SPECIALIST MATHEMATICS

Units 3 & 4 – Written examination 1

*(*TSSM's *2012 trial exam updated for the current study design)*

SOLUTIONS

Question 1

Since $\tan^{-1}(x-20) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then

$$\frac{1}{2}\tan^{-1}(x-20)+\frac{2\pi}{3} \in \left(\frac{1}{2}\times-\frac{\pi}{2}+\frac{2\pi}{3}, \frac{1}{2}\times\frac{\pi}{2}+\frac{2\pi}{3}\right)=\left(\frac{5\pi}{12}, \frac{11\pi}{12}\right)$$ [M1]

Therefore,

$\frac{1}{2}\tan^{-1}(x-20)+\frac{2\pi}{3}=p$ has no solutions when $p \in \left(-\infty, \frac{5\pi}{12}\right] \cup \left[\frac{11\pi}{12}, \infty\right)$ [A1]

Question 2

a. $y=\frac{(x-1)^2}{2x} \Rightarrow y=\frac{x^2-2x+1}{2x}=\frac{x}{2}-1+\frac{1}{2x}$ [A1]

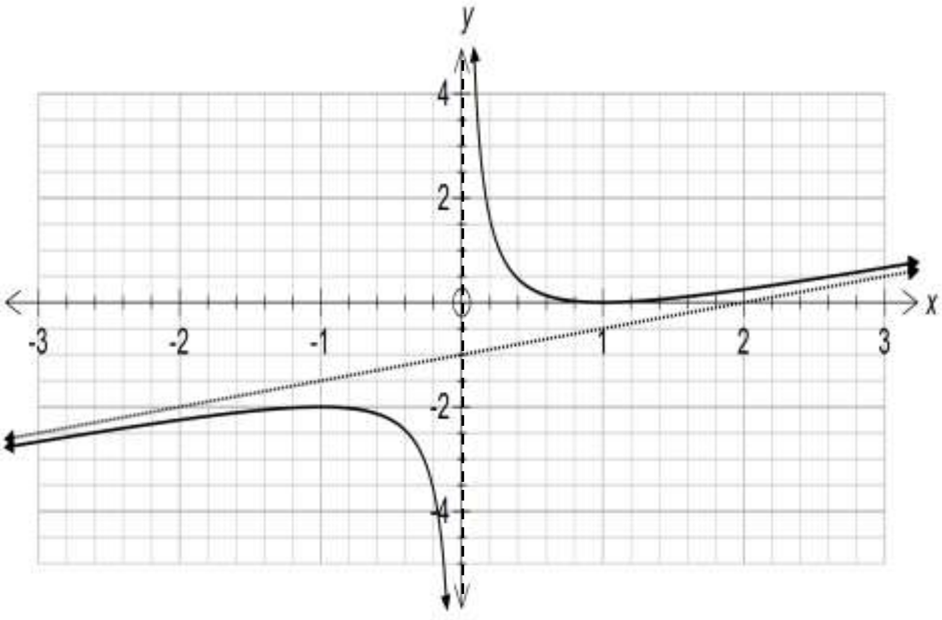
b. The x-intercept is $(1, 0)$ and the asymptotes are $y=\frac{x}{2}-1$ and $x=0$ [A1]

$$\frac{dy}{dx}=\frac{1}{2}-\frac{1}{2x^2}=\frac{x^2-1}{2x^2}$$

At a stationary point, $\frac{x^2-1}{2x^2}=0 \Rightarrow x=\pm 1$ [M1]

Therefore, the minimum and maximum turning points are $(1, 0)$ and $(-1, -2)$ respectively. [A1]

c.



[A2]

Question 3

$z = 2\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$ and $w = 4\text{cis}\left(\frac{\pi}{6}\right)$ [M1]

From de Moivre's theorem,

$$Arg\left(\frac{z}{w}\right)^3 = \frac{\text{cis}\left(-\frac{3\pi}{4}\right)}{\text{cis}\left(\frac{\pi}{2}\right)} = \text{cis}\left(-\frac{5\pi}{4}\right)$$ [M1]

Since $Arg\left(\frac{z}{w}\right)^3 \in (-\pi, \pi] \Rightarrow Arg\left(\frac{z}{w}\right)^3 = 2\pi - \frac{5\pi}{4} = \frac{3\pi}{4}$ [A1]

Question 4

$$\left.\begin{aligned} f'(x) &= \frac{1}{\sqrt{1-\left(\frac{1}{2x}\right)^2}} \times \frac{d}{dx}\left(\frac{1}{2x}\right) \\ &= -\frac{\frac{1}{2x^2}}{\sqrt{\frac{4x^2-1}{4x^2}}} \\ &= -\sqrt{\frac{1}{x^2\left(4x^2-1\right)}} \\ &= \frac{-1}{|x|\sqrt{4x^2-1}},\ x \in \left(-\infty, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right) \end{aligned}\right\}$$ [M2]

Since $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

$$f'(x) = \begin{cases} \dfrac{1}{x\sqrt{4x^2-1}}, & x \in \left(-\infty, -\dfrac{1}{2}\right) \\ \dfrac{-1}{x\sqrt{4x^2-1}}, & x \in \left(\dfrac{1}{2}, \infty\right) \end{cases}$$ [A2]

Question 5

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right)=\frac{x}{1-x^2}$$

$$\frac{1}{2}v^2=\int\frac{x}{1-x^2}\,dx$$ [M1]

$$u=1-x^2$$

$$-\frac{1}{2}\frac{du}{dx}=x$$

$$\frac{1}{2}v^2=-\frac{1}{2}\int\frac{1}{u}\,du$$

$$v^2=-\log_e\left|1-x^2\right|+c$$ [M1]

Given $v(2)=-1$

$$1=-\log_e\left|-3\right|+c$$

$$c=1+\log_e\left|-3\right|$$

$$v^2=1+\log_e\left|-3\right|-\log_e\left|1-x^2\right|$$

$$=1+\log_e\left|\frac{3}{x^2-1}\right|$$ [M1]

And since $v(2)=-1$ then $v(x)=-\sqrt{1+\log_e\left|\frac{3}{x^2-1}\right|}$ [A1]

Question 6

a. $\overrightarrow{OB}=\underset{\sim}{a}+\underset{\sim}{b}$ and $\overrightarrow{AC}=\underset{\sim}{b}-\underset{\sim}{a}$ [A1]

b. If $\overrightarrow{OB}\perp\overrightarrow{AC}$

It follows that

$$(\underset{\sim}{a}+\underset{\sim}{b})\cdot(\underset{\sim}{b}-\underset{\sim}{a})=0$$

$$\underset{\sim}{a}\cdot\underset{\sim}{b}-\underset{\sim}{a}\cdot\underset{\sim}{a}+\underset{\sim}{b}\cdot\underset{\sim}{b}-b\cdot\underset{\sim}{a}=0$$

$$-\underset{\sim}{a}\cdot\underset{\sim}{a}+\underset{\sim}{b}\cdot\underset{\sim}{b}=0$$

$$-|\underset{\sim}{a}|^2+|\underset{\sim}{b}|^2=0$$

$$|\underset{\sim}{a}|^2=|\underset{\sim}{b}|^2$$

$$|\underset{\sim}{a}|=|\underset{\sim}{b}|$$ [M2]

Therefore, $|\underset{\sim}{a}|=|\underset{\sim}{b}|$ as required [A1]

Question 7

$$\frac{d}{dx}\left(x^2y+2y^2\right)=0$$

$$2xy+x^2\frac{dy}{dx}+4y\frac{dy}{dx}=0$$

$$2xy+\frac{dy}{dx}\left(x^2+4y\right)=0$$

$$\frac{dy}{dx}=-\frac{2xy}{x^2+4y}$$

[M1]

When $x=1$,

$$y+2y^2=6$$

$$2y^2+y-6=0$$

$$\left(2y-3\right)\left(y+2\right)=0$$

$$y=\frac{3}{2} \text{ since } y>0$$

[M1]

Therefore, $\frac{dy}{dx}=-\frac{2\times1\times\frac{3}{2}}{1+4\times\frac{3}{2}}=-\frac{3}{7}$ and so the gradient of the normal is $\frac{7}{3}$ [M1]

It follows that the equation of the normal is

$y-\frac{3}{2}=\frac{7}{3}\left(x-1\right)\Rightarrow y=\frac{7}{3}x-\frac{5}{6}$ [A1]

Question 8

$$\int_0^{\frac{\pi}{4}} \frac{\left(\sec^2\theta - 1\right)}{\cos^2\theta}\, d\theta = \int_0^{\frac{\pi}{4}} \frac{\tan^2\theta}{\cos^2\theta}\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \tan^2\theta \sec^2\theta \, d\theta$$

[M1]

Let $u = \tan\theta$

$$\frac{du}{d\theta} = \sec^2\theta$$

$\theta = 0 \Rightarrow u = 0$

$\theta = 0 \Rightarrow u = 1$

[M1]

Therefore,

$$\int_0^{\frac{\pi}{4}} \frac{\left(\sec^2\theta - 1\right)}{\cos^2\theta}\, d\theta = \int_0^{1} u^2 \frac{du}{d\theta}\, d\theta$$

$$= \int_0^1 u^2 du$$

$$= \left[\frac{u^3}{3}\right]_0^1$$

$$= \frac{1}{3}$$

[M1] [A1}

Question 9

At the point of intersection, $x = e^2 \Rightarrow y = \log_e e^2 = 2$ [M1]

Also $y = \log_e x \Rightarrow x = e^y$

The required volume, V is calculated as

$$V = \pi\int_0^2 \left(e^2\right)^2 dy - \pi\int_0^2 \left(e^y\right)^2 dy$$

$$= \pi\int_0^2 e^4 dy - \pi\int_0^2 e^{2y} dy$$

$$= \pi\left[e^4 y\right]_0^2 - \frac{\pi}{2}\left[e^{2y}\right]_0^2$$

$$= 2\pi e^4 - \frac{\pi}{2}\left(e^4 - 1\right)$$

$$= \frac{\pi}{2}\left(3e^4 + 1\right) units^3$$

[M2] [A1}

Question 10

a.

$$T - 4g\sin(30°) = 4a$$
$$5g - T = 5a \rightarrow T = 5g - 5a$$
$$5g - 5a - 2g = 4a$$ [M2]
$$9a = 3g$$
$$a = \frac{g}{3} m/s^2$$

[A1]

b.

$$T = 5g - 5a = 5g - \frac{5g}{3} = \frac{10g}{3} \ N$$