2012 VCAA Specialist Math Exam 2 Solutions

Note: Some steps can be done by CAS to save time

SECTION 1

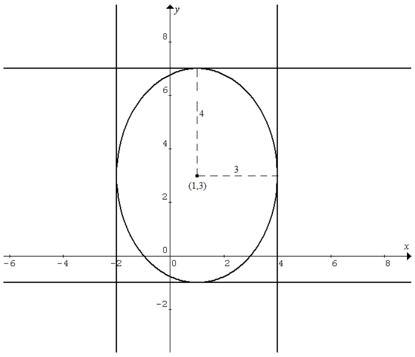
1	2	3	4	5	6	7	8	9	10	11
D	C	E	D	C	C	B	E	B	A	E

12	13	14	15	16	17	18	19	20	21	22
A	B	B	D	A	D	C	E	C	E	D

Q1 $y=\frac{1}{2x^2-x-6}=\frac{1}{(2x+3)(x-2)}=\frac{1}{2\left(x+\frac{3}{2}\right)(x-2)}$

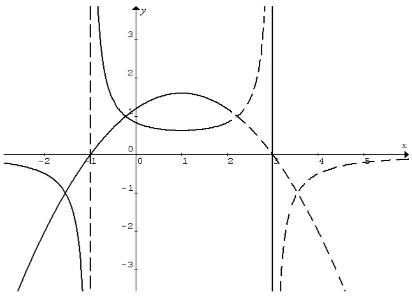
Straight line asymptotes are: $y=0$, $x=-\frac{3}{2}$ and $x=2$. D

Q2

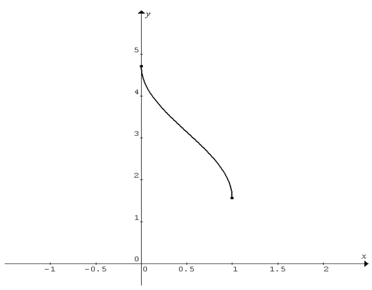


Equation of the ellipse: $\frac{(x-1)^2}{9}+\frac{(y-3)^2}{16}=1$ C

Q3 See graph below. E



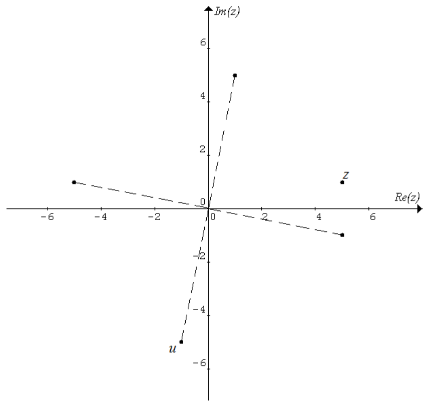
Q4 Find the domain and range from the graph of $f(x)=\arccos(2x-1)+\frac{\pi}{2}$. D



Q5 $z=\sqrt{2}cis\left(-\frac{4\pi}{5}\right)$, $w=z^9=\left(\sqrt{2}\right)^9 cis\left(-\frac{4\pi}{5}\times 9\right)$

$=16\sqrt{2}cis\left(\frac{4\pi}{5}-8\pi\right)=16\sqrt{2}cis\left(\frac{4\pi}{5}\right)$ C

Q6 $u=i^3\bar{z}$ is the reflection of z about the x-axis followed by an anticlockwise rotation through $3\times\frac{\pi}{2}$ about the origin. The result is the same as reflecting z about the y-axis and then rotating anticlockwise through $\frac{\pi}{2}$ about the origin. C



Q7 $|z+2i|=|z|$, $|z-(-2i)|=|z|$ defines a set of points such that each point is equidistant from $(0,-2)$ and $(0,0)$. This set of points forms a straight line which is the perpendicular bisector of the line segment joining $(0,-2)$ and $(0,0)$.
The equation is $y=-1$, i.e. $\text{Im}(z)=-1$. B

Q8 $(z-\bar{z})(z+\bar{z})=2bi\times 2a=4abi$ E

Q9

$x_0=0 \quad y_0=1 \quad \frac{dy}{dx}=\cos 0=1$

$x_1=0.1 \quad y_1\approx 1+0.1\times 1\approx 1.1 \quad \frac{dy}{dx}=\cos 0.1\approx 0.995$

$x_2=0.2 \quad y_2\approx 1.1+0.1\times 0.995\approx 1.1995$

$\frac{dy}{dx}$ is a decreasing function in the region under consideration,
.: 1.1995 is an overestimate of y at $x=0.2$. B

Q10 $\frac{dy}{dx}=xy$

At $(1,1)$, $\frac{dy}{dx}=1$; at $(1,-1)$, $\frac{dy}{dx}=-1$; at $(-1,1)$, $\frac{dy}{dx}=-1$;

at $(-1,-1)$, $\frac{dy}{dx}=1$; at $(0,1)$, $\frac{dy}{dx}=0$; at $(1,0.3)$, $\frac{dy}{dx}=0.3$ A

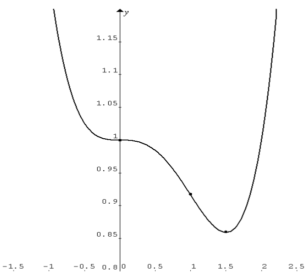
Q11 Let $\frac{d^2y}{dx^2} = x^2 - x = 0$, $x(x-1) = 0$, .: a *possible* point of inflection at $x = 0$ or $x = 1$.

$\frac{d^2y}{dx^2} = x^2 - x$, .: $\frac{dy}{dx} = \frac{x^3}{3} - \frac{x^2}{2} + c = x^2\left(\frac{x}{3} - \frac{1}{2}\right) + c$

Given $\frac{dy}{dx} = 0$ at $x = 0$, .: $c = 0$ and $\frac{dy}{dx} = 0$ at $x = \frac{3}{2}$ also.

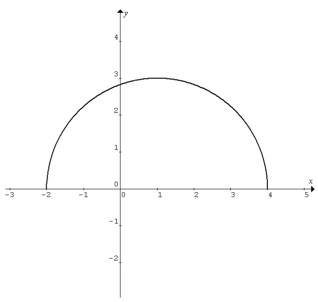
x	<0	0	$0 < x < \frac{3}{2}$	$\frac{3}{2}$	>0
$\frac{dy}{dx}$	$-$	0	$-$	0	$+$
Nature		point of inflection		local minimum	

There must be another point of inflection between $x = 0$ and $x = \frac{3}{2}$. See graph below. E



Q12 $y = \sqrt{9 - (x-1)^2}$ is the positive half of the circle $(x-1)^2 + y^2 = 9$ which has a radius of 3 units. A sphere is formed when it is rotated about the *x*-axis. See graph below.

$Volume = \frac{4}{3}\pi \times 3^3 = 4\pi(3)^2$ A



Q13 $\int_0^{\frac{\pi}{3}} \sin^3 x \cos^4 x dx = \int_0^{\frac{\pi}{3}} \sin^2 x \cos^4 x \sin x dx$

$= \int_0^{\frac{\pi}{3}} \left(1 - \cos^2 x\right)\cos^4 x \sin x dx$

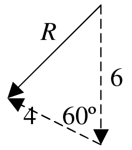
$= \int_1^{\frac{1}{2}} \left(u^6 - u^4\right) du$

Let $u = \cos x$,
$-\frac{du}{dx} = \sin x$.
When $x = 0$, $u = 1$.
When $x = \frac{\pi}{3}$, $u = \frac{1}{2}$.

B

Q14 *Resultant force* R

$= \sqrt{4^2 + 6^2 - 2(4)(6)\cos 60°} = 2\sqrt{7}$ N B



Q15 Let $\tilde{a}.\tilde{b} = 0$

$\left(2\tilde{i} + m\tilde{j} - 3\tilde{k}\right).\left(m^2\tilde{i} - \tilde{j} + \tilde{k}\right) = 0$

.: $2m^2 - m - 3 = 0$, $(2m-3)(m+1) = 0$,

.: $m = \frac{3}{2}$, -1 D

Q16 *Distance between* $P(-2,4,3)$ and $Q(1,-2,1)$

$= \sqrt{(1-(-2))^2 + (-2-4)^2 + (1-3)^2} = 7$ A

Q17 $\tilde{u} = 2\tilde{i} - 2\tilde{j} + \tilde{k}$, $\tilde{v} = 3\tilde{i} - 6\tilde{j} + 2\tilde{k}$

$|\tilde{u}| = \sqrt{2^2 + (-2)^2 + 1^2} = 3$, .: $\hat{u} = \frac{1}{3}\left(2\tilde{i} - 2\tilde{j} + \tilde{k}\right)$

Scalar resolute of $\tilde{v}$ *in the direction of* $\tilde{u}$ $= \tilde{v}.\hat{u}$

$= \left(3\tilde{i} - 6\tilde{j} + 2\tilde{k}\right).\frac{1}{3}\left(2\tilde{i} - 2\tilde{j} + \tilde{k}\right) = \frac{1}{3}(6 + 12 + 2) = \frac{20}{3}$

Vector resolute of $\tilde{v}$ *in the direction of* $\tilde{u}$ $= (\tilde{v}.\hat{u})\hat{u}$

$= \frac{20}{3} \times \frac{1}{3}\left(2\tilde{i} - 2\tilde{j} + \tilde{k}\right) = \frac{20}{9}\left(2\tilde{i} - 2\tilde{j} + \tilde{k}\right)$ D

Q18 x_0 is a red herring. C

Q19 Given $v = x$, $x = -1$ when $t = 3$.

$\frac{dx}{dt} = x$, $\frac{dt}{dx} = \frac{1}{x}$, $t = \int_{-1}^{x} \frac{1}{x} dx + 3 = \left[\log_e|x|\right]_{-1}^{x} + 3 = \log_e|x| + 3$

.: $|x| = e^{t-3}$, .: $x = -e^{t-3}$ to satisfy $x = -1$ when $t = 3$. E

Q20 *Resultant force* $=$ *mass* $\times$ *acceleration*

$mg - 3g = (m+3) \times 4.9$, $mg - 3g = (m+3) \times \frac{g}{2}$

.: $m - 3 = \frac{m+3}{2}$, .: $m = 9.0$ C

Q21 $v = x^2$, $\frac{1}{2}v^2 = \frac{1}{2}x^4$, $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dx}\left(\frac{1}{2}x^4\right) = 2x^3$

When $x = 2$, $a = 16$ and $F = ma = 3 \times 16 = 48$N E

Q22 $2T\sin 60° - 12g = 0$, .: $\sqrt{3}T - 12g = 0$,

$T = \frac{12g}{\sqrt{3}} = 4\sqrt{3}g$ D

SECTION 2

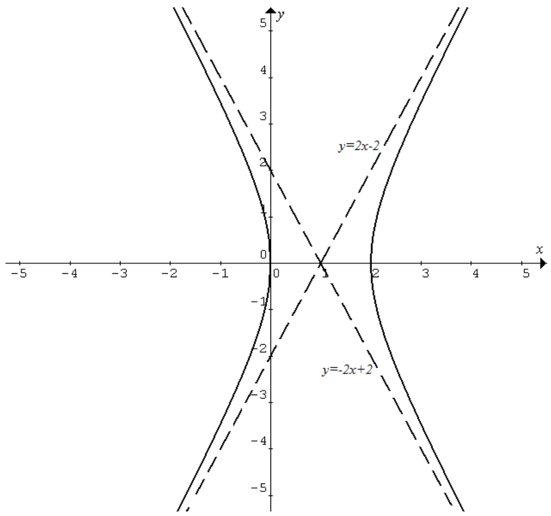
Q1a Given $x = \operatorname{cosec}\theta + 1$, $y = 2\cot\theta$

$1-\cos^2\theta = \sin^2\theta$, $\dfrac{1-\cos^2\theta}{\sin^2\theta} = 1$, $\dfrac{1}{\sin^2\theta} - \dfrac{\cos^2\theta}{\sin^2\theta} = 1$,

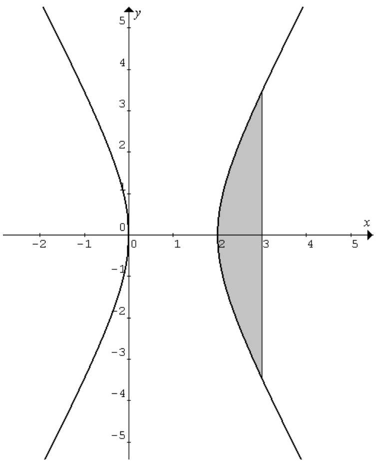
.: $\operatorname{cosec}^2\theta - \cot^2\theta = 1$, .: $(x-1)^2 - \left(\dfrac{y}{2}\right)^2 = 1$

.: $(x-1)^2 - \dfrac{y^2}{4} = 1$

Q1b



Q1c



$V_{solid} = \pi\int_2^3 y^2 dx = 4\pi\int_2^3 \left[(x-1)^2 - 1\right] dx = 4\pi\left[\dfrac{(x-1)^3}{3} - x\right]_2^3 = \dfrac{16\pi}{3}$

Q1d At $\theta = \dfrac{7\pi}{6}$, $x = \operatorname{cosec}\dfrac{7\pi}{6} + 1 = -1$, $y = 2\cot\dfrac{7\pi}{6} = 2\sqrt{3}$

By implicit differentiation of $(x-1)^2 - \dfrac{y^2}{4} = 1$,

$2(x-1) - \dfrac{y}{2}\times\dfrac{dy}{dx} = 0$, .: $\dfrac{dy}{dx} = \dfrac{4(x-1)}{y} = -\dfrac{4}{\sqrt{3}} = -\dfrac{4\sqrt{3}}{3}$.

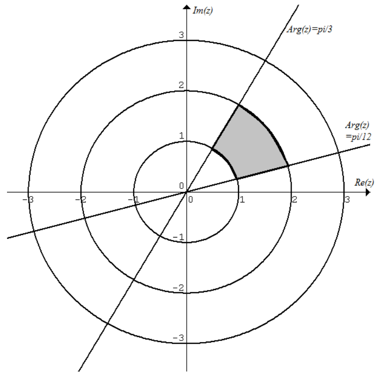
Q2a $\sin\dfrac{\pi}{12} = \sqrt{1-\cos^2\dfrac{\pi}{12}} = \sqrt{1-\left(\dfrac{\sqrt{\sqrt{3}+2}}{2}\right)^2}$

$= \sqrt{1-\dfrac{\sqrt{3}+2}{4}} = \sqrt{\dfrac{2-\sqrt{3}}{4}} = \dfrac{\sqrt{2-\sqrt{3}}}{2}$

Q2bi $z_1 = \cos\dfrac{\pi}{12} + i\sin\dfrac{\pi}{12}$ or simply $z_1 = cis\dfrac{\pi}{12}$

Q2bii $z_1^{\,4} = cis\left(4\times\dfrac{\pi}{12}\right) = cis\dfrac{\pi}{3}$

Q2c



Q2d $Area = \left(\pi\times 2^2 - \pi\times 1^2\right)\times\dfrac{\frac{\pi}{3}-\frac{\pi}{12}}{2\pi} = 3\pi\times\dfrac{1}{8} = \dfrac{3\pi}{8}$

Q2ei $z_1 = \cos\dfrac{\pi}{12} + i\sin\dfrac{\pi}{12}$, $z_1^{\,n} = \cos\dfrac{n\pi}{12} + i\sin\dfrac{n\pi}{12}$

$\operatorname{Re}\left(z_1^{\,n}\right) = 0$, $\cos\dfrac{n\pi}{12} = 0$

.: $\dfrac{n\pi}{12} = \pm\dfrac{\pi}{2}$, $\pm\dfrac{3\pi}{2}$, $\pm\dfrac{5\pi}{2}$, ..., i.e. $\pm$ odd integral multiple of $\dfrac{\pi}{2}$, which can be expressed as $\dfrac{n\pi}{12} = (2k+1)\dfrac{\pi}{2}$ where $k \in Z$.

.: $n = (2k+1)6$.

Q2eii For the values of n found in part i, $\operatorname{Re}\left(z_1^{\,n}\right) = 0$,

.: $z_1^{\,n} = i\sin\dfrac{n\pi}{12} = i\sin\dfrac{(2k+1)\pi}{2} = i(\pm 1) = \pm i$ for $k \in Z$.

Q3a $V = 17\tan^{-1}\dfrac{\pi T}{6}$, $T \ge 0$

As $T \to \infty$, $\tan^{-1}\dfrac{\pi T}{6} \to \dfrac{\pi}{2}^-$, .: $V \to \dfrac{17\pi}{2}^-$ (≈ 26.7) m s^{-1}

Q3b $a = \dfrac{dV}{dT} = 17\times\dfrac{\frac{\pi}{6}}{1+\left(\frac{\pi T}{6}\right)^2} \approx 0.3$ m s^{-2} when $T = 10$

Q3c When $V = 25$, $17\tan^{-1}\frac{\pi T}{6} = 25$, $\frac{\pi T}{6} = \tan\frac{25}{17}$, $T \approx 19$ s

Q3di $\frac{dv}{dt} = -\frac{1}{100}(145 - 2t)$, $v = 25$ when $t = 0$

$\therefore\ v = \int_0^t -\frac{1}{100}(145-2t)dt + 25 = \left[-\frac{1}{100}\left(145t - t^2\right)\right]_0^t + 25$

$= 0.01t^2 - 1.45t + 25$

Q3dii When $v = 0$, $0.01t^2 - 1.45t + 25 = 0$, $t \geq 0$

$\therefore\ t = 20$ s

Q3ei Distance (m) travelled during each stage:

Stage 1 distance $= \int_0^{19}\left(17\tan^{-1}\frac{\pi T}{6}\right)dT \approx 400$ by CAS

Stage 2 distance $= 25 \times 120 = 3000$

Stage 3 distance $= \int_0^{20}\left(0.01t^2 - 1.45t + 25\right)dt \approx 237$ by CAS

Q3eii *Total distance* ≈ 3637 m

Q4a Let $\theta = \pi(1.3t - 0.1)$ for $0 \leq t \leq 0.154$ hours.

$\tilde{r} = (6800\sin(\pi(1.3t-0.1)))\tilde{i} + (6800\cos(\pi(1.3t-0.1)) - 6400)\tilde{j}$.

$\therefore\ \tilde{r} = (6800\sin\theta)\tilde{i} + (6800\cos\theta - 6400)\tilde{j}$

At point *P*, the $\tilde{i}$ component is zero, $6800\sin\theta = 0$, $\theta = 0$.

$\therefore\ h = 6800\cos 0 - 6400 = 400$ km

Q4b $\tilde{v} = \frac{d\tilde{r}}{dt} = \frac{d\tilde{r}}{d\theta} \times \frac{d\theta}{dt} = 6800 \times 1.3\pi\left((\cos\theta)\tilde{i} - (\sin\theta)\tilde{j}\right)$

$\tilde{a} = \frac{d\tilde{v}}{dt} = \frac{d\tilde{v}}{d\theta} \times \frac{d\theta}{dt} = 6800 \times (1.3\pi)^2\left((-\sin\theta)\tilde{i} - (\cos\theta)\tilde{j}\right)$

$\therefore\ \tilde{a}(t) = -6800 \times (1.3\pi)^2\left((\sin(\pi(1.3t-0.1)))\tilde{i} + (\cos(\pi(1.3t-0.1)))\tilde{j}\right)$

$\hat{v}.\hat{a} = \left((\cos\theta)\tilde{i} - (\sin\theta)\tilde{j}\right).\left((-\sin\theta)\tilde{i} - (\cos\theta)\tilde{j}\right)$

$= -\sin\theta\cos\theta + \sin\theta\cos\theta = 0$

$\therefore\ \tilde{a} \perp \tilde{v}$

Q4c $\tilde{v} = 6800 \times 1.3\pi\left((\cos\theta)\tilde{i} - (\sin\theta)\tilde{j}\right)$

Speed $= |\tilde{v}| = 6800 \times 1.3\pi\sqrt{\cos^2\theta + (-\sin\theta)^2}$

$= 6800 \times 1.3\pi \approx 27772$ km/h

Q4d Parametric equations:

$x = 6800\sin\theta$, $y = 6800\cos\theta - 6400$

Since $\sin^2\theta + \cos^2\theta = 1$, $\therefore$ the cartesian equation of the path is

$\left(\frac{x}{6800}\right)^2 + \left(\frac{y+6400}{6800}\right)^2 = 1$, i.e. $x^2 + (y+6400)^2 = 6800^2$.

It is a circular path of radius 6800 km centred at $(0, -6400\text{ km})$.

Q4e $\tilde{r} = (6800\sin\theta)\tilde{i} + (6800\cos\theta - 6400)\tilde{j}$

$|\tilde{r}| = \sqrt{(6800\sin\theta)^2 + (6800\cos\theta - 6400)^2} = 1000$

$\therefore\ (6800\sin\theta)^2 + (6800\cos\theta - 6400)^2 = 1000^2$ and can be simplified to $(\sin\theta)^2 + \left(\cos\theta - \frac{16}{17}\right)^2 = \left(\frac{5}{34}\right)^2$

$\therefore\ \sin^2\theta + \cos^2\theta - \frac{32}{17}\cos\theta + \frac{256}{289} = \frac{25}{1156}$

$1 - \frac{32}{17}\cos\theta + \frac{256}{289} = \frac{25}{1156}$, $\therefore\ \cos\theta \approx 0.99035$

$\therefore\ \theta \approx -0.139$ or 0.139 by CAS

$\therefore\ \pi(1.3t - 0.1) \approx -0.139$ or 0.139, $\therefore\ t \approx 0.04$ or 0.11 hours

Q5a $u = 0$, $s = 10$, $v = 6$, find *t*.

$s = \frac{1}{2}(u+v)t$, $t = \frac{10}{3}$ s

Q5b $s = -6$, $u = 10$, $a = -9.8$, find *t*.

$s = ut + \frac{1}{2}at^2$, $t \approx 2.5$ s

Q5c Assume that the end of the slide is at ground level.

Total time of travel allowed for the chocolate $= \frac{10}{3} + 4 = \frac{22}{3}$ s

$a = -9.8$, $s = -6$, $t = \frac{22}{3}$, find *u*.

$s = ut + \frac{1}{2}at^2$, $u = 35.1$ m s^{-1}

Q5di $a = -\frac{1}{10}\sqrt{196 - v^2}$,

$\frac{dv}{dt} = -\frac{1}{10}\sqrt{196 - v^2}$, $\therefore\ \frac{dt}{dv} = 10 \times \frac{-1}{\sqrt{196 - v^2}}$,

$\frac{t}{10} = \int \frac{-1}{\sqrt{14^2 - v^2}}dv + c$, $\frac{t}{10} = \cos^{-1}\left(\frac{v}{14}\right) + c$

When $t = 0$, $v = 7$, $\therefore\ 0 = \cos^{-1}\left(\frac{7}{14}\right) + c$, $c = -\frac{\pi}{3}$

$\therefore\ \frac{t}{10} = \cos^{-1}\left(\frac{v}{14}\right) - \frac{\pi}{3}$, $\therefore\ v = 14\cos\left(\frac{t}{10} + \frac{\pi}{3}\right)$

Q5dii When $v = 0$, $\frac{t}{10} = \cos^{-1}(0) - \frac{\pi}{3} = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$, $\therefore\ t = \frac{5\pi}{3}$.

Q5diii $v = 14\cos\left(\frac{t}{10} + \frac{\pi}{3}\right)$, $\frac{dx}{dt} = 14\cos\left(\frac{t}{10} + \frac{\pi}{3}\right)$

$x = \int_0^{\frac{5\pi}{3}} 14\cos\left(\frac{t}{10} + \frac{\pi}{3}\right)dt \approx 18.8$ m by CAS

Distance required ≈ 18.8 m

Please inform mathline@itute.com re conceptual and/or mathematical errors