THE SCHOOL FOR EXCELLENCE
UNIT 3 & 4 SPECIALIST MATHEMATICS 2010
COMPLIMENTARY WRITTEN EXAMINATION 1 - SOLUTIONS

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MARKING SCHEME

- **(A** $4\times\frac{1}{2}\downarrow$ **)** means four answer half-marks rounded **down** to the next integer. Rounding occurs at the end of a part of a question.
- **M1** = 1 **M**ethod mark.
- **A1** = 1 **A**nswer mark (it **must** be this or its equivalent).
- **H1** = 1 consequential mark (**H**is/**H**er mark…correct answer from incorrect statement or slip).

QUESTION 1

(a) $u = e^x \therefore du = e^x dx,\ u^2 = e^{2x}$ **M1**

If $x = 0,\ \therefore u = 1$ and if $x = \ln 3,\ \therefore u = 3$ **M1**

$$\therefore \int_0^{\ln 3} \frac{e^x}{e^{2x}+9}\ dx = \int_1^3 \frac{du}{u^2+9}$$

(b) $\int_1^3 \frac{du}{u^2+9} = \left[\frac{1}{3}\tan^{-1}\left(\frac{u}{3}\right)\right]_1^3$ **M1**

$$= \left(\tfrac{1}{3}\tan^{-1} 1\right) - \left(\tfrac{1}{3}\tan^{-1}\tfrac{1}{3}\right)$$

$= \frac{1}{3}\times\frac{\pi}{4} - \frac{1}{3}\tan^{-1}\left(\frac{1}{3}\right)$ **M1**

$= \dfrac{\pi - 4\tan^{-1}\left(\frac{1}{3}\right)}{12}$ as required.

Total = 4 marks

QUESTION 2

(a) $\overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ},\quad \overrightarrow{RP} = \overrightarrow{OP} - \overrightarrow{OR}$ **A1**

(b) $\overrightarrow{OP} \perp \overrightarrow{QR} \ \therefore \overrightarrow{OP} \bullet \overrightarrow{QR} = 0$

$$\therefore \overrightarrow{OP} \bullet \left(\overrightarrow{OR} - \overrightarrow{OQ}\right) = 0$$

$\therefore \overrightarrow{OP} \bullet \overrightarrow{OR} = \overrightarrow{OP} \bullet \overrightarrow{OQ}$ **M1**

$\overrightarrow{OQ} \perp \overrightarrow{RP} \ \therefore \overrightarrow{OQ} \bullet \overrightarrow{RP} = 0$

$$\therefore \overrightarrow{OQ} \bullet \left(\overrightarrow{OP} - \overrightarrow{OR}\right) = 0$$

$\therefore \overrightarrow{OQ} \bullet \overrightarrow{OP} = \overrightarrow{OQ} \bullet \overrightarrow{OR}$ **M1**

$$\therefore \overrightarrow{OP} \bullet \overrightarrow{OR} = \overrightarrow{OQ} \bullet \overrightarrow{OR}$$

$$\therefore \overrightarrow{OR} \bullet \left(\overrightarrow{OP} - \overrightarrow{OQ}\right) = 0$$

$$\therefore \overrightarrow{OR} \bullet \overrightarrow{QP} = 0$$

Therefore $\overrightarrow{OR}$ perpendicular to $\overrightarrow{QP}$, as required. **M1**

Total = 4 marks

QUESTION 3

(a) $x = 1 \;\therefore e^{y} - y^{2}\log_{e}1 = e$

$\therefore e^{y} = e$

$\therefore y = 1$

$\therefore a = 1$ **A1**

(b)

$$\frac{d}{dx}\left(e^{xy}\right) = e^{xy}\left(x\frac{dy}{dx} + y\right)$$

$$\frac{d}{dx}\left(y^{2}\log_{e}x\right) = \frac{y^{2}}{x} + 2y\frac{dy}{dx}\log_{e}x$$ **M1**

$$\frac{d}{dx}(e) = 0$$

$$\therefore e^{xy}\left(x\frac{dy}{dx} + y\right) - \frac{y^{2}}{x} - 2y\frac{dy}{dx}\log_{e}x = 0$$

Substitute $x = 1$ and $y = 1$:

$$e\left(\frac{dy}{dx} + 1\right) - 1 = 0$$ **M1**

$$\frac{dy}{dx} + 1 = \frac{1}{e}$$

$$\frac{dy}{dx} = \frac{1}{e} - 1$$

A1

Total = 4 marks

QUESTION 4

(a)

$$\frac{d}{dx}\left(\tfrac{1}{2}v^2\right)=\frac{2x}{x^2+1}$$

$$\tfrac{1}{2}v^2=\int\frac{2x}{x^2+1}\,dx=\ln\left|x^2+1\right|+c$$ **M1**

Substitute $v=2,\ x=1\ \therefore\ 2=\ln 2+c\ \therefore\ c=2-\ln 2$ **M1**

$$\therefore \tfrac{1}{2}v^2=\ln\left|x^2+1\right|-\ln 2+2$$

$$\therefore v^2=2+\ln\left|\frac{x^2+1}{2}\right|$$

$$\therefore v=\sqrt{2+\ln\left|\frac{x^2+1}{2}\right|}$$

Positive root only since velocity must be positive . **A1**

(b) $x=5\ \ \therefore v=\sqrt{2+\ln 13}$

Therefore $a=13$ and $b=2$. **A1**

Total = 4 marks

QUESTION 5

(a) $z=\tfrac{1}{2}cis\left(\tfrac{2\pi}{3}\right)=\tfrac{1}{2}\left(\cos(\tfrac{2\pi}{3})+i\sin(\tfrac{2\pi}{3})\right)$

$w=\tfrac{1}{2}cis\left(\tfrac{\pi}{4}\right)=\tfrac{1}{2}\left(\cos(\tfrac{\pi}{4})+i\sin(\tfrac{\pi}{4})\right)$ **A2**

(b) $zw=\tfrac{1}{2}\times\tfrac{1}{2}\times cis\left(\tfrac{2\pi}{3}+\tfrac{\pi}{4}\right)=\tfrac{1}{4}cis\left(\tfrac{11\pi}{12}\right)=\tfrac{1}{4}\left(\cos(\tfrac{11\pi}{12})+i\sin(\tfrac{11\pi}{12})\right)$ **M1**

(c) (i) $zw=\dfrac{(-1+i\sqrt{3})(\sqrt{2}+i\sqrt{2})}{16}=\dfrac{-\sqrt{2}-i\sqrt{2}+i\sqrt{6}-\sqrt{6}}{16}=\left(\dfrac{-\sqrt{2}-\sqrt{6}}{16}\right)+i\left(\dfrac{\sqrt{6}-\sqrt{2}}{16}\right)$ **A1**

(ii) $\cos\left(\tfrac{11\pi}{12}\right)=\dfrac{-\sqrt{2}-\sqrt{6}}{4}$ and $\sin\left(\tfrac{11\pi}{12}\right)=\dfrac{-\sqrt{2}+\sqrt{6}}{4}$ **A1**

Total = 5 marks

QUESTION 6

(a) $\frac{dx}{dt} = -60kmh^{-1}$ **A1**

$\frac{dy}{dt} = -70kmh^{-1}$

(b) $z^2 = x^2 + y^2, \quad x = 0.8 \ and \ \ y = 0.6 \ \therefore z = 1$ **M1**

$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$ **M1**

$z\frac{dz}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$

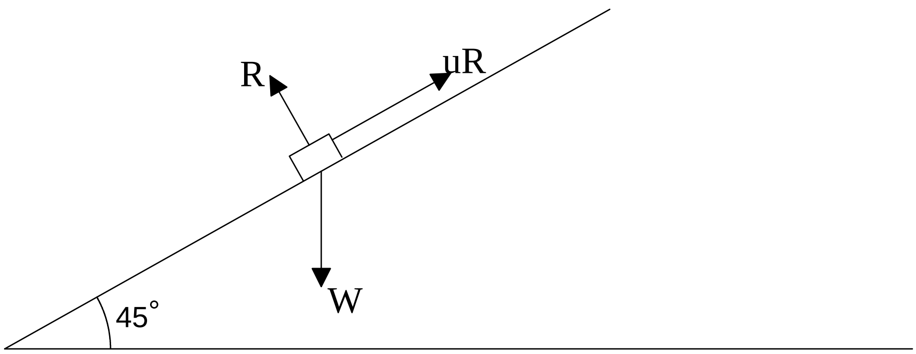
$\frac{dz}{dt} = 0.8 \times (-60) + 0.6 \times (-70) = -90kmh^{-1}$

Therefore z is decreasing at the rate of 90 kmh^{-1}. **A1**

Total = 4 marks

QUESTION 7

(a)



A1

(b) Resolving parallel to plane: $W\sin 45^0 - \mu R = 0 \ \therefore W\sin 45^0 = \mu R$

Resolving perpendicular to plane: $R - W\cos 45^0 = 0 \ \therefore R = W\cos 45^0$

$W\sin 45^0 = \mu W\cos 45^0$

$\mu = \frac{\sin 45^0}{\cos 45^0} = 1$ **A1**

(c) (i) $\mu R = 1 \times 2g = 19.6 \ Newtons$ but only require friction of *9.8 Newtons* to stop the motion. Therefore friction = *9.8 Newtons* **South.** **A1**

(ii) Acceleration = 0 **A1**

(d) $\sum \vec{F} = m\vec{a} \therefore 29.4 - 19.6 = 2a \;\; \therefore a = 4.9 ms^{-2}$

$\because$ *uniform acceleration*, $v^2 = u^2 + 2as,\; u = 0,\; v = 7,\; a = 4.9$

$\therefore s = \dfrac{49}{9.8} = 5 \text{ metres}$

Therefore object travels 5 metres. **A1**

Total = 5 marks

QUESTION 8

$\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j} \;\; \therefore \underset{\sim}{v} = \dfrac{dx}{dt}\underset{\sim}{i} + \dfrac{dy}{dt}\underset{\sim}{j}$ **M1**

$\dfrac{dx}{dt} = \frac{1}{3},\; \dfrac{dy}{dt} = \dfrac{dy}{dx} \times \dfrac{dx}{dt} = \left(6x - 3x^2\right) \times \frac{1}{3} = 2x - x^2$

If velocity is horizontal, then $\dfrac{dy}{dt} = 0$.

$\therefore 2x - x^2 = 0$

$\therefore x(2 - x) = 0$

$\therefore x = 0 \;\; or \;\; x = 2$

M1

$\underset{\sim}{v} = \frac{1}{3}\underset{\sim}{i} + (2x - x^2)\underset{\sim}{j}$

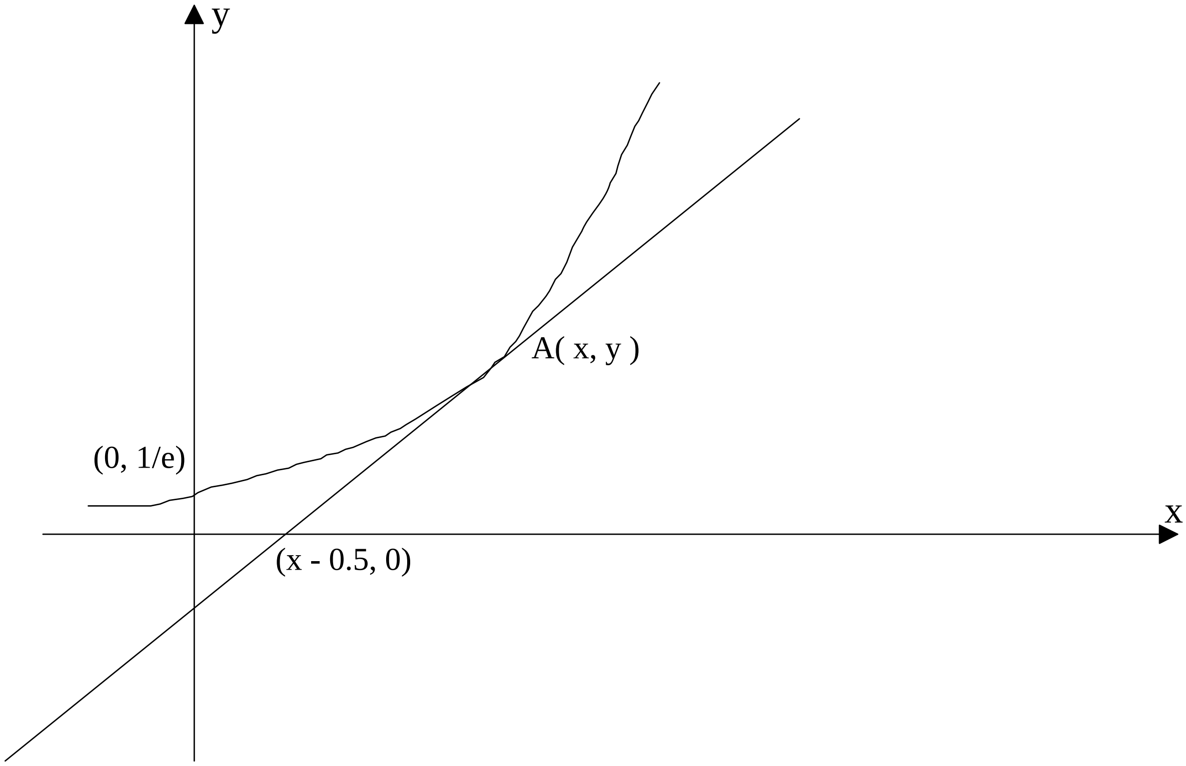
$\therefore \underset{\sim}{a} = \dfrac{dv}{dt} = 0\underset{\sim}{i} + \dfrac{d}{dt}(2x - x^2)\underset{\sim}{j} = \dfrac{d}{dx}(2x - x^2) \times \dfrac{dx}{dt}\underset{\sim}{j} = (2 - 2x) \times \frac{1}{3}\underset{\sim}{j} = \left(\frac{2}{3} - \frac{2x}{3}\right)\underset{\sim}{j}$

If *x = 0,* then $\underset{\sim}{a} = \frac{2}{3}\underset{\sim}{j}$

If *x = 2,* then $\underset{\sim}{a} = -\frac{2}{3}\underset{\sim}{j}$ **A1**

Total = 3 marks

QUESTION 9



Gradient of tangent is $\frac{y}{\frac{1}{2}} = 2y = \frac{dy}{dx}$

$\therefore \frac{dx}{dy} = \frac{1}{2y}$ **M1**

$\therefore x = \frac{1}{2}\ln|y| + c$

$when \quad x = 0, \; y = \frac{1}{e} \;\; \therefore c = -\frac{1}{2}\ln\left|\frac{1}{e}\right| = -\frac{1}{2} \times -1 = \frac{1}{2}$

M1

$\therefore x = \frac{1}{2}\ln|y| + \frac{1}{2}$

$\therefore 2x - 1 = \ln|y|$

$\therefore y = e^{2x-1} \quad hence \;\; equation \; of \;\; curve \; is \;\; f(x) = e^{2x-1}.$

A1

Total = 3 marks

QUESTION 10

(a) Sum of angles in $\Delta ABC = \alpha + \alpha + \alpha + 2\alpha = 5\alpha$

$\therefore 5\alpha = 180^0$

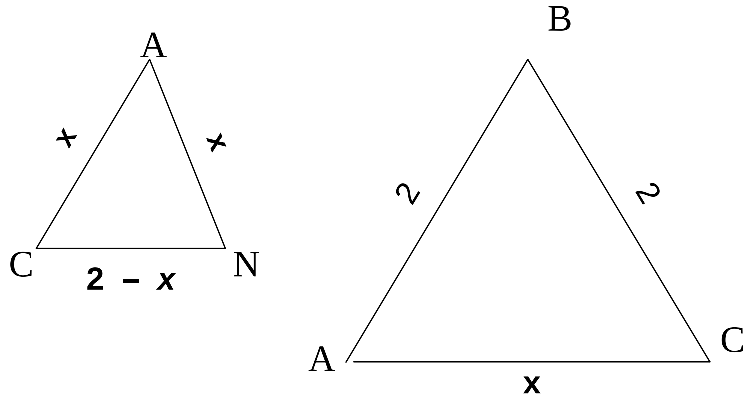
$\therefore \alpha = 36^0$ **M1**

(b) ΔABC has two angles of 72^0, therefore isosceles.

In $\Delta ANC,\ \angle ANC = 180^0 - 3\alpha = 180^0 - 108^0 = 72^0$

Therefore ΔANC also has two angles of 72^0. **M1**

(c) ΔABC is similar to ΔANC because 3 pairs of equal corresponding angles.



Using ratios of corresponding sides:

$\frac{x}{2-x} = \frac{2}{x} \quad \therefore x^2 = 4 - 2x \quad \therefore x^2 + 2x - 4 = 0$

$\therefore x = -1 \pm \sqrt{5}$

But $x > 0$, therefore $x = -1 + \sqrt{5}$. **A1**

M is midpoint of AB, because ΔABN is isosceles. Therefore AM = 1

$AN = AC = x = -1 + \sqrt{5}$

and $\therefore \cos\alpha = \frac{1}{-1+\sqrt{5}} = \frac{1}{-1+\sqrt{5}} \times \frac{1+\sqrt{5}}{1+\sqrt{5}} = \frac{1+\sqrt{5}}{4}$

$\therefore \cos\left(\frac{\pi}{5}\right) = \frac{1+\sqrt{5}}{4}$

A1

Total = 4 marks

END OF SOLUTIONS TO EXAMINATION 1