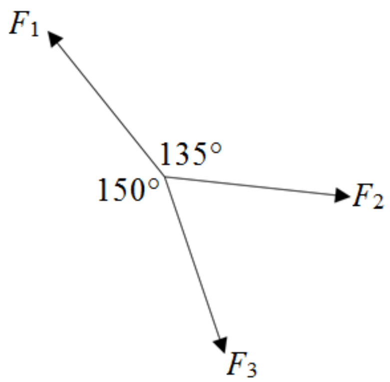


If $v = 4 - t^2$ for $0 \le t < 3$, after 12 seconds the distance of the particle from its starting point is

A. 3 m
B. 10 m
C. $\frac{1}{2}$ m
D. $\frac{17}{6}$ m
E. 43 m

Question 20

Three co-planar forces act on a particle which is in equilibrium, as shown.



The magnitude of F_1 is 100 newtons. The magnitudes of F_2 and F_3, correct to the nearest integer, are

A. F_2 = 50 newtons and F_3 = 50 newtons
B. F_2 = 52 newtons and F_3 = 73 newtons
C. F_2 = 58 newtons and F_3 = 71 newtons
D. F_2 = 87 newtons and F_3 = 71 newtons
E. F_2 = 50 newtons and F_3 = 87 newtons

Question 21

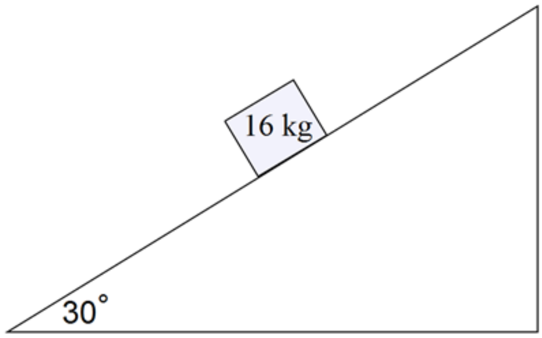
A particle of mass $\frac{4}{5}$ kg is moving in a straight line such that its speed, $v\,\text{ms}^{-1}$ at time t seconds, is given by $v = 3t^2 - 2t + 5,\ t \geq 0$.

When the particle has travelled a distance of 125 m, its momentum, in kg ms^{-1}, is

A. 5
B. 56
C. $\frac{16}{3}$
D. 40
E. 100

Question 22

The coefficient of friction between a block of mass 16 kg and a plane inclined at 30° to the horizontal is $\frac{1}{4}$.



The least force parallel to the inclined plane required to pull the block up the plane is given by

A. $8g$
B. $8\sqrt{3}\,g$
C. $\left(8+\sqrt{3}\right)g$
D. $\left(8+2\sqrt{3}\right)g$
E. $\left(8+8\sqrt{3}\right)g$

SECTION 2

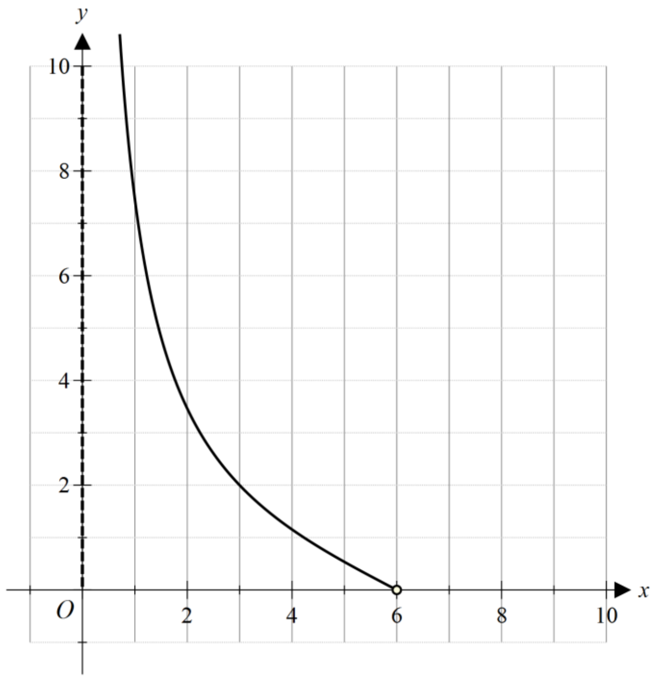
Instructions for Section 2

Answer **all** questions in the spaces provided.
Unless otherwise specified an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g m/s^2, where g = 9.8.

Question 1 **(9 marks)**

Consider the function $f:\{x:0<x<6\}\to R,\ f(x)=2\cot\left(\frac{\pi}{12}x\right)$.

A part of the graph of f is shown below.



a. On the set of axes above, sketch the graph of f^{-1}, the inverse function of f. Label the endpoint of f^{-1} with its coordinates and any asymptote with its equation. 2 marks

b. The rule of f^{-1} can be expressed as $f^{-1}(x)=k\tan^{-1}\left(\frac{2}{x}\right)$, where k is a positive real constant.

Find the exact value of k. 1 mark

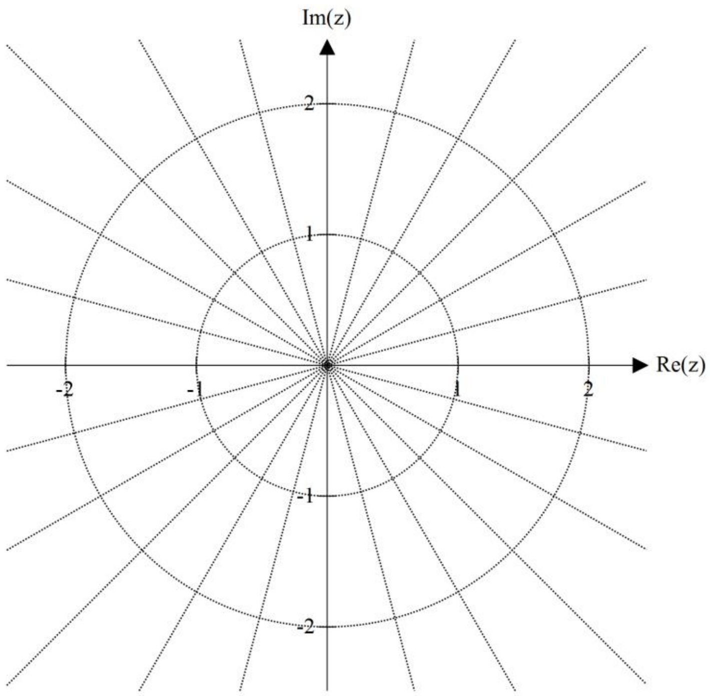
c. Show, using calculus, that the derivative of $\log_e\left(x^2+4\right)+x\tan^{-1}\left(\frac{2}{x}\right)$ is $\tan^{-1}\left(\frac{2}{x}\right)$. 3 marks

d. **Hence**, show using calculus that the area of the region bounded by the graph of f^{-1} and the *x*-axis for $0<x\le 2$ is $\dfrac{12\log_e\left(2\right)+6\pi}{\pi}$. 3 marks

Question 2 (10 marks)

In the complex plane, a set of points $S = \left\{ z : |z| = \sqrt{2} \right\}$.

a. Sketch S on the argand diagram below. 2 marks



Let T be a subset of the complex plane with equation $|z+i| = \left| z + \left(\frac{1}{2} - i\frac{\sqrt{3}}{2} \right) \right|$.

b. Show that the cartesian equation of T is given by $y = \left(2 - \sqrt{3}\right)x$. 2 marks

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c. If $\text{Arg}(z) = \phi$ is the equation of the part of T in the first quadrant, find the exact value of ϕ in radians. 1 mark

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d. Sketch T on the argand diagram above. 1 mark

e. Let $w^4 = 2 + 2\sqrt{3}\,i$.

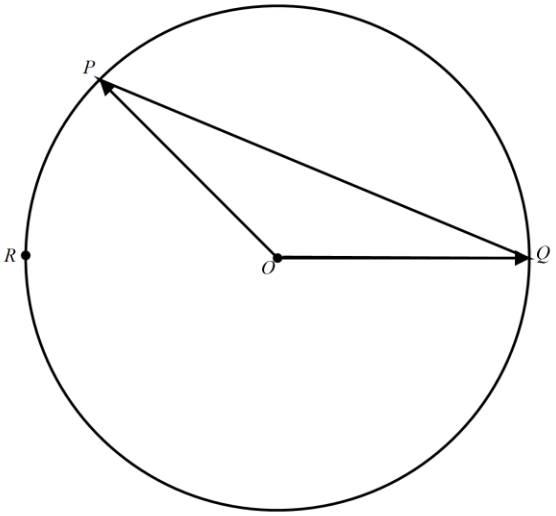
i. Express w^4 in polar form. 1 mark

ii. Find, in polar form, $\left\{w : w^4 = 2 + 2\sqrt{3}\,i\right\}$; that is, find all fourth roots of $2 + 2\sqrt{3}\,i$, in polar form. 2 marks

iii. On the argand diagram above, plot $\left\{w : w^4 = 2 + 2\sqrt{3}\,i\right\}$. Label the plotted points in the first, second, third and fourth quadrants as w_1, w_2, w_3 and w_4 , respectively. 1 mark

Question 3 (9 marks)

The diagram below shows a triangle *OPQ* which has *O* is at the centre of a circle and *P* and *Q* on the circumference of the circle. Point *R*, also on the circumference of the circle, is collinear with points *O* and *Q*.



a.

i. If $\overrightarrow{OQ} = h\underset{\sim}{i}$, where $h > 0$, and $\overrightarrow{OP} = -4\underset{\sim}{i} - \sqrt{5}\,\underset{\sim}{j} + 2\underset{\sim}{k}$, show that $h = 5$. 1 mark

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ii. Find $\overrightarrow{PR}$. 1 mark

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iii. Use a vector method to show that angle *QPR* is a right angle. 2 marks

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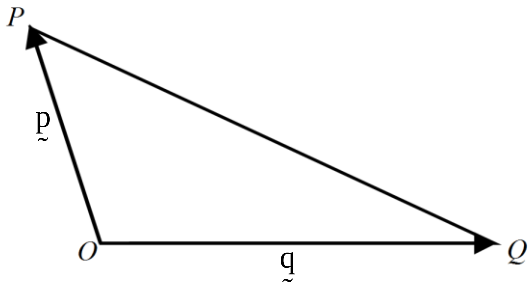
A different triangle OPQ, where O is the origin, is shown in the diagram below, with vectors $\overrightarrow{OP} = \underset{\sim}{p}$ and $\overrightarrow{OQ} = \underset{\sim}{q}$.



The points M and N are the midpoints of the line segments OP and OQ, respectively.

b. Let U be the point of intersection of the line segments PN and QM.

i. Express $\overrightarrow{PN}$ in terms of $\underset{\sim}{p}$ and $\underset{\sim}{q}$. 1 mark

ii. Express $\overrightarrow{QM}$ in terms of $\underset{\sim}{p}$ and $\underset{\sim}{q}$. 1 mark

iii. Given that $\overrightarrow{PU} = a\,\overrightarrow{PN}$ and $\overrightarrow{QU} = b\,\overrightarrow{QM}$, where a and b are scalars, use a vector method to find the values of a and b. 3 marks

Question 4 (16 marks)

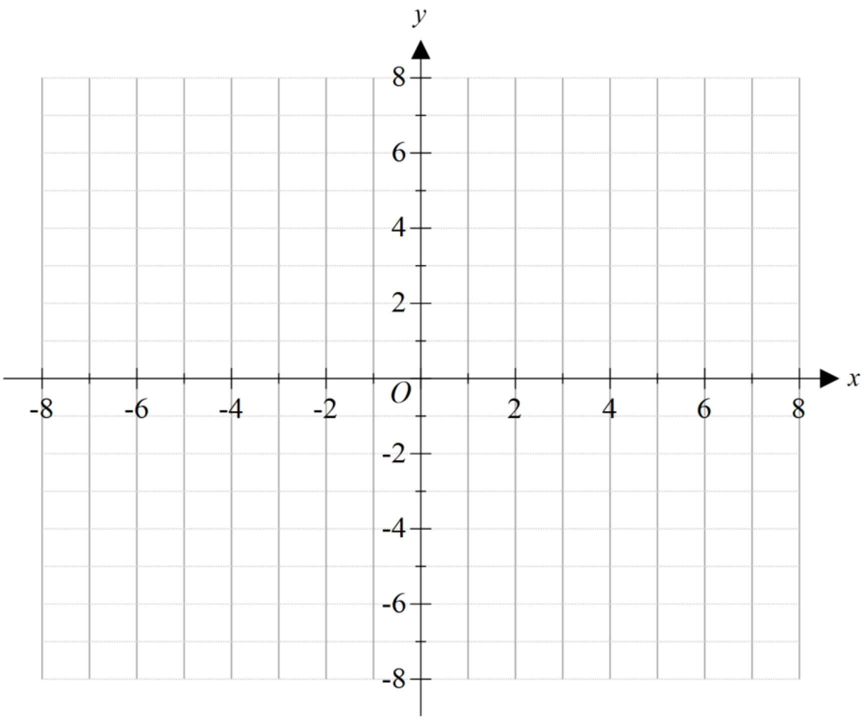
The position vector of a particle from an origin O is given by $\underset{\sim}{r} = \dfrac{2}{\cos(t)}\underset{\sim}{i} + \left(\sqrt{3}\tan(t) + 3\right)\underset{\sim}{j}$,

where $\dfrac{3\pi}{2} < t < \dfrac{5\pi}{2}$. The path of the particle traces out a curve on the cartesian plane.

a. Show that the domain of the curve is $\{x : x \in [2, \infty)\}$. 1 mark

b. Show that the curve is the right-hand branch of the hyperbola with cartesian equation $\dfrac{x^2}{4} - \dfrac{(y-3)^2}{3} = 1$. 3 marks

c. On the set of axes below, sketch the graph of the curve traced out by the particle. Label any asymptotes with their equations. 2 marks



A large storage tank at a paint factory is modelled by the volume of revolution formed when the region bounded by the curve, the coordinate axes and the line $y = 5$ is rotated about the y-axis, where the values on the axes are in metres.
The tank contains $V\ \text{m}^3$ of a liquid solvent.

d. Evaluate the exact maximum volume of solvent that the tank can hold, in m^3. 2 marks

Solvent is pumped into the tank at a constant rate of 2 m^3/hour. It can be shown that the volume V m^3 and depth y m of solvent in the tank are related by $\frac{dV}{dy} = \frac{4\pi}{3}\left(y^2 - 6y + 12\right)$

e. Find the exact rate at which the depth of solvent is increasing at the instant when the volume of solvent in the tank is 24π m^3. 3 marks

A concentrated varnish is made in the tank by dissolving 10 tonne of a resin in the solvent. The volume of varnish solution in the tank is adjusted to exactly 100 m^3. The varnish is stirred constantly while in the tank.

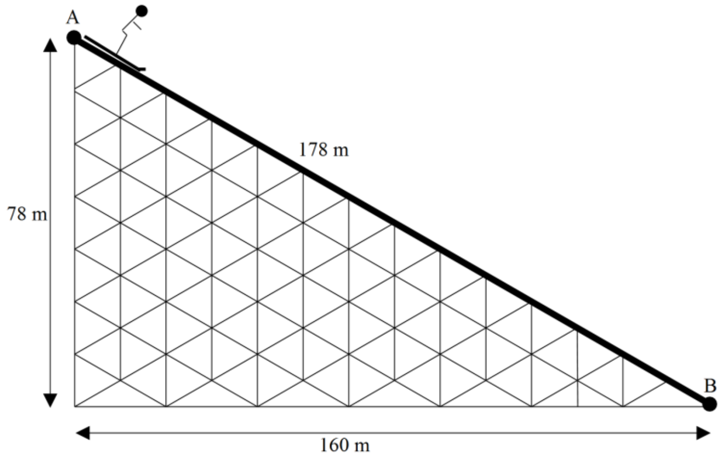
f. To adjust the concentration of resin in the tank, a dilute solution containing 0.02 tonne of resin per m^3 of solution is pumped into the vat at a constant rate of 2 m^3/hour. Varnish is pumped out of the tank at the same rate.

i. Let N tonne be the mass of resin dissolved in the tank t hours after the pumping started. Write down a differential equation relating N and t. 2 marks

ii. Pumping was stopped when the varnish in the tank reached the desired concentration of 0.05 tonne of resin per m^3 of the varnish solution. When pumping was stopped, what was the value of t, correct to the nearest hour? 3 marks

Question 5 (14 marks)

At a training session for thrill-seekers, Xue slides down a ski ramp on a snowboard. A section of the ramp, from point *A* to point *B*, is shown in the diagram above. The distance *AB* is 178 metres.
The mass of Xue and her equipment is 75 kg, and she began sliding down ramp from rest at point *A*.
The coefficient of friction between the snowboard and the ramp is $\frac{1}{5}$. Assume that any other forces acting on Xue are negligible as she travels down the ramp.



a. On the diagram above, clearly label the forces acting on Xue. 1 mark

b. Find the magnitude of Xue's acceleration as she slides down the ramp. Express the answer in ms^{-2}, correct to two decimal places. 2 marks

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c. As Xue passes point *B*, she is travelling at a speed of $k\,\text{ms}^{-1}$. Determine the value of *k* correct to the nearest integer. 2 marks

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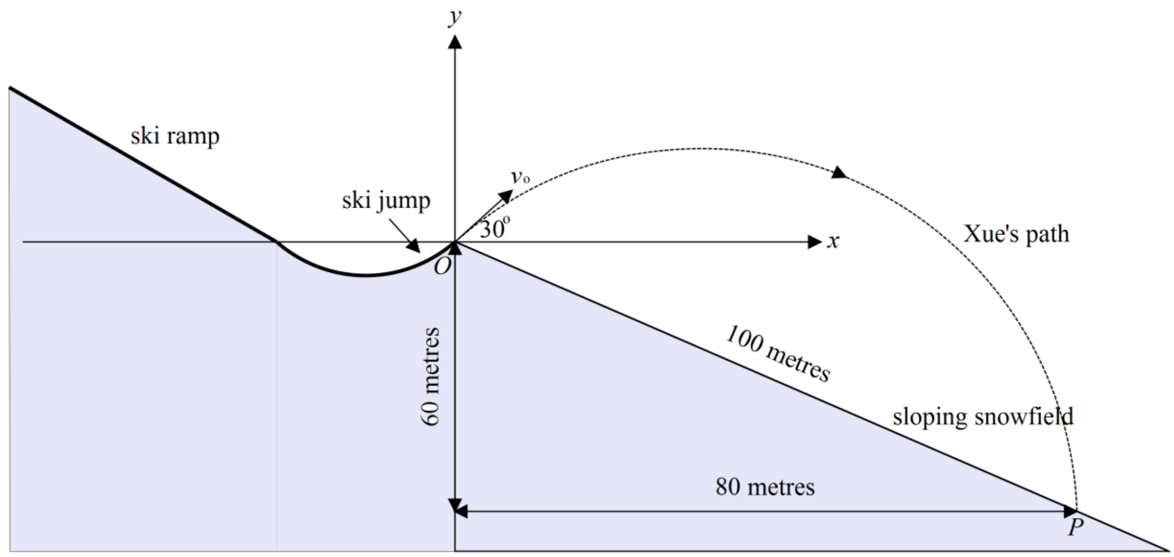
d. How long does it take Xue to reach point B? Express the answer in seconds, correct to one decimal place. 1 mark

In another training session, Xue travels down a ski ramp on her snowboard and then travels up a short ski jump. Xue leaves the ski jump at O, the origin of a cartesian coordinate system, with a speed of v_0 ms^{-1} at an angle of 30° to the horizontal, as shown on the diagram below.



Xue lands at a point P on the snowfield of constant slope, 100 metres down-hill from O.

e. The position vector of P is $80\underset{\sim}{i} - 60\underset{\sim}{j}$, where all measurements are in metres, and $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are unit vectors in the positive directions of the x and y axes, respectively. Assume that air resistance is negligible during Xue's jump from O to P.

i. The position vector of Xue as she jumps from O to P is given by $\alpha\underset{\sim}{i} + \beta\underset{\sim}{j}$. Write expressions for α and β in terms v_0, t seconds after leaving O. 3 marks

ii. Find the value of v_0, and the time taken for Xue to jump from point *O* to point *P*, in seconds. Express the answers correct to two decimal places. 2 marks

After landing at *P*, Xue continues to slide down the snowfield of constant slope. When she reaches a point *Q* her speed is $22\,\text{ms}^{-1}$. At *Q* she uses the snowboard to brake with a retarding force of $b(t)$ newtons parallel to the slope, where $b(t) = 150t + 75, t \geq 0$, and *t* is the time in seconds from when she passed point *Q*. Assume that $b(t)$ is the only retarding force acting on Xue.

f. How long after passing point *Q* does Xue come to rest? Express the answer in seconds, correct to one decimal place. 3 marks

END OF QUESTION AND ANSWER BOOKLET

