SPECIALIST MATHEMATICS

Units 3 & 4 – Written examination 2

2010 Trial Examination

SOLUTIONS

SECTION 1: Multiple-choice questions (1 mark each)

Question 1

Answer: B

Explanation:

By completing the square, the equation of $x^2 - 4y^2 - 4mx - 8my = 4$ becomes

$(x-2m)^2 - 4m^2 - 4(y+m)^2 + 4m^2 = 4$

$(x-2m)^2 - 4(y+m)^2 = 4$

$\dfrac{(x-2m)^2}{4} - (y+m)^2 = 1$

This is a hyperbola with centre (2*m, -m*) and the semi-axes *a* = 2 and *b* = 1

Question 2

Answer: A

Explanation:

For $f(x)$ to have non-stationary point of inflection at $x=1$, second derivative at this point must be zero. Because $f(x)$ is an increasing function, the first derivative must be positive.

$f'(x) = 2Ax - 1 + \dfrac{B}{x}, \quad f''(x) = 2A - \dfrac{B}{x^2}$

When $x=1$, $2A - 1 + B > 0, \quad 2A - B = 0$

After substituting $B = 2A$ into $2A - 1 + B > 0$, we have $A > \dfrac{1}{4}$ and $B > \dfrac{1}{2}$

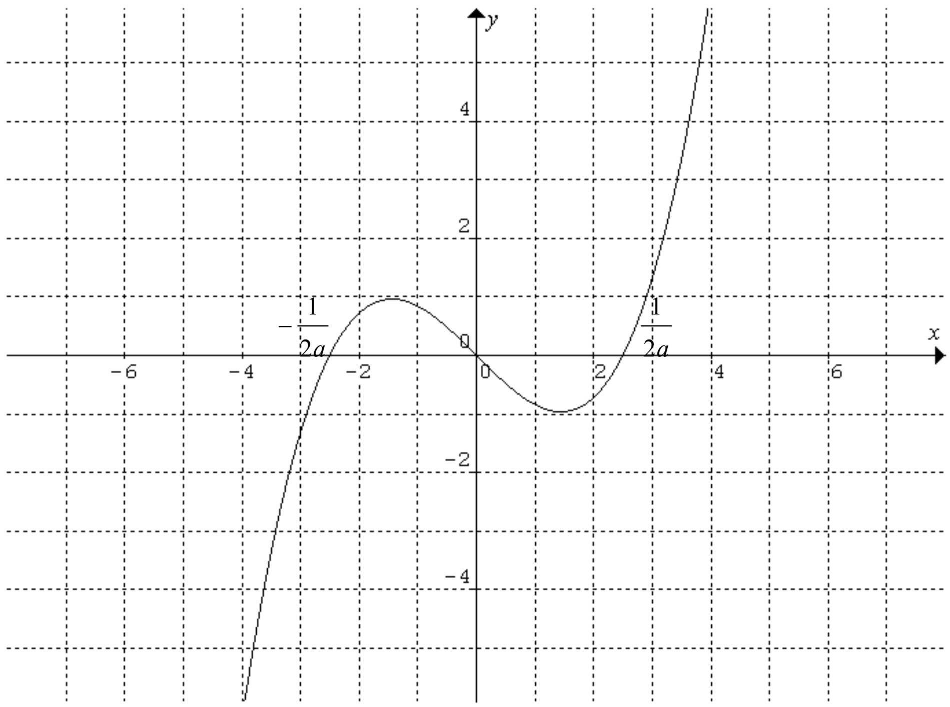
Question 3

Answer: D

Explanation:

$f(x)=\sqrt{4a^2x^3-x}$, $a \in R^+$, is defined when $4a^2x^3-x \geq 0$.

Solving the inequality graphically, $x \in \left[-\frac{1}{2a},0\right] \cup \left[\frac{1}{2a},+\infty\right)$.



Question 4

Answer: D

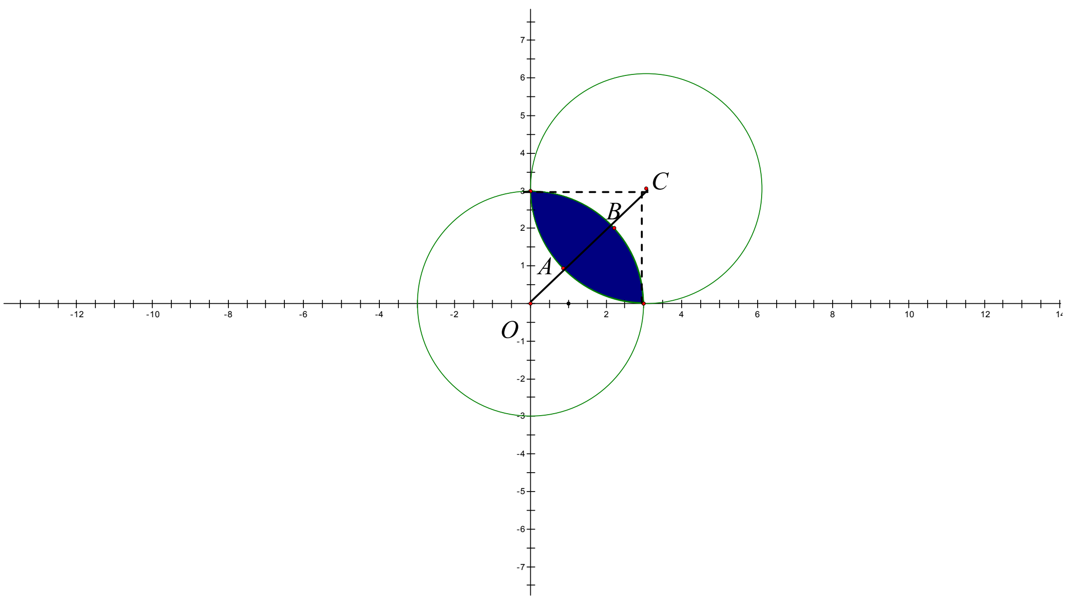
Explanation:

For the stationary point of inflection, both $f'(x)$ and $f''(x)$ must equal zero which is true at (0, 0).

Question 5

Answer: B

Explanation:
Alternative A is obviously incorrect as for $S = \{z : |z| \le 3\} \cap \{z : |z - 3 - 3i| \le 3\}$, z belongs to the shaded region , and therefore $0 \le Arg(z) \le \frac{\pi}{2}$.



Further, for every z from the shaded region, $|z| \le 3$. The minimum value of $|z| = OA$.
As $OA = BC = 3\sqrt{2} - 3$, it follows that $3(\sqrt{2} - 1) \le |z| \le 3$.

Question 6

Answer: E

Explanation:

Substituting $z = x + iy$ into $|z| - z = 1 + 2i$ and rearranging

$\sqrt{x^2 + y^2} - x - iy = 1 + 2i$

$\sqrt{x^2 + y^2} = (x + 1) + (2 + y)i$

Equating real and imaginary parts, $2 + y = 0$, $\sqrt{x^2 + y^2} = x + 1$ and solving for x and y gives $x = \frac{3}{2}, y = -2$. Therefore, $z = \frac{3}{2} - 2i$. The only correct alternative is $|z| = \frac{5}{2}$.

Question 7

Answer: C

Explanation:

The polynomial has real coefficients. By the conjugate root theorem, $z = 1 + ai$ is also a solution. The quadratic factor is
$z^2 - (1 + ai + 1 - ai)z + (1 + ai)(1 - ai) = z^2 - 2z + 1 + a^2$

Question 8

Answer: E

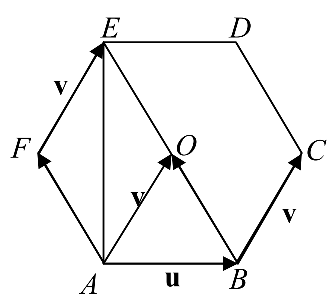
Explanation:

$\mathbf{a} + n\mathbf{b} = 6\mathbf{i} + (1 + 3n)\mathbf{j} + (1 - n)\mathbf{k}$, $\mathbf{c} = -2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$
The scalar product of these two vectors must be 0.
$(\mathbf{a} + n\mathbf{b}) \bullet \mathbf{c} = -12 + 3(1 + 3n) + 5(1 - n) = 0 \Rightarrow n = 1$

Question 9

Answer: C

Explanation:



$\overrightarrow{AF} = \overrightarrow{BO} = \overrightarrow{BA} + \overrightarrow{AO} = -\mathbf{u} + \mathbf{v}$

$\overrightarrow{AE} = \overrightarrow{AF} + \overrightarrow{FE} = -\mathbf{u} + \mathbf{v} + \mathbf{v} = -\mathbf{u} + 2\mathbf{v}$

Question 10

Answer: A

Explanation

Let $\alpha = \cos^{-1}\dfrac{1}{\sqrt{1+a^2}}$ and $\beta = \cos^{-1}\dfrac{a}{\sqrt{1+a^2}}$.

Then $\cos\alpha = \dfrac{1}{\sqrt{1+a^2}}$ and $\cos\beta = \dfrac{a}{\sqrt{1+a^2}} \Rightarrow \tan\alpha = a$ and $\tan\beta = \dfrac{1}{a}$.

Using the addition formula for tangent, $\tan(\alpha - \beta) = \dfrac{a - \frac{1}{a}}{1+1} = \dfrac{a^2 - 1}{2a}$

Question 11

Answer: D

Explanation:

$y = \ln\frac{1}{1+x} = -\ln(1+x)$ $\qquad x\frac{dy}{dx} + p = e^y$

Substituting $\frac{dy}{dx} = -\frac{1}{1+x}$ into $x\frac{dy}{dx} + p = e^y$ yields

$\frac{-x}{1+x} + p = e^{\ln\frac{1}{1+x}}$

$\frac{-x}{1+x} + p = \frac{1}{1+x} \Rightarrow p = \frac{x}{1+x} + \frac{1}{1+x} = 1$

Question 12

Answer: A

Explanation:

Using the chain rule:

$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$

$x = a\cos t,\ \frac{dx}{dt} = -a\sin t$

$y = b\sin t,\ \frac{dy}{dt} = b\cos t$

The gradient of the tangent is $\frac{dy}{dx} = -\frac{b\cos t}{a\sin t}$.

For $t = \frac{3\pi}{4}$, the gradient of the normal is $\frac{a}{b}\tan\frac{3\pi}{4} = -\frac{a}{b}$

Question 13

Answer: B

Explanation:

Let $u = x+2$, then $x = u-2$ and $dx = du$

$$\int_0^3 \frac{2x-1}{\sqrt{x+2}}dx = \int_2^5 \frac{2u-5}{\sqrt{u}}du$$

$$= \int_2^5 \left(2\sqrt{u} - \frac{5}{\sqrt{u}}\right)du$$

Question 14

Answer: E

Explanation:

The slope field given in this question appears only in the first and third quadrant. The corresponding differential equation is defined only when *x* and *y* are both positive or both negative.

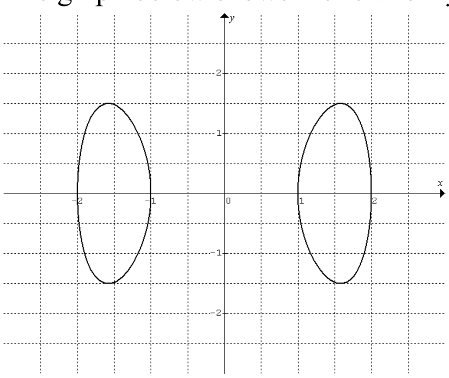
$\frac{dy}{dx} = \frac{y}{x}\ln\left(\frac{y}{x}\right)$ is defined for $\frac{y}{x} > 0$.

Question 15

Answer: D

Explanation:

The graph below shows the relation $y^2 = -x^4 + 5x^2 - 4$.



$$\text{enclosed area} = 4\int_1^2 \sqrt{-x^4 + 5x^2 - 4}\,dx = 4.646$$

Question 16

Answer: C

Explanation:

$$\int \frac{\sin^2 x \cos^5 x}{1+\cos 2x}\,dx = \int \frac{\sin^2 x \cos^5 x}{2\cos^2 x}dx$$
$$= \frac{1}{2}\int \sin^2 x \cos^3 x dx$$
$$= \frac{1}{2}\int \sin^2 x(1-\sin^2 x)\cos x dx$$

The substitution $\sin x = u, \cos x dx = du$ gives $\frac{1}{2}\int (u^2 - u^4)du$.

Question 17

Answer: C

Explanation:

$t = 0, x = 50 \times 10 = 500 grams$

$\frac{dx}{dt} = Rate_{in} \times Concentration_{in} - Rate_{out} \frac{x}{V_0 + (Rate_{in} - Rate_{out})t}$

$\frac{dx}{dt} = 3 \times 8 - \frac{2x}{50 + t}$

$\frac{dx}{dt} = 24 - \frac{2x}{50 + t}, \quad t = 0, x = 500$

Question 18

Answer: A

Explanation:

$x_{n+1} = x_n + h, \quad y_{n+1} = y_n + hf(x_n, y_n), \; x_0 = 1, y_0 = 0, h = 0.1, \quad f(x, y) = x - \log_e(x + y)$

$x_0 = 1 \qquad y_1 = 0 + 0.1(1 - \log_e 1) = 0.1$

$x_1 = 1.1, \quad y_2 = 0.1 + 0.1(1.1 - \log_e 1.2)$

$= 0.1 + 0.11 - 0.1\log_e 1.2$

$= 0.21 - 0.1\log_e 1.2$

Question 19

Answer: A

Explanation:

When $t = 0, v_1 = 0$ and $v_2 = -10.$

$v_1 = 3t^2 \Rightarrow s_1 = \int 3t^2 dt = t^3$

$v_2 = (6t^2 - 10) \Rightarrow s_2 = \int (6t^2 - 10)dt = 2t^3 - 10t$

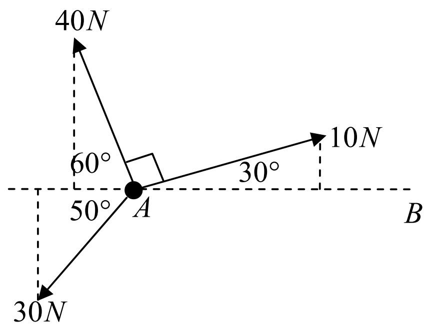
$t = 10, s_1 = 1000, s_2 = 1900$

$s_2 - s_1 = 900m$

Question 20

Answer: D

Explanation:

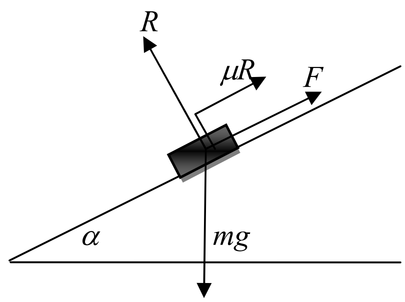


Sum of perpendicular components = $40\sin 60° - 30\sin 50° + 10\sin 30° = 16.66$ (2 dec.pl).

Question 21

Answer: B

Explanation:



Resolving horizontally : $F + \mu R - mg\sin\alpha = 0$
Resolving vertically : $R - mg\cos\alpha = 0 \Rightarrow R = mg\cos\alpha$
Therefore, $F = mg\sin\alpha - \mu mg\cos\alpha = mg(\sin\alpha - \mu\cos\alpha)$.

Question 22

Answer: B

Explanation:

$$v = \sqrt[3]{2 - x^2} = (2 - x^2)^{\frac{1}{3}} \Rightarrow \frac{dv}{dx} = \frac{-2x}{3\sqrt[3]{(2 - x^2)^2}}$$

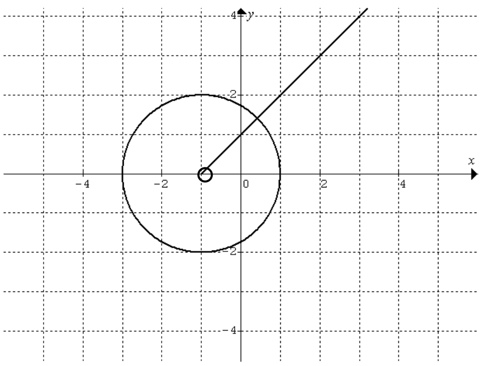
$$a = v\frac{dv}{dx} = \sqrt[3]{2 - x^2} \times \frac{-2x}{3\sqrt[3]{\left(2 - x^2\right)^2}}$$

$$= \frac{-2x}{3\sqrt[3]{2 - x^2}} = \frac{-2x}{3v}$$

SECTION 2

Question 1

a. Region $A = \left\{ z : Arg(1+z) = \frac{\pi}{4} \right\}$ represents a ray with gradient of 1 with the domain $(-1, +\infty)$, while region $B = \{ z : |1+z| = 2 \}$ is a circle with centre at (-1, 0) and radius 2.



Ray with correct domain (excluding -1) and gradient A2
Circle with correct centre and radius A2

b. $y = x + 1, x > -1$ A1

$(x+1)^2 + y^2 = 4$ A1

c. Method 1

If $|1+z_0| = 2$ and $Arg(1+z_0) = \frac{\pi}{4}$, then the polar form of 1+z_0 is $2cis\frac{\pi}{4} = 2\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$. M1

$z = -1 + \sqrt{2} + i\sqrt{2}$, as required. M1

Method 2

Solving the Cartesian equations simultaneously yields

$(x+1)^2 + (x+1)^2 = 4 \Rightarrow x = -1 \pm \sqrt{2}$ M1

As $x > -1$, $x = -1 + \sqrt{2}, y = \sqrt{2}$ and therefore, $z_0 = -1 + \sqrt{2} + i\sqrt{2}$. M1

d.

i $(1+z_0)^6 = 2^6 cis\frac{6\pi}{4}$ M1

$= 64\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$ A1

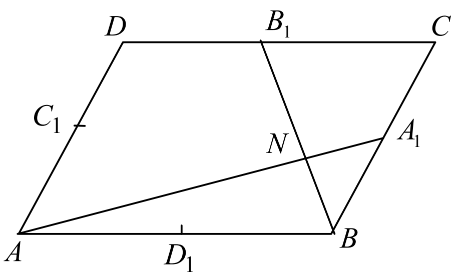
$= -64i$

ii $\sqrt{1+z_0} = \left(2cis\frac{\pi}{4}\right)^{\frac{1}{2}} = \sqrt{2}cis\frac{\pi}{8}$ and $\sqrt{2}cis\frac{9\pi}{8} = \sqrt{2}cis\left(-\frac{7\pi}{8}\right)$ A2

Question 2

a. $\overrightarrow{AA_1} = \mathbf{a} + \frac{\mathbf{1}}{\mathbf{2}}\mathbf{b}$, $\overrightarrow{BB_1} = \mathbf{b} - \frac{1}{2}\mathbf{a}$ A2

b.



$\overrightarrow{AN} = n\,\overrightarrow{AA_1} = n\left(\mathbf{a} + \frac{1}{2}\mathbf{b}\right)$ (1) , $\overrightarrow{BN} = m\,\overrightarrow{BB_1} = m\left(\mathbf{b} - \frac{1}{2}\mathbf{a}\right)$ A1

Also, $\overrightarrow{AN} = \overrightarrow{AB} + \overrightarrow{BN}$

$\overrightarrow{AN} = \mathbf{a} + m\mathbf{b} - \frac{m}{2}\mathbf{a}$ (2) M1

Equating (1) and (2) yields $\mathbf{a} + m\mathbf{b} - \frac{m}{2}\mathbf{a} = n\mathbf{a} + \frac{1}{2}n\mathbf{b}$

$$\left(1 - \frac{m}{2}\right)\mathbf{a} + m\mathbf{b} = n\mathbf{a} + \frac{n}{2}\mathbf{b}$$

$$1 - \frac{m}{2} = n,\ m = \frac{n}{2}$$ M1

Solving the last two equations simultaneously gives $n = \frac{4}{5},\ m = \frac{2}{5}$.

Therefore, $\overrightarrow{AN} = \frac{4}{5}\overrightarrow{AA_1},\ \overrightarrow{NA_1} = \frac{1}{5}\overrightarrow{AA_1} \Rightarrow AN : NA_1 = 4:1$ A1

c. $\overrightarrow{AA_1} \bullet \overrightarrow{BB_1} = \left(\mathbf{a} + \frac{\mathbf{1}}{\mathbf{2}}\mathbf{b}\right) \bullet \left(\mathbf{b} - \frac{\mathbf{1}}{\mathbf{2}}\mathbf{a}\right)$

M1

$$= \mathbf{a} \bullet \mathbf{b} - \frac{1}{2}|\mathbf{a}|^2 + \frac{1}{2}|\mathbf{b}|^2 - \frac{1}{4}\mathbf{a} \bullet \mathbf{b}$$

$$= \frac{3}{4}\mathbf{a} \bullet \mathbf{b} - \frac{1}{2}|\mathbf{a}|^2 + \frac{1}{2}|\mathbf{b}|^2 \quad (1)$$ M1

$\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos 60°$

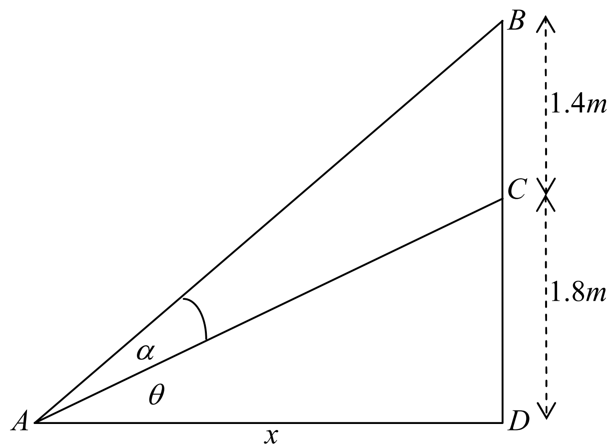
$= k \times \frac{3}{2}k \times \frac{1}{2} = \frac{3}{4}k^2$ (2) A1

Substituting (2) and $|\mathbf{a}| = k$, $|\mathbf{a}| = \frac{2}{3}|\mathbf{b}|$ into equation (1), we have

$\overrightarrow{AA_1} \bullet \overrightarrow{BB_1} = \frac{9}{16}k^2 - \frac{1}{2}k^2 + \frac{9}{8}k^2 = \frac{19}{16}k^2$ A1

Question 3

a. Let $\angle CAD = \theta$.



From the triangle ACD, $x = \dfrac{1.8}{\tan\theta} \Rightarrow \tan\theta = \dfrac{1.8}{x}$, A1

From the triangle ABD, $x = \dfrac{3.2}{\tan(\alpha+\theta)} \Rightarrow \tan(\alpha+\theta) = \dfrac{3.2}{x}$. A1

Further, $\dfrac{\tan\alpha + \dfrac{1.8}{x}}{1 - \tan\alpha \times \dfrac{1.8}{x}} = \dfrac{3.2}{x}$ M1

$\dfrac{1.8 + x\tan\alpha}{x - 1.8\tan\alpha} = \dfrac{3.2}{x}$

$1.8x + x^2\tan\alpha = 3.2x - 5.76\tan\alpha$ M1

$\tan\alpha(x^2 + 5.76) = 1.4x \Rightarrow \tan\alpha = \dfrac{1.4x}{x^2 + 5.76}$

b.

i. $f(x) = \dfrac{1.4x}{x^2 + 5.76}$

$f'(x) = \dfrac{1.4\left(x^2 + 5.76\right) - 1.4x \times 2x}{(x^2 + 5.76)^2}$ M1

$= \dfrac{1.4\left(x^2 + 5.76 - 2x^2\right)}{(x^2 + 5.76)^2}$ M1

$= \dfrac{1.4\left(5.76 - x^2\right)}{(x^2 + 5.76)^2}$, as required.

ii. The angle α will have a maximum value when $\tan\alpha$ has a maximum and therefore when *f(x)* has a maximum.

M1

$f'(x) = 0$ when $x = \pm\sqrt{5.76} = \pm 2.4$. As $x > 0$, x=2.4 M1

$\alpha = \tan^{-1}\dfrac{1.4\times 2.4}{2.4^2 + 5.76} = 16.26°$ A1

c. $\dfrac{d\alpha}{dt} = \dfrac{d\alpha}{dx}\dfrac{dx}{dt}$

$\dfrac{dx}{dt} = -1.2ms^{-1}$

$\alpha = \tan^{-1}\dfrac{1.4x}{x^2 + 5.76}$, when $x = 4$, $\dfrac{d\alpha}{dx} = -1.626978...$degrees per second (calculator) M1

$\dfrac{d\alpha}{dt} = -1.626978...\times -1.2 = 1.952374.. \approx 2$ degrees per second. pos. rate A1

correct value A1

Question 4

a. $y = \dfrac{e^{mx} + e^{-mx} - 2}{2m}$

$\dfrac{dy}{dx} = \dfrac{me^{mx} - me^{-mx}}{2m}$

$= \dfrac{e^{mx} - e^{-mx}}{2}$ M1

$\left(\dfrac{dy}{dx}\right)^2 = \dfrac{(e^{mx} - e^{-mx})^2}{4}$

$= \dfrac{e^{2mx} - 2e^{-mx}e^{mx} + e^{-2mx}}{4}$ M1

$= \dfrac{e^{2mx} - 2 + e^{-2mx}}{4}$, as required.

b.

$arc\ \ length = \int_a^b \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}\ dx$

$l = 2\int_0^d \sqrt{1 + \dfrac{e^{2mx} - 2 + e^{-2mx}}{4}} dx$ A1

$= 2\int_0^d \sqrt{\dfrac{e^{2mx} + 2 + e^{-2mx}}{4}} dx$

$= 2\int_0^d \sqrt{\dfrac{(e^{mx} + e^{-mx})^2}{4}} dx$ M1

$= 2\int_0^d \dfrac{e^{mx} + e^{-mx}}{2} dx$

$= \int_0^d (e^{mx} + e^{-mx}) dx$ A1

$= \dfrac{1}{m}\left[e^{mx} - e^{mx}\right]_0^d = \dfrac{e^{md} - e^{-md}}{m}$, as required. M1

c.

i. When $h = 20,\ x = d$ and $m = \dfrac{1}{32}$, we have $20 = \dfrac{e^{\frac{d}{32}} + e^{-\frac{d}{32}} - 2}{\frac{1}{16}}$

The solution can be found by using a graphics calculator $d = 34.135438....$
The distance between the poles is 68.27 metres (2 dp). A1

ii The length of the cable is $81.975607\ldots \approx 81.98$ metres. A1

d. $100 = \dfrac{e^{40m} - e^{-40m}}{m}$ M1

$m = 0.02956814.... \approx 0.03$ A1

Question 5

a. **i.** $\mathbf{r}_A = (1 + \cos(kt))\mathbf{i} + (2 + 2\sin(kt))\mathbf{j}$

$x = 1 + \cos(kt) \Rightarrow \cos(kt) = x - 1$

$y = 2 + 2\sin(kt) \Rightarrow \sin(kt) = \dfrac{y-2}{2}$ A1

Squaring both equations and then adding them results in the given equation.

$\cos^2(kt) = (x-1)^2, \quad \sin^2(kt) = \left(\dfrac{y-2}{2}\right)^2$

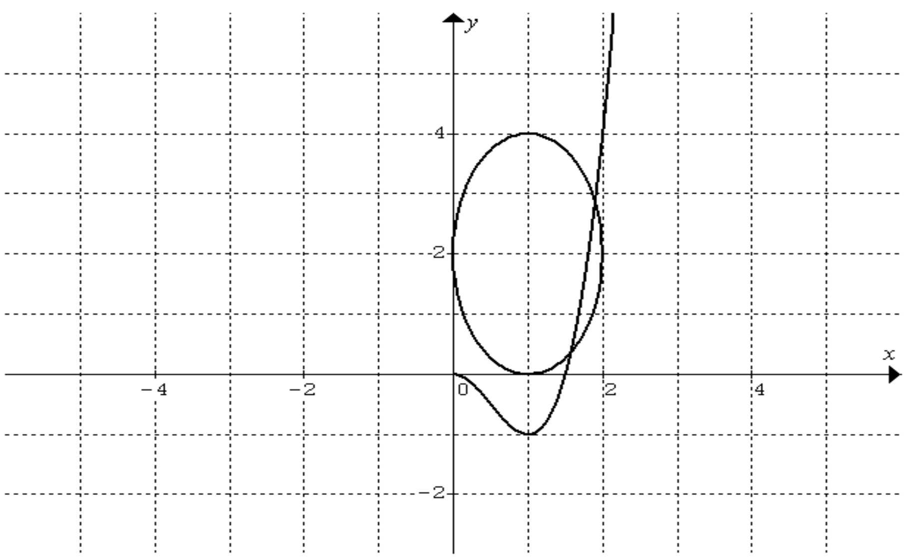
$\cos^2(kt) + \cos^2(kt) = 1, \quad (x-1)^2 + \dfrac{(y-2)^2}{4} = 1$ M1

ii. $\mathbf{r}_B = t\,\mathbf{i} + (2t^3 - 3t^2)\mathbf{j}$

$x = t, y = 2t^3 - 3t^2 \Rightarrow y = 2x^3 - 3x^2, \; x \geq 0$ A1

(domain must be given)

b.



Both graphs correct, including the domain for particle *B* A2

c. The velocity of particle A is $\dot{\mathbf{r}}_A = (-k\sin(kt))\mathbf{i} + (2k\cos(kt))\mathbf{j}$ M1

and the speed is

$$|\dot{\mathbf{r}}_A| = \sqrt{k^2\sin^2(kt) + 4k^2\cos^2(kt)}$$

$$= k\sqrt{1+3\cos^2(kt)}$$ M2

The maximum of $1+3\cos^2(kt)$ is 4. Therefore, the maximum speed of particle A is $2k$ A1

d. The points of intersection are (1.573, 0.361) and (1.899, 2.876) A2

e. The particles will collide when they are at the same position at the same time.

For the first point, $t = x = 1.573$, so $1+\cos(1.573k) = 1.573 \Rightarrow k = 0.610808...$ M1

Now, for $k = 0.610808...$, $y = 2 + 2\sin(0.610808 \times 1.573) = 3.6393...$, so this is not a point of collision.

For the second point , $t = x = 1.899$ M1

$1+\cos(1.899k) = 1.899 \Rightarrow k = 0.23871277...$ and

$y = 2 + 2\sin(0.23871277 \times 1.899) = 2.8758.....$, so this is the first point of collision.

Therefore, the minimum value of k is 0.239(3dp). A1