Neap:

Trial Examination 2012

VCE Specialist Mathematics Units 3 & 4

Written Examination 2

Suggested Solutions

SECTION 1

1	A	B	C	**D**	E
2	A	B	**C**	D	E
3	A	B	C	D	**E**
4	A	B	**C**	D	E
5	A	B	C	**D**	E
6	A	B	**C**	D	E
7	A	B	C	**D**	E
8	A	B	C	D	**E**
9	A	B	C	**D**	E
10	A	B	C	**D**	E
11	A	B	**C**	D	E
12	A	B	C	**D**	E
13	A	B	C	D	**E**
14	**A**	B	C	D	E
15	**A**	B	C	D	E
16	A	B	C	**D**	E
17	**A**	B	C	D	E
18	**A**	B	C	D	E
19	A	B	C	D	**E**
20	A	**B**	C	D	E
21	A	B	C	**D**	E
22	**A**	B	C	D	E

Question 1 D

Using the Proper Fraction command with a CAS,

$$\frac{2x^2 - x + 3}{4 - x} = -\frac{31}{x-4} - 2x - 7$$

oblique asymptote occurs as $x \to \pm\infty$ $\therefore$ oblique asymptote $y = -2x - 7$

Question 2 C

The ellipse has centre $(-5, 3)$, a horizontal semi-axis of 5, and a vertical semi-axis of 3.

Therefore $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ becomes $\frac{(x-(-5))^2}{5^2} + \frac{(y-3)^2}{3^2} = 1$

$$\therefore 225\left(\frac{(x+5)^2}{25} + \frac{(y-3)^2}{9}\right) = 1$$

$\therefore (9(x+5)^2 + 25(y-3)^2) = 225$ gives **C**.

Question 3 E

A hyperbola of the form $\frac{x^2}{9} - \frac{y^2}{16} = 1$ has x-intercepts of $(-3, 0)$ and $(3, 0)$, and therefore domain $x \in R \setminus (-3, 3)$.

Translating this hyperbola 2 units to the right, $\frac{(x-2)^2}{9} - \frac{y^2}{16} = 1$ will give domain $x \in R \setminus \{-3+2, 3+2\}$, i.e. $x \in R\setminus\{-1, 5\}$.

Question 4 C

The range of $y = \sin^{-1}(x)$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

The range of $y = a\sin^{-1}(x)$ is $\left[-\frac{\pi}{2} \times a, \frac{\pi}{2} \times a\right]$

The range of $y = a\sin^{-1}(x) + c$ is $\left[-\frac{a\pi}{2} + c, \frac{a\pi}{2} + c\right]$, which is equivalent to $\left[\frac{-a\pi - 2c}{2}, \frac{a\pi + 2c}{2}\right]$

Question 5 **D**

If $z = r\text{cis}(\theta)$, then $\dfrac{\bar{z}}{i} = \dfrac{r\text{cis}(-\theta)}{\text{cis}\left(\frac{\pi}{2}\right)} = r\text{cis}\left(-\theta - \frac{\pi}{2}\right) = r\text{cis} + \left[-\left(\theta + \frac{\pi}{2}\right)\right]$.

This represents a rotation of $\frac{\pi}{2}(90°)$ counter-clockwise followed by a reflection of z in the real axis giving alternative **D**.

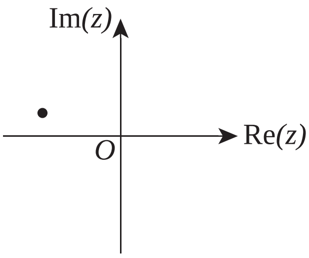
Or let $z = -a + bi$ where $a > 0$, $b > 0$

$$\therefore \frac{\bar{z}}{i} = \frac{-a - bi}{i}$$

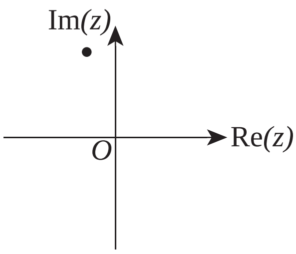
$$= \frac{-a - bi}{i} \times \frac{i}{i}$$

$$= ai - b$$

So



becomes



Question 6 **C**

If $z^5 = ai$

$$z^5 = a\text{cis}\left(\frac{\pi}{2}\right)$$

$$z = \sqrt[5]{a}\,\text{cis}\left(\frac{\pi}{10} + \frac{2k\pi}{5}\right),\ k = 0, 1, 2, 3, 4$$

$$= \sqrt[5]{a}\,\text{cis}\left(\frac{\pi}{10} \text{ or } \frac{5\pi}{10} \text{ or } \frac{9\pi}{10} \text{ or } \frac{13\pi}{10} \text{ or } \frac{17\pi}{10}\right)$$

$$= \sqrt[5]{a}\,\text{cis}\left(\frac{\pi}{10} \text{ or } \frac{\pi}{2} \text{ or } \frac{9\pi}{10} \text{ or } \frac{-7\pi}{10} \text{ or } \frac{-3\pi}{10}\right)$$

We require $\text{Arg}(z) > 0$. This leaves $\frac{\pi}{10}, \frac{\pi}{2}, \frac{9\pi}{10}$.

There are 3 solutions.

Question 7 **D**

As all coefficients are real, $z = 1 + i$ is also a solution.

$$(z - 3)(z - 1 + i)(z - 1 - i) = 0$$

$$z^3 - 5z^2 + 8z - 6 = 0$$

$$z^3 - 5z^2 + 8z = 6$$

Question 8 E

$$|z - ai| - |z + a| = 0$$

$$|z - ai| = |z + a|$$

$$|x + (y - a)i| = |(x + a) + yi|$$

$$\sqrt{x^2 + (y - a)^2} = \sqrt{(x + a)^2 + y^2}$$

$$x^2 + y^2 - 2ay + a^2 = x^2 + 2ax + a^2 + y^2$$

$$-2ay = 2ax$$

$$y = -x$$

i.e. a straight line through origin, gradient = –1

Question 9 D

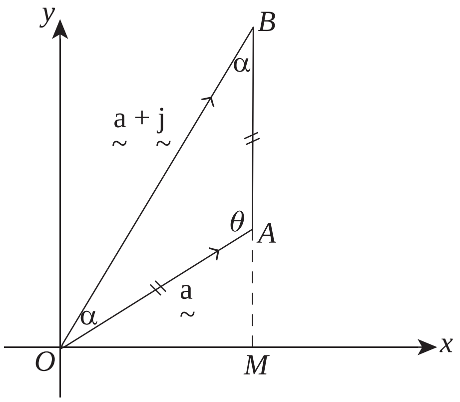
Using the Algebra Solve command with a CAS suggests $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ or 2π over $x \in [0, 2\pi]$

However, $\cos(x) \neq 0$

$$\therefore x \neq \frac{\pi}{2} \text{ or } \frac{3\pi}{2}$$

$$\therefore x = 0 \text{ or } \pi \text{ or } 2\pi, \text{ which sum to } 3\pi.$$

Question 10 D



$\underset{\sim}{a} = \frac{1}{\sqrt{2}}(\underset{\sim}{i} + \underset{\sim}{j}) \Rightarrow |\underset{\sim}{a}| = \frac{1}{\sqrt{2}}\sqrt{1^2 + 1^2} = 1$

Now $\overrightarrow{OB} = \underset{\sim}{a} + \underset{\sim}{j} \Rightarrow \overrightarrow{AB} = \underset{\sim}{j}$. Thus $|\overrightarrow{AB}| = 1$

So triangle *BOA* is isosceles with $\overrightarrow{OA} = \overrightarrow{AB}$

In triangle *OAM*, $|\overrightarrow{OM}| = |\overrightarrow{AM}| = \frac{1}{\sqrt{2}}$, thus $\angle AOM = \angle OAM = \frac{\pi}{4}$

So $\theta = \frac{3\pi}{4}$, giving $2\alpha = \frac{\pi}{4} \Rightarrow \alpha = \frac{\pi}{8}$

Using the sine rule, area $OAB = \frac{1}{2} \times |\overrightarrow{OA}| \times |\overrightarrow{AB}| \times \sin(\theta) = \frac{1}{2} \times 1 \times 1 \times \sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{4}$

Question 11 C

Let unit vectors in the East, North and up directions be represented by $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$, respectively.

The resultant force, $\underset{\sim}{R}$, is given by $\underset{\sim}{R} = 1200\underset{\sim}{i} + 500\underset{\sim}{j} + 1700\underset{\sim}{k}$

The force in the horizontal plane, $\underset{\sim}{H}$, is given by $\underset{\sim}{H} = 1200\underset{\sim}{i} + 500\underset{\sim}{j}$

The angle, $\theta°$, is determined from $\cos(\theta) = \frac{\underset{\sim}{R} \cdot \underset{\sim}{H}}{|\underset{\sim}{R}||\underset{\sim}{H}|}$

$\underset{\sim}{R} \cdot \underset{\sim}{H} = (1200\underset{\sim}{i} + 500\underset{\sim}{j} + 1700\underset{\sim}{k}) \cdot (1200\underset{\sim}{i} + 500\underset{\sim}{j}) = 1200^2 + 500^2 = 1300^2$

$|\underset{\sim}{R}| = \sqrt{1200^2 + 500^2 + 1700^2} = 2140.9$ and $|\underset{\sim}{H}| = \sqrt{1200^2 + 500^2} = 1300$

Thus $\cos(\theta) = \frac{1300^2}{2140.9 \times 1300} = \frac{1300}{2140.9}$, giving $\theta = 52.6°$

Question 12 D

$\underset{\sim}{a} = 4\underset{\sim}{i} + 2\underset{\sim}{j} + 4\underset{\sim}{k}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$, so $\underset{\sim}{a} - \underset{\sim}{b} = \underset{\sim}{i} + \underset{\sim}{k}$.

If the angle between $\underset{\sim}{a}$ and $\underset{\sim}{a} - \underset{\sim}{b}$ is θ, then $\cos(\theta) = \dfrac{(\underset{\sim}{a} - \underset{\sim}{b}) \cdot \underset{\sim}{a}}{|\underset{\sim}{a} - \underset{\sim}{b}||\underset{\sim}{a}|} = \dfrac{(\underset{\sim}{i} + \underset{\sim}{k}) \cdot (4\underset{\sim}{i} + 2\underset{\sim}{j} + 4\underset{\sim}{k})}{\sqrt{1^2 + 1^2}\sqrt{4^2 + 2^2 + 4^2}}$

$$\cos(\theta) = \frac{4 + 4}{\sqrt{2} \cdot \sqrt{36}} = \frac{8}{6\sqrt{2}} = \frac{2\sqrt{2}}{3}$$

$$\theta = \arccos\left(\frac{2\sqrt{2}}{3}\right)$$

Question 13 E

$x = \tan(\theta) \Rightarrow \dfrac{dx}{d\theta} = \sec^2(\theta)$ and $y = \tan(2\theta) \Rightarrow \dfrac{dy}{d\theta} = 2\sec^2(2\theta)$

Hence $\dfrac{dy}{dx} = \dfrac{dy}{d\theta} \times \dfrac{d\theta}{dx} = 2\sec^2(2\theta) \times \dfrac{1}{\sec^2(\theta)}$

Thus $\dfrac{dy}{dx} = \dfrac{2\cos^2(\theta)}{\cos^2(2\theta)}$

At $\theta = \dfrac{\pi}{6}$, $\dfrac{dy}{dx} = \dfrac{2\cos^2\left(\frac{\pi}{6}\right)}{\cos^2\left(\frac{\pi}{3}\right)} = \dfrac{2\left(\frac{\sqrt{3}}{2}\right)^2}{\left(\frac{1}{2}\right)^2} = \dfrac{\frac{6}{4}}{\frac{1}{4}} = 6$

So the gradient of the normal equals $-\dfrac{1}{6}$

$\theta = \dfrac{\pi}{6}$, $x = \tan\left(\dfrac{\pi}{6}\right) = \dfrac{1}{\sqrt{3}}$ and $y = \tan\left(\dfrac{\pi}{3}\right) = \sqrt{3}$

The equation of the normal is $y - \sqrt{3} = -\dfrac{1}{6}\left(x - \dfrac{1}{\sqrt{3}}\right) \Rightarrow y = -\dfrac{1}{6}x + \dfrac{1}{6\sqrt{3}} + \sqrt{3}$

Multiplying both sides by 18 gives

$18y = -3x + \dfrac{18}{6\sqrt{3}} + 18\sqrt{3}$, i.e. $3x + 18y = 19\sqrt{3}$

Question 14 A

$\int_0^{\frac{\pi}{4}} \frac{\cos(2x)}{\sqrt{1+2\sin(2x)}}dx$ can be simplified in a number of ways, but the denominator in the alternatives suggests

$\sqrt{1+u}$ or $\sqrt{1+2u}$ where $u = \sin(2x)$ or $u = 2\sin(2x)$

Let $u = 2\sin(2x) \Rightarrow du = 4\cos(2x)dx$ when $x = 0$, $u = 0$, $x = \frac{\pi}{2}$, $u = 2$

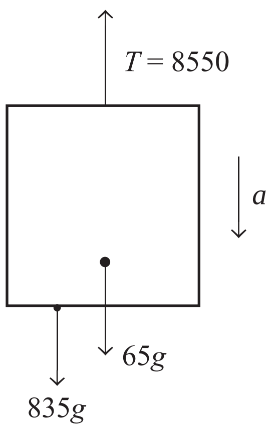
Thus $\int_0^{\frac{\pi}{4}} \frac{\cos(2x)}{\sqrt{1+2\sin(2x)}}dx$ becomes $\frac{1}{4}\int_0^{\frac{\pi}{4}} \frac{4\cos(2x)}{\sqrt{1+2\sin(2x)}}dx = \frac{1}{4}\int_0^{2} \frac{du}{\sqrt{1+u}}$

Question 15 A

Motion downwards is treated as the positive direction.

$900g - 8550 = 900a$

$a = 0.3 \text{ m/s}^2$



Now consider the girl of mass 65 kg having a reaction force R exerted on her by the lift floor:

$65g - R = 65 \times 0.3$

$R = 617.5 \text{ N}$

Question 16 D

Let $\ddot{\underset{\sim}{r}}(t) = 2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$, which gives $\dot{\underset{\sim}{r}}(t) = 2t\underset{\sim}{i} + t\underset{\sim}{j} - 2t\underset{\sim}{k} + \underset{\sim}{c}_1$

As $\dot{\underset{\sim}{r}}(0) = 4\underset{\sim}{i} - \underset{\sim}{j}$, $\underset{\sim}{c}_1 = 4\underset{\sim}{i} - \underset{\sim}{j}$, so $\dot{\underset{\sim}{r}}(t) = (2t+4)\underset{\sim}{i} + (t-1)\underset{\sim}{j} - 2t\underset{\sim}{k}$

$$\underset{\sim}{r}(t) = (t^2 + 4t)\underset{\sim}{i} + \left(\frac{t^2}{2} - t\right)\underset{\sim}{j} - t^2\underset{\sim}{k} + \underset{\sim}{c}_2$$

At $t = 0$, $r = 0$, so $\underset{\sim}{c}_2 = 0$, giving $\underset{\sim}{r}(t) = (t^2 + 4t)\underset{\sim}{i} + \left(\frac{t^2}{2} - t\right)\underset{\sim}{j} - t^2\underset{\sim}{k}$

$$\underset{\sim}{r}(4) = 32\underset{\sim}{i} + 4\underset{\sim}{j} - 16\underset{\sim}{k}$$

The distance the object is from O is $|r(4)| = \sqrt{32^2 + 4^2 + 16^2} = 36$

Question 17 A

The formula sheet gives the volume of a pyramid as $V = \frac{1}{3}Ah$.

The cross sectional area, A, is found by using the sine rule, $\text{Area} = \frac{1}{2}bc\sin(A)$, as each face of the tetrahedron is an equilateral triangle of side L.

Thus $A = \frac{1}{2}L^2\sin(60°) = \frac{1}{2}L^2 \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}L^2}{4}$

$V = \frac{1}{3}Ah = \frac{1}{3} \times \frac{\sqrt{3}L^2}{4} \times \frac{\sqrt{6}L}{3} = \frac{\sqrt{18}L^3}{36} = \frac{\sqrt{2}L^3}{12}$

We require $\frac{dL}{dt} = \frac{dL}{dV} \times \frac{dV}{dt}$

Now $\frac{dV}{dL} = \frac{\sqrt{2}L^2}{4} \Rightarrow \frac{dL}{dV} = \frac{4}{\sqrt{2}L^2}$

$\frac{dL}{dt} = \frac{4}{\sqrt{2}L^2} \times 6 = \frac{24}{\sqrt{2}L^2}$

If the area of a face is $\sqrt{3}$ cm^2 then $\sqrt{3} = \frac{\sqrt{3}L^2}{4} \Rightarrow L = 2$

$\therefore \frac{dL}{dt} = \frac{24}{4\sqrt{2}} = \frac{6}{\sqrt{2}} = 3\sqrt{2}$

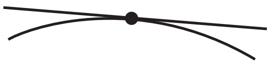
Question 18 A

Using the Euler approximation $y_{n+1} = y_n + h\left.\frac{dy}{dx}\right|_{x = x_n}$ we have $y_1 = y_0 + h\left.\frac{dy}{dx}\right|_{x = x_1}$

Now $y_0 = f(4) = 1$, $h = -0.2$, $\left.\frac{dy}{dx}\right|_{x = x_1} = \left.\frac{dy}{dx}\right|_{x = 4} = f'(4) = -1$ using the graph.

Thus $y_1 = 1 + (-0.2)(-1) = 1.2$

Since $f'(x)$ is decreasing near $x = 4$, the graph of $y = f(x)$ must be concave near $x = 4$, meaning the tangent line drawn from $x = 4$ must lie above the graph of $y = f(x)$, as shown below.



Thus the Euler approximation to $f(3.8)$ overestimates the value of $f(3.8)$.

Question 19 E

The statement $|v(t)| \neq v(t)$ implies that for some of the time, the body has a speed which does not equal its velocity. Thus the particle for part of the time interval [0, 20] is travelling with a negative velocity.

So $\int_0^{20} v(t)dt$ represents the displacement of the particle over the time interval [0, 20].

Note that in alternative **B**, the position of the particle at $t = 20$ represents its displacement from the origin. The particle cannot be assumed to have started from the origin.

Question 20 B

Given $\frac{dv}{dt} = \frac{k}{v}$, then $3 = \frac{k}{1} \Rightarrow k = 3$ so that $\frac{dv}{dt} = \frac{3}{v}$

$\frac{dt}{dv} = \frac{v}{3} \Rightarrow t = \int \frac{v}{3}dv \Rightarrow \frac{v^2}{6} + c$

Using initial conditions, $t = 0$, $v = -2$, $c = -2$, thus $t = \frac{v^2}{6} - 2 \Rightarrow v^2 = 6t + 4$

$v = \pm\sqrt{6t + 4}$, as initial velocity is to the left.

$\therefore v = -\sqrt{6t + 4}$

Question 21 D

As the acceleration is constant, $s = ut + \frac{1}{2}at^2$ gives $3 = \frac{1}{2}a(2\sqrt{3})^2 \Rightarrow a = \frac{1}{2}$

equations of motion:

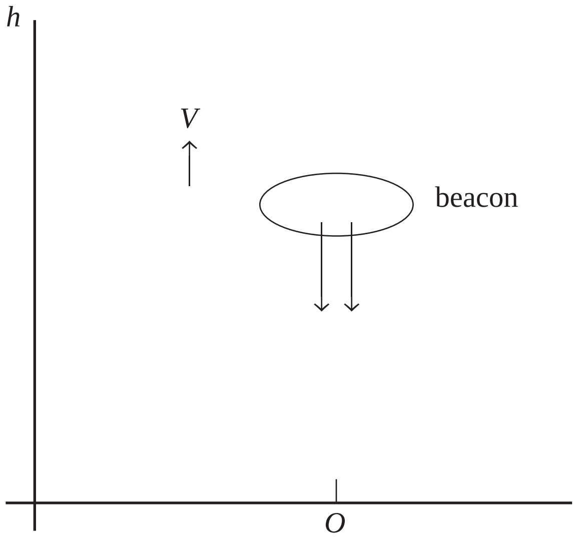
$2g - T = 2 \times \frac{1}{2} \Rightarrow T = 2g - 1$

$T - xg = x \times \frac{1}{2} \Rightarrow T = xg + \frac{1}{2}x$

Thus $2g - 1 = xg + \frac{1}{2}x \Rightarrow 4g - 2 = 2xg + x$

$\therefore 2(2g - 1) = x(2g + 1) \Rightarrow x = \frac{2(2g - 1)}{2g + 1}$

Question 22 A



Applying Newton's second law:
The resultant force is in the opposite direction to the motion and

$F_{resultant} = -25g - 25kV^2 = -25(g + kV^2)$

Using $F = ma$ we get $-25(g + kV^2) = 25a$

Thus $a = -(g + kV^2) \Rightarrow V\frac{dV}{dh} = -(g + kV^2)$, i.e. $V\frac{dV}{dh} + kV^2 + g = 0$

SECTION 2

Question 1

a. **i.** $p(z) = z^3 - 2(1 - \sqrt{3}i)z^2 - 4(1 + \sqrt{3}i)z + 8$

$p(2) = 8 - 8(1 - \sqrt{3}i) - 8(1 + \sqrt{3}i) + 8$

So $p(2) = 8 - 8 + 8\sqrt{3}i - 8 - 8\sqrt{3}i + 8 = 0$ A1

ii. Using CAS, $\dfrac{p(z)}{z-2} = z^2 - 4 + 2\sqrt{3}zi$

Thus $b = 2\sqrt{3}i$ and $c = -4$ A1

iii. By completing the square, $z^2 + 2\sqrt{3}i - 4 = 0$ can be written as

$z^2 + 2\sqrt{3}zi + (\sqrt{3}i)^2 = 4 + (\sqrt{3}i)^2$ M1

Thus $(z + \sqrt{3}i)^2 = 4 + 3i^2 = 1$, giving $h = \sqrt{3}i$ and $k = 1$ A1

Solving $(z + \sqrt{3}i)^2 = 1 \Rightarrow z + \sqrt{3}i = \pm 1$

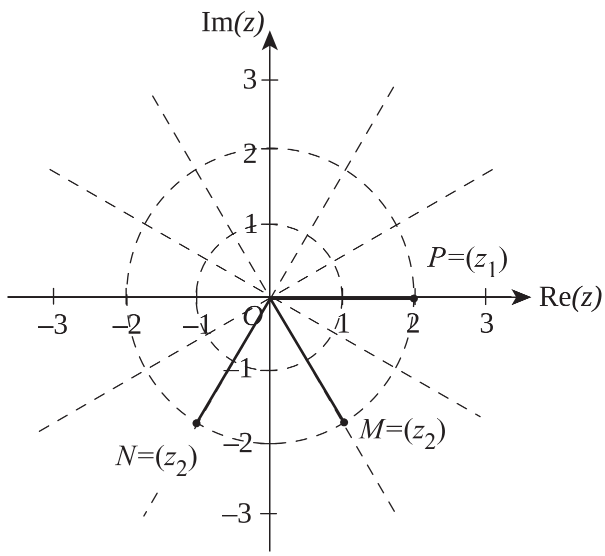
Thus $z = \pm 1 - \sqrt{3}i$ which gives the 3 solutions to $p(z) = 0$ as

$z_1 = 2$, $z_2 = 1 - \sqrt{3}i$, $z_3 = -1 - \sqrt{3}i$ as required. A1

iv. $|z_1| = |z_2| = |z_3| = 2$ A1

b. **i.** $z_3 = 2cis\left(-\dfrac{2\pi}{3}\right)$ A1

ii.



points in correct position A1

correct labelling A1

c. $\overrightarrow{NP} = \overrightarrow{OP} - \overrightarrow{ON} = 2\underset{\sim}{i} - (-\underset{\sim}{i} - \sqrt{3}\underset{\sim}{j}) = 3\underset{\sim}{i} + \sqrt{3}\underset{\sim}{j}$

$\overrightarrow{OM} = \underset{\sim}{i} - \sqrt{3}\underset{\sim}{j}$ A1

$\overrightarrow{NP} \cdot \overrightarrow{OM} = (3\underset{\sim}{i} + \sqrt{3}\underset{\sim}{j}) \cdot (i - \sqrt{3}\underset{\sim}{j}) = 3 - 3 = 0$ M1

Thus the diagonals of the quadrilateral *ONMP* are perpendicular.
Also, as *ONMP* is a parallelogram, and its diagonals are at right angles, then it is a rhombus. A1

Question 2

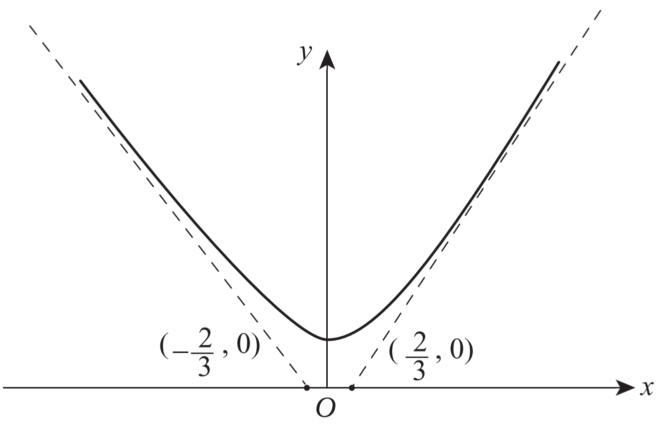
a. (0, 2) A1

b. Asymptotes $(y - k) = \pm\frac{b}{a}(x - h)$

Here $(y + 1) = \pm\frac{3}{2}(x - 0)$

$2(y + 1) = 3x$ or $2(y + 1) = -3x$ M1

$3x - 2y = 2$ or $3x + 2y = -2$ A1



two asymptotes shown with $x = -\frac{2}{3}$ and $x = \frac{2}{3}$ as the *x*-intercepts A1

c. $V = \pi\int_{y_1}^{y_2} x^2 \, dy$ M1

$= \pi\int_{2}^{10} \left(4\left(\frac{(y+1)^2}{9} - 1\right)\right) dy$ M1

$= 506$ mL A1

d. mass of a glass $= 2.5 \times$ volume of glass

$$= 2.5\pi\left(\int_0^{10}\left(\frac{2y+2}{3}\right)^2 dy - \int_2^{10} 4\left(\frac{(y+1)^2}{9} - 1\right)\right) dy$$ M1 A1

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Question 3

a. **i.** $V(12) = 12\,960\pi$ A1

time to fill = $\frac{12\,960\pi}{6000} = \frac{54\pi}{25}$ hours A1

ii. Using the chain rule

$\frac{dh}{dt} = \frac{dh}{dv} \times \frac{dv}{dt}$ and $V = \pi(6h^3 + 216h)$

$$\frac{dh}{dt} = \frac{1}{\pi(18h^2 + 216)} \times 6000, \text{ since } \frac{dv}{dh} = \pi(18h^2 + 216)$$ M1

$$= \frac{1000}{3\pi(h^2 + 12)}$$ A1

At $h = 6$, $\frac{dh}{dt} = \frac{125}{18\pi}$ m/hr A1

b. **i.** $\frac{dh}{dt} = \frac{dh}{dv} \times \frac{dv}{dt}$ M1

$$= \frac{1}{\pi(18h^2 + 216)} \times \frac{-36\pi}{\sqrt{h}}$$

$$= \frac{-2}{\sqrt{h}(h^2 + 12)}$$ A1

ii. Using $\frac{dt}{dh} = \frac{\sqrt{h}(h^2 + 12)}{-2}$

gives $t = \int_{12}^{1} \frac{\sqrt{h}(h^2 + 12)}{-2}\, dh$ M1

$$= \frac{4128\sqrt{3} - 29}{7}$$

= 1017.27 hours A1

Question 4

a. Terminal velocity is found by solving $0.3v^2 + 80 = 120 \times 9.8$ M1

Solving on CAS gives $v = 60.4$

So the terminal velocity of Melanie is 60.4 m/s, correct to one decimal place. A1

b. Applying Newton's second law we have

$ma = mg - (0.3v^2 + 80)$ A1

$a = g - \frac{0.3v^2 + 80}{120}$

Thus $\frac{dv}{dt} = 9.8 - \frac{v^2}{400} - \frac{2}{3}$ M1

$= \left(\frac{49}{5} - \frac{2}{3}\right) - \frac{v^2}{400}$

As required, $\frac{dv}{dt} = \frac{137}{15} - \frac{v^2}{400}$ A1

c. **i.** Using $\frac{dv}{dt} = \frac{137}{15} - \frac{v^2}{400}$. Using CAS we have $\frac{dv}{dt} = \frac{10\,960 - 3v^2}{1200}$

Thus $\frac{dt}{dv} = \frac{1200}{10\,960 - 3v^2}$ M1

$v = 150$ km/h $= 150 \times \frac{10}{36}$ m/s $= \frac{125}{3}$ m/s

The required definite integral is $t = \int_0^{\frac{125}{3}} \frac{1200}{10\,960 - 3v^2}\, dv$ A1

ii. Evaluating the integral above using CAS $t = 5.604$

Melanie's panic set in after 5.6 seconds. A1

d. $\frac{dv}{dt} = \frac{137}{15} - \frac{v^2}{400} = \frac{10\,960 - 3v^2}{1200}$

Thus we have $v\frac{dv}{dx} = \frac{10\,960 - 3v^2}{1200} \Rightarrow \frac{dv}{dx} = \frac{10\,960 - 3v^2}{1200v}$ M1

$\frac{dx}{dv} = \frac{1200v}{10\,960 - 3v^2} \Rightarrow x = \int_0^{60.4} \frac{1200v}{10\,960 - 3v^2}\, dv = 200\log_e\left(\frac{68\,500}{97}\right)$ A1

e. We need to solve $\frac{dv}{dt} = -9.8 - 0.6v$ with $v(0) = -60.4$, the velocity of Melanie at the instant the parachute is opened.

$\frac{dv}{dt} = -9.8 - 0.6v \Rightarrow \frac{dt}{dv} = -\frac{1}{0.6v + 9.8} = -\frac{1}{0.6\left(v + \frac{49}{3}\right)}$

This gives $(-0.6t) + c = \int \frac{1}{v + \frac{49}{3}}\, dv$ M1

i.e. $(-0.6t) + c = \log_e\left(\left|v + \frac{49}{3}\right|\right)$

$v + \frac{49}{3} = Ce^{-0.6t}$ and using $v(0) = -60.4$ we have $C = -44.07$ M1

This gives $v = -\left(44.07e^{-0.6t} + \frac{49}{3}\right)$ as the required approximate expression for velocity. A1

f. We need to solve $\int_0^T v(t)dt = -3500 + 200\log_e\left(\frac{68\ 500}{97}\right)$ M1

Using CAS to solve $\int_0^T -\left(49.07e^{-0.6t} + \frac{49}{3}\right)dt = -2188$ gives $T = 129.46$.

Thus it will take Melanie approximately 129 seconds to reach the ground from when the parachute is opened. A1

g. $\frac{dv}{dt} = -kv - g \Rightarrow \frac{dt}{dv} = -\frac{1}{k\left(v + \frac{g}{k}\right)}$

$t = -\frac{1}{k}\log_e\left(\left|v + \frac{g}{k}\right|\right) + c$, giving $-kt + c = \log_e\left(\left|v + \frac{g}{k}\right|\right)$ M1

Thus $v = Ce^{-kt} - \frac{g}{k}$

As $v(0) = -50$ we have $v = \left(\frac{g}{k} - 50\right)e^{-kt} - \frac{g}{k}$ A1

As $t \to \infty, v \to -\frac{g}{k}$

This is the velocity with which the parachutist is descending. Now $\frac{g}{k} < 5.5 \Rightarrow k > \frac{98}{55}$ A1

Question 5

a. $y = 2$ when $x = 0$

This gives $y = \frac{-g\sec^2(\theta)}{2v^2}x^2 + \tan(\theta)x + 2$

At $v = 16$, $\theta = 45$, $g = -9.8$

$y = \frac{-9.8\sec^2(45°)}{2(16)^2}x^2 + \tan(45°)x + 2$

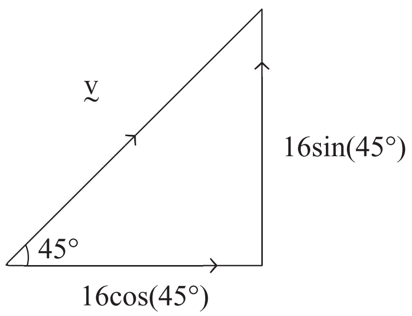
$y = \frac{-49}{1445}x^2 + x + 2$ A1

Solving $\frac{-49}{1445}x^2 + x + 2 = 3$ M1

gives $x = 1.04$ or 25.0809

The point $(25.08, 3)$ is inside the line segment $y = 3$ where $24.25 < x < 25.25$ A1

b.



$\underset{\sim}{v} = 16\cos(45°)\underset{\sim}{i} + 16\sin(45°)\underset{\sim}{j}$

$= 8\sqrt{2}\underset{\sim}{i} + 8\sqrt{2}\underset{\sim}{j}$ A1

c. **i.** $\underset{\sim}{a} = 0\underset{\sim}{i} - g\underset{\sim}{j}$

$v(t) = \int \underset{\sim}{a}\ dt$

$= c_1\underset{\sim}{i} + (c_2 - gt)\underset{\sim}{j}$ M1

given $v(0) = 8\sqrt{2}\underset{\sim}{i} + 8\sqrt{2}\underset{\sim}{j}$, $c_1 = 8\sqrt{2}$ and $c_2 = 8\sqrt{2}$

$\underset{\sim}{v}(t) = 8\sqrt{2}\underset{\sim}{i} + (8\sqrt{2} - gt)\underset{\sim}{j}$ A1

$\underset{\sim}{r}(t) = \int v(t)\ dt$

$= d_1 + 8\sqrt{2}t\underset{\sim}{i} + \left(d_2 + 8\sqrt{2}t - \frac{1}{2}gt^2\right)\underset{\sim}{j}$ M1

given $\underset{\sim}{r}(0) = 0\underset{\sim}{i} + 2\underset{\sim}{j}$ $d_1 = 0$ and $d_2 = 2$

$\underset{\sim}{r}(t) = 8\sqrt{2}t\underset{\sim}{i} + \left(2 + 8\sqrt{2}t - \frac{1}{2}gt^2\right)\underset{\sim}{j}$ A1

ii. Solving $8\sqrt{2}t = 25.0809...$ M1

$t = \frac{25.0809...}{8\sqrt{2}}$

$\therefore\ v\left(\frac{25.0809}{8\sqrt{2}}\right) = 8\sqrt{2}\underset{\sim}{i} + \left(8\sqrt{2} - g\left(\frac{25.0809}{8\sqrt{2}}\right)\right)\underset{\sim}{j}$

$\therefore$ speed $= |\underset{\sim}{v}| = \sqrt{(8\sqrt{2})^2 + \left(8\sqrt{2} - g\frac{25.0809}{8\sqrt{2}}\right)^2}$

$= 15.375$

$= 15.4$ m/s A1