Neap:

Trial Examination 2010

VCE Specialist Mathematics Units 3 & 4

Written Examination 1

Suggested Solutions

Question 1

Method 1:

Factorising in two groups of two we obtain:

$z^3 - \sqrt{2}z^2 + z - \sqrt{2} = z^2(z - \sqrt{2}) + 1(z - \sqrt{2})$

So $(z^2 + 1)(z - \sqrt{2}) = 0$. M1

Hence $z = \pm i,\ \sqrt{2}$. A1

Method 2:

Let $P(z) = z^3 - \sqrt{2}z^2 + z - \sqrt{2}$.

Showing that either $P(\sqrt{2}) = 0$ or $P(i) = 0$ or $P(-i) = 0$. M1

Using a suitable method to obtain $(z^2 + 1)(z - \sqrt{2}) = 0$.

Hence $z = \pm i,\ \sqrt{2}$. A1

Question 2

a. Parametric equations are $x(t) = -3\sin\left(\frac{t}{2}\right)$ and $y(t) = 4\cos\left(\frac{t}{2}\right) - 1$.

$-\frac{x}{3} = \sin\left(\frac{t}{2}\right)$ and $\frac{y+1}{4} = \cos\left(\frac{t}{2}\right)$ A1

Squaring both equations we obtain:

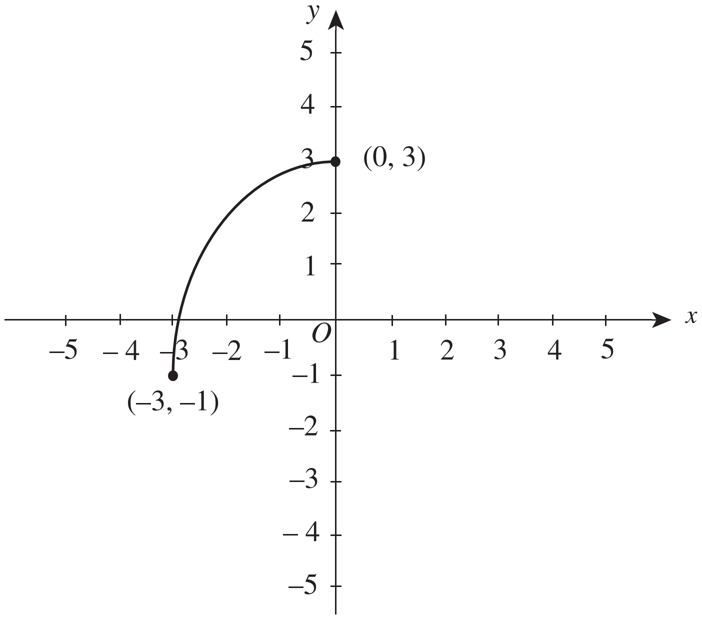
$\frac{x^2}{9} = \sin^2\left(\frac{t}{2}\right)$ and $\frac{(y+1)^2}{16} = \cos^2\left(\frac{t}{2}\right)$

Adding both equations: M1

$\frac{x^2}{9} + \frac{(y+1)^2}{16} = \sin^2\left(\frac{t}{2}\right) + \cos^2\left(\frac{t}{2}\right)$

As $\sin^2\left(\frac{t}{2}\right) + \cos^2\left(\frac{t}{2}\right) = 1$, we obtain $\frac{x^2}{9} + \frac{(y+1)^2}{16} = 1$. A1

b.



Correct shape A1

Correct end points A1

Question 3

Let $u = 1 + x$ and so $du = dx$.

When $x = -1$, $u = 0$.

When $x = 0$, $u = 1$.

So $15\int_{-1}^{0} x\sqrt{1+x}\, dx = 15\int_{0}^{1} (u-1)\sqrt{u}\, du$. M1

$$= 15\int_{0}^{1}\left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)du$$

$$= 15\left[\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right]_0^1$$ A1

$$= 15\left(\left(\frac{2}{5} - \frac{2}{3}\right) - 0\right)$$

$$= -4$$ A1

Question 4

If $m\underset{\sim}{u} + n\underset{\sim}{v} + p\underset{\sim}{w} = \underset{\sim}{0}$ only when $m = n = p = 0$, then $\underset{\sim}{u}$, $\underset{\sim}{v}$ and $\underset{\sim}{w}$ are linearly independent.

$m(\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}) + n(2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}) + p(\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}) = \underset{\sim}{0}$ A1

For the above vector equation to be satisfied, the coefficients of $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ must all be zero.

$m + 2n + p = 0$ (1)

$m + n + 2p = 0$ (2)

$-m - 2n + p = 0$ (3)

For stating the above three equations A1

(1) + (3) gives $2p = 0$ i.e. $p = 0$.

Substitute $p = 0$ into (1) and (2) to obtain:

$m + 2n = 0$ (4)

$m + n = 0$ (5)

(4) – (5) gives $n = 0$ and so $m = 0$.

Attempting to solve the above system of equations. M1

So $m = n = p = 0$ and thus $\underset{\sim}{u}$, $\underset{\sim}{v}$ and $\underset{\sim}{w}$ are linearly independent. A1

Question 5

Let A be the area of the shaded region.

$$A = \int_0^{\frac{\pi}{4}} \sin(2x) - 2\sin^2(x)\ dx$$ A1

Using $\cos(2x) = 1 - 2\sin^2(x)$ i.e. $-2\sin^2(x) = \cos(2x) - 1$ we obtain:

$$\text{Area} = \int_0^{\frac{\pi}{4}} \sin(2x)\ dx + \int_0^{\frac{\pi}{4}} \cos(2x) - 1\ dx \text{ (or equivalent)}$$ M1

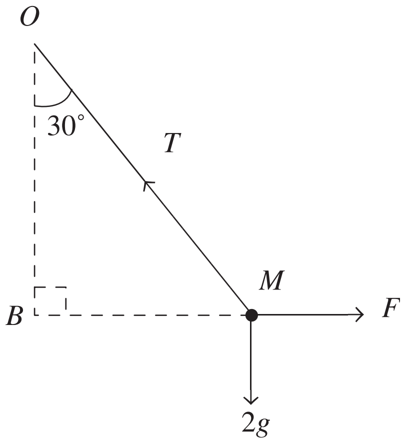
$$= \left[-\frac{1}{2}\cos(2x)\right]_0^{\frac{\pi}{4}} + \left[\frac{1}{2}\sin(2x)\right]_0^{\frac{\pi}{4}} - [x]_0^{\frac{\pi}{4}}$$ A1

$$= -\frac{1}{2}\cos\left(\frac{\pi}{2}\right) + \frac{1}{2}\cos(0) + \frac{1}{2}\sin\left(\frac{\pi}{2}\right) - \frac{1}{2}\sin(0) - \frac{\pi}{4} + 0$$

$$= 0 + \frac{1}{2} + \frac{1}{2} - 0 - \frac{\pi}{4}$$

$$= 1 - \frac{\pi}{4} \text{ (square units)}$$ A1

Question 6



The forces are in equilibrium i.e. $\Sigma \underset{\sim}{F} = \underset{\sim}{0}$.

$OM = 4$ (m) and $BM = 2$ (m) so that $\sin(B\hat{O}M) = \frac{1}{2}$ and $B\hat{O}M = 30°$.

Horizontally: $F - T\sin(30°) = 0$ A1

Vertically: $T\cos(30°) - 2g = 0$ A1

Substituting $T = \frac{2g}{\cos(30°)}$ into $F = T\sin(30°)$ gives $F = \frac{2\sqrt{3}g}{3}$ (newtons). A1

Question 7

a. $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 3x^2 - 12x$

$$\frac{1}{2}v^2 = \int(3x^2 - 12x)dx$$

$$\frac{1}{2}v^2 = x^3 - 6x^2 + c$$ M1

At $x = 0$, $v = 4\sqrt{2}$ and so $c = 16$.

Hence $\frac{1}{2}v^2 = x^3 - 6x^2 + 16$ and so $v^2 = 2(x^3 - 6x^2 + 16)$. A1

b. At $x = 2$, $v = 2(2^3 - 6 \times 2^2 + 16) = 0$ i.e. the particle's velocity is zero.

At $x = 2$, $a = -12$ i.e. the particle's acceleration is –12 m/s^2.

At $x = 2$, $v = 0$ and $a = -12$. A1

Hence the particle moves towards O from $x = 2$. A1

Question 8

If $\frac{dy}{dx} = f(x)$, $x_0 = a$ and $y_0 = b$, then $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + hf(x_n)$.

Here $x_0 = 2$, $y_0 = 1$ and $h = 0.1$.

For $x = 2.1$, using $y_1 = y_0 + hf(x_0)$, we obtain $y_1 = 1 + 0.1e^{-2}$. A1

For $x = 2.2$, using $y_2 = y_1 + hf(x_1)$, we obtain $y_2 = y_1 + 0.1e^{-2.1}$.

So $y_2 = y_1 + 0.1e^{-2.1}$. A1

Question 9

Attempting implicit differentiation i.e. $\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$. M1

So, $\frac{dy}{dx} = \frac{b^2 x}{a^2 y}$. A1

The equation of the tangent line at the point $P(x_1, y_1)$ is given by $y - y_1 = \frac{b^2 x_1}{a^2 y_1}(x - x_1)$. A1

$$\frac{y_1}{b^2}(y - y_1) = \frac{x_1}{a^2}(x - x_1)$$

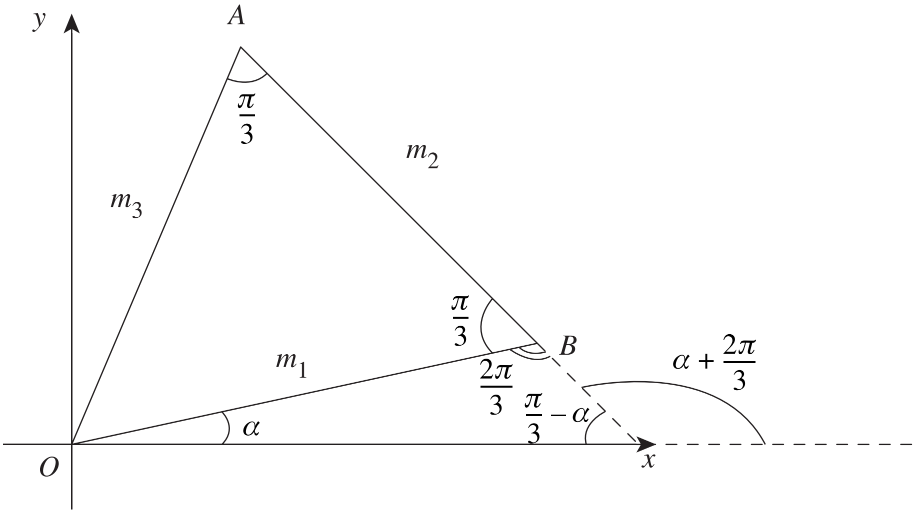
$$\frac{y_1 y}{b^2} - \frac{(y_1)^2}{b^2} = \frac{x_1 x}{a^2} - \frac{(x_1)^2}{a^2} \qquad \text{A1}$$

$$\frac{(x_1)^2}{a^2} - \frac{(y_1)^2}{b^2} = \frac{x_1 x}{a^2} - \frac{y_1 y}{b^2}$$

Given that $\frac{(x_1)^2}{a^2} - \frac{(y_1)^2}{b^2} = 1$, the equation of the tangent line becomes $\frac{x_1 x}{a^2} - \frac{y_1 y}{b^2} = 1$. A1

Question 10

a.



$m_2 = \tan\left(\alpha + \frac{2\pi}{3}\right)$ A1

Using $\tan(x + y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}$ we obtain: M1

$$m_2 = \tan\left(\alpha + \frac{2\pi}{3}\right) = \frac{\tan(\alpha) + \tan\left(\frac{2\pi}{3}\right)}{1 - \tan(\alpha)\tan\left(\frac{2\pi}{3}\right)}$$ A1

As $\tan\left(\frac{2\pi}{3}\right) = -\sqrt{3}$, $m_2 = \frac{\tan(\alpha) - \sqrt{3}}{1 + \sqrt{3}\tan(\alpha)}$. A1

b. $m_1 m_2 = \frac{\tan(\alpha)(\tan(\alpha) - \sqrt{3})}{1 + \sqrt{3}\tan(\alpha)}$, $m_2 m_3 = \frac{\tan^2(\alpha) - 3}{1 - 3\tan^2(\alpha)}$, $m_3 m_1 = \frac{\tan(\alpha)(\tan(\alpha) + \sqrt{3})}{1 - \sqrt{3}\tan(\alpha)}$ A1

$m_1 m_2 + m_2 m_3 + m_3 m_1$

$$= \frac{\tan(\alpha)(1 - \sqrt{3}\tan(\alpha))(\tan(\alpha) - \sqrt{3}) + (\tan^2(\alpha) - 3) + \tan(\alpha)(1 + \sqrt{3}\tan(\alpha))(\tan(\alpha) + \sqrt{3})}{1 - 3\tan^2(\alpha)}$$

M1

$$= \frac{-\sqrt{3}\tan^3(\alpha) + 4\tan^2(\alpha) - \sqrt{3}\tan(\alpha) + \tan^2(\alpha) - 3 + \sqrt{3}\tan^3(\alpha) + 4\tan^2(\alpha) + \sqrt{3}\tan(\alpha)}{1 - 3\tan^2(\alpha)}$$

$$= \frac{9\tan^2(\alpha) - 3}{1 - 3\tan^2(\alpha)}$$ A1

$$= \frac{-3(1 - 3\tan^2(\alpha))}{1 - 3\tan^2(\alpha)}$$

$= -3$ A1