2010 VCAA Specialist Math Exam 2 Solutions

Section 1

1	2	3	4	5	6	7	8	9	10	11
D	E	A	B	E	B	D	C	E	D	C

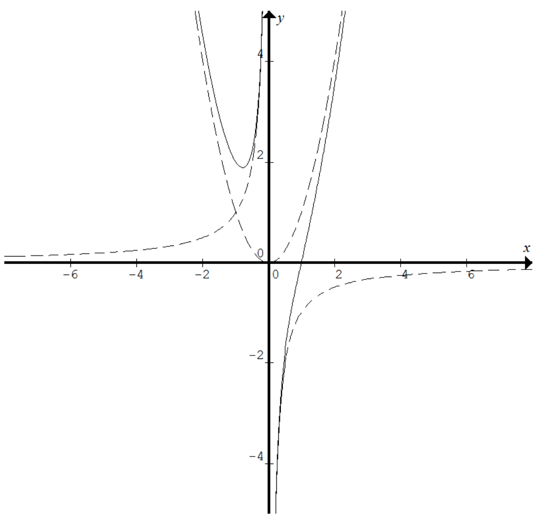
12	13	14	15	16	17	18	19	20	21	22
E	B	A	E	C	C	B	A	D	C	D

Q1 $\frac{(x-3)^2}{1^2}+\frac{(y+5)^2}{2^2}=1$, $4(x-3)^2+(y+5)^2=4$ D

Q2 Change $2x^2-4x-y^2-4y=4$ to $\frac{(x-1)^2}{1^2}-\frac{(y+2)^2}{\left(\sqrt{2}\right)^2}=1$.

Gradients of asymptotes are $\pm\sqrt{2}$, $\sqrt{2}\times\left(-\sqrt{2}\right)\neq -1$, .: the asymptotes are not perpendicular. E

Q3 $f(x)=\frac{x^k+a}{x}=x^{k-1}+\frac{a}{x}$ where $k-1$ is even and $a<0$.



A

Q4 $x=\sin(t)$, $y=\cos(2t)=1-2\sin^2(t)$, .: $y=1-2x^2$ B

Q5 $y=2\sec^2(x)$, $y=5|\tan(x)|$ and $-\frac{\pi}{2}<x<\frac{\pi}{2}$.

Consider $0\le x<\frac{\pi}{2}$, $y=2\sec^2(x)$, $y=5\tan(x)$

Let $2\sec^2(x)=5\tan(x)$, .: $2\tan^2(x)-5\tan(x)+2=0$,

$(2\tan(x)-1)(\tan(x)-2)=0$,

.: $x=\tan^{-1}\left(\frac{1}{2}\right)$ and $y=\frac{5}{2}$, or $x=\tan^{-1}(2)$ and $y=10$.

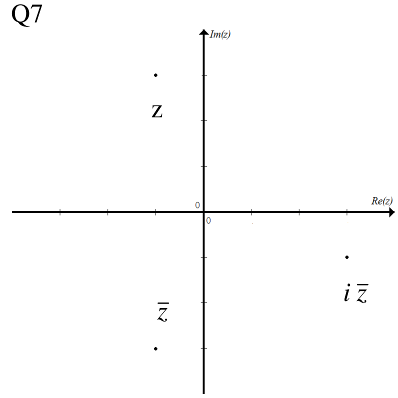
Both functions are symmetrical about the y-axis. E

Quicker by CAS/calculator.

Q6 $z=cis\left(\frac{5\pi}{6}\right)=-\frac{\sqrt{3}}{2}+\frac{1}{2}i$, .: $z-i=-\frac{\sqrt{3}}{2}-\frac{1}{2}i$,

.: $\text{Im}(z-i)=-\frac{1}{2}$. B

Q7 D



Q8 $P(z)=z(z-(3+i))(z-(2-i))=z^3-5z^2+(7-i)z$ and it has only one complex coefficient. C

Q9 $z=4cis\left(\frac{2\pi}{3}\right)$,

$z^5=4^5cis\left(\frac{10\pi}{3}\right)=4^5cis\left(\frac{10\pi}{3}-4\pi\right)=4^5cis\left(-\frac{2\pi}{3}\right)$,

.: $Arg\left(z^5\right)=-\frac{2\pi}{3}$ E

Q10 Let $z=x+yi$, $(z+\bar{z})^2-(z-\bar{z})^2=16$,

.: $(2x)^2-(2yi)^2=16$, .: $x^2+y^2=4$ D

Q11 As V increases (i.e. move up), the slope (i.e. $\frac{dV}{dt}$) becomes 'more' negative.

$\frac{dV}{dt}=-kV^2$, where $k>0$, fits the description. C

Q12 $y_1\approx y_0+h\times\left.\frac{dy}{dx}\right|_{x=x_0}$, $y_1\approx 2+0.1\times\frac{0+2}{1}=2.20$ E

Q13 $\frac{dx}{dt}=-0.15x$, $\frac{dt}{dx}=-\frac{1}{0.15}\frac{1}{x}$

.: $-0.15t=\int\frac{1}{x}dx$ where $x>0$

.: $-0.15t=\log_e(x)+c$

Let $x=a$ initially (i.e. $t=0$), .: $c=-\log_e(a)$

.: $-0.15t=\log_e\left(\frac{x}{a}\right)$

When $x=\frac{a}{2}$, $t=\frac{20}{3}\log_e(2)$ B

Q14 Let $u = x^2 - 1$, $\frac{du}{dx} = 2x$, .: $x = \frac{1}{2}\frac{du}{dx}$

$$\int_0^2 \frac{x}{\sqrt{x^2-1}}dx = \int_0^2 \frac{\frac{1}{2}\frac{du}{dx}dx}{\sqrt{u}} = \frac{1}{2}\int_{-1}^{3}\frac{du}{\sqrt{u}} = \frac{1}{2}\int_{-1}^{3} u^{-\frac{1}{2}}du \qquad \text{A}$$

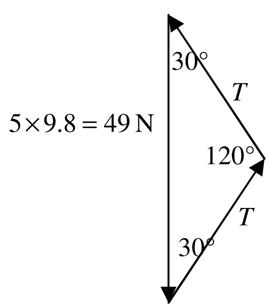
Q15 $\tilde{a} = 3\tilde{i} - \tilde{k}$, $\hat{b} = \frac{1}{3}\left(2\tilde{i} - \tilde{j} - 2\tilde{k}\right)$

.: $\tilde{a}.\hat{b} = \frac{1}{3}(6+0+2) = \frac{8}{3}$ E

Q16 $\tilde{d} = 5\tilde{i} - \tilde{j} + \sqrt{10}\tilde{k}$, $\left|\tilde{d}\right|^2 = \tilde{d}.\tilde{d} = 25+1+10 = 36$ C

Q17 $\cos\theta = \frac{\tilde{a}.\tilde{b}}{\left|\tilde{a}\right|\left|\tilde{b}\right|} = \frac{1}{2}$, $\theta = \frac{\pi}{3}$ C

Q18 Join the force vector head to tail to form the following triangle.



$\frac{T}{\sin 30°} = \frac{49}{\sin 120°}$, $T \approx 28.3$ N B

Q19 Take northerly direction as positive.
$a = +2$, $v = +30$, $s = +100$, use $v^2 = u^2 + 2as$ to find the initial velocity $u = +10\sqrt{5}$. A

Q20 $\frac{dv}{dt} = \frac{v}{\log_e(v)}$

$\frac{dt}{dv} = \frac{\log_e(v)}{v}$

.: $t = \int \frac{\log_e(v)}{v}dv$

Let $u = \log_e(v)$, .: $\frac{du}{dv} = \frac{1}{v}$

$t = \int u\,du = \frac{u^2}{2} + c = \frac{(\log_e(v))^2}{2} + c$ and $v = 5$ at $t = 0$

.: $c = -\frac{(\log_e(5))^2}{2}$ and $t = \frac{(\log_e(v))^2}{2} - \frac{(\log_e(5))^2}{2}$

.: $(\log_e(v))^2 = 2t + (\log_e(5))^2$

.: $\log_e(v) = \sqrt{2t + (\log_e(5))^2}$, satisfying $v = 5$ at $t = 0$

.: $v = e^{\sqrt{2t+(\log_e(5))^2}}$ D

Q21 Resultant force on the system is $Mg - 5g$.

Newtons second law: $Mg - 5g = (M+5)\frac{7}{5}$, .: $M = 6\frac{2}{3}$ C

Q22 Given $x = x_0$, $v = v_0$ and $x = x_1$, $v = v_1$

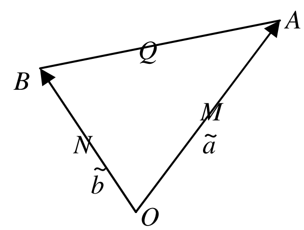
$a = \frac{1}{2}\frac{d}{dx}\left(v^2\right) = \frac{F(x)}{m}$

.: $\frac{d}{dx}\left(v^2\right) = \frac{2F(x)}{m}$, $v^2 = \frac{2}{m}\int_{x_0}^{x} F(x)dx + {v_0}^2$

.: ${v_1}^2 = \frac{2}{m}\int_{x_0}^{x_1} F(x)dx + {v_0}^2$.: $v_1 = \sqrt{\frac{2}{m}\int_{x_0}^{x_1} F(x)dx + {v_0}^2}$ D

Section 2

Q1ai



M, *N* and *Q* are midpoints. $\overrightarrow{MA} = \frac{1}{2}\tilde{a}$

Q1aii $\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = \tilde{a} - \tilde{b}$

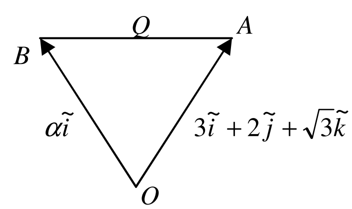
Q1aiii $\overrightarrow{AQ} = \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}\left(-\overrightarrow{BA}\right) = \frac{1}{2}\left(\tilde{b} - \tilde{a}\right)$

Q1b $\overrightarrow{MN} = \overrightarrow{ON} - \overrightarrow{OM} = \frac{1}{2}\tilde{b} - \frac{1}{2}\tilde{a} = \frac{1}{2}\left(\tilde{b} - \tilde{a}\right)$

.: $\overrightarrow{MN} = \overrightarrow{AQ}$, i.e. $\overline{MN}$ and $\overline{AQ}$ are parallel and equal in length

.: $MNQA$ is a parallelogram.

Q1c



$\left|\overrightarrow{OB}\right| = \left|\overrightarrow{OA}\right|$, .: $\alpha^2 = 16$, and given $\alpha > 0$, .: $\alpha = 4$

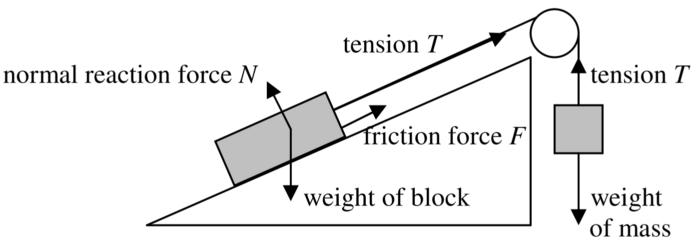
Q1di $\overrightarrow{OQ} = \frac{4\tilde{i} + \left(3\tilde{i} + 2\tilde{j} + \sqrt{3}\tilde{k}\right)}{2} = \frac{1}{2}\left(7\tilde{i} + 2\tilde{j} + \sqrt{3}\tilde{k}\right)$

Q1dii $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = 4\tilde{i} - \left(3\tilde{i} + 2\tilde{j} + \sqrt{3}\tilde{k}\right) = \tilde{i} - 2\tilde{j} - \sqrt{3}\tilde{k}$

$\overrightarrow{OQ}.\overrightarrow{AB} = \frac{1}{2}(7-4-3) = 0$, and since $\overrightarrow{OQ} \neq \tilde{0}$ and $\overrightarrow{AB} \neq \tilde{0}$,

.: $\overrightarrow{OQ}$ is perpendicular to $\overrightarrow{AB}$.

Q2a



Weight of block $= mg$, *weight of mass* $= 10g$, *friction* $= \mu N$

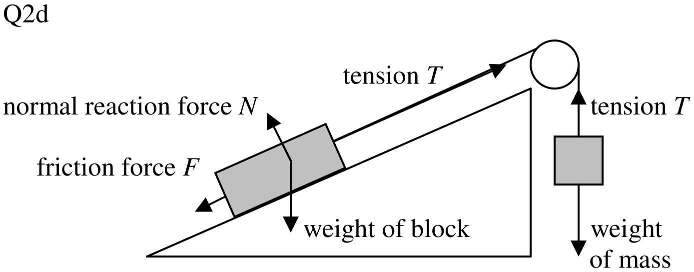
Q2b The block is *on the point* of moving down the plane, $a = 0$, resultant force on each object is zero.
On the 10-kg mass: $T - 10g = 0$
On the m-kg block:
$N - mg\cos 30° = 0$ and $mg\sin 30° - \mu N - T = 0$

Q2c Given $\mu = \frac{1}{4}$
From the first two equations in Q2b, $T = 10g$,
$N = mg\cos 30° = \frac{\sqrt{3}mg}{2}$.

Substitute into the third equation, $\frac{mg}{2} - \frac{1}{4} \times \frac{\sqrt{3}mg}{2} - 10g = 0$

.: $\frac{m}{2} - \frac{\sqrt{3}m}{8} - 10 = 0$, $m = \frac{80}{4 - \sqrt{3}}$.

Q2d



Weight of block $= mg$, *weight of mass* $= 20g$, *friction* $= \mu_1 N$

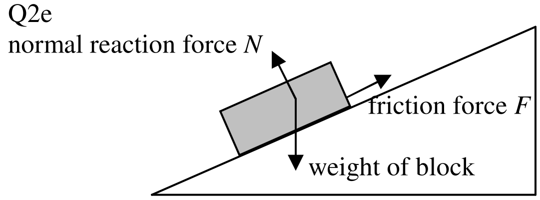
where $N = \frac{mg\sqrt{3}}{2}$, $T = 20g$

Now the block is *on the point* of moving up the plane, $a = 0$, resultant force on each object is zero.

.: on the block $20g - \frac{\mu_1 mg\sqrt{3}}{2} - \frac{mg}{2} = 0$,

.: $40 - \mu_1 m\sqrt{3} - m = 0$, $\mu_1 = \frac{40 - m}{m\sqrt{3}} = \frac{\frac{40}{m} - 1}{\sqrt{3}} = \frac{\frac{4-\sqrt{3}}{2} - 1}{\sqrt{3}} \approx 0.077$

Q2e



Weight of block $= mg$, *friction* $= \mu_1 N$

$a = \frac{R}{m} = \frac{mg\sin 30° - \mu_1 N}{m} = \frac{mg\sin 30° - \mu_1 mg\cos 30°}{m}$
$= g(\sin 30° - 0.077\cos 30°) \approx 4.2465$
$u = 0$, $t = 3$, .: $v = u + at \approx 4.2465 \times 3 \approx 12.7$ ms^{-1}

Q3a $P = 20000\left(4 - 3e^{0.01t}\right)$,
when $t = 0$, $P = 20000(4 - 3) = 20000$.

$\frac{dP}{dt} = 20000\left(-0.03e^{0.01t}\right) = -600e^{0.01t}$

$\frac{P}{100} - 800 = 200\left(4 - 3e^{0.01t}\right) - 800 = -600e^{0.01t}$

.: $\frac{dP}{dt} = \frac{P}{100} - 800$

Q3b When $P = 0$, $20000\left(4 - 3e^{0.01t}\right) = 0$

$4 - 3e^{0.01t} = 0$, $0.01t = \log_e\left(\frac{4}{3}\right)$, $t \approx 29$ years

Q3c $\frac{dt}{dP} = \frac{100}{P - 100k}$, and $k < 0$.: $P - 100k > 0$.

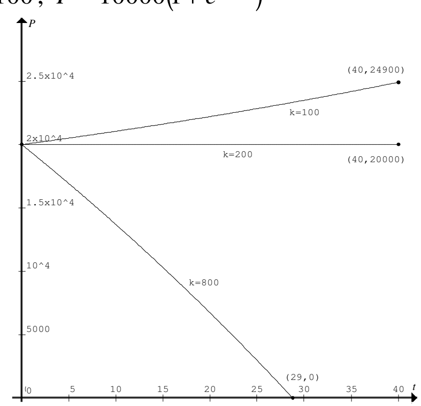
$t = 100\int \frac{1}{P - 100k}\,dP$, .: $0.01t = \log_e(P - 100k) + c$

With $P = 20000$ when $t = 0$, .: $c = -\log_e(20000 - 100k)$

.: $0.01t = \log_e\left(\frac{P - 100k}{20000 - 100k}\right)$, $\frac{P - 100k}{20000 - 100k} = e^{0.01t}$

.: $P = (20000 - 100k)e^{0.01t} + 100k$

Q3d i. $k = 800$, $P = 20000\left(4 - 3e^{0.01t}\right)$
ii. $k = 200$, $P = 20000$
iii. $k = 100$, $P = 10000\left(1 + e^{0.01t}\right)$



Q3ei When $t = 12$, $P = 22550$

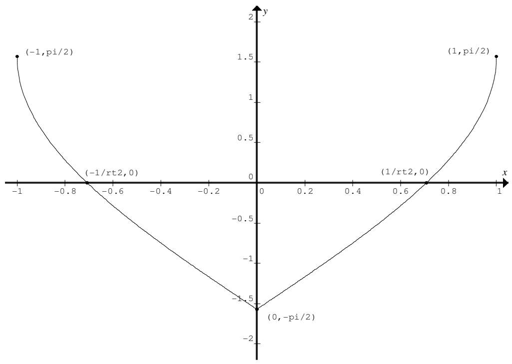
.: $(20000 - 100k)e^{0.12} + 100k = 22550$

$k = \dfrac{200e^{0.12} - 225.5}{e^{0.12} - 1}$, $k \approx -0.004939$

Q3eii k is the number of people leaving per year minus the number of people arriving per year.
If the number of people arriving per year is greater than the number of people leaving per year, k is a negative value as shown in part i.

.: $\dfrac{dP}{dt} = \dfrac{P}{100} - k$ is a positive rate, i.e. P increases with t.

Q4a



Q4bi $y = \sin^{-1}(2x^2 - 1)$, $x^2 = \dfrac{1}{2}(\sin(y) + 1)$

$$V = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \pi x^2 dy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\pi}{2}(\sin(y) + 1)dy$$

Q4bii $V = \dfrac{\pi}{2}\left[-\cos(y) + y\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \dfrac{\pi^2}{2}$

Q4c $f(x) = \sin^{-1}(2x^2 - 1)$,

$$f'(x) = \frac{4x}{\sqrt{1 - (2x^2 - 1)^2}} = \frac{4x}{\sqrt{(1 - (2x^2 - 1))(1 + (2x^2 - 1))}}$$

$$= \frac{4x}{\sqrt{4x^2(1 - x^2)}} = \frac{2x}{|x|\sqrt{1 - x^2}}.$$

Since $1 - x^2 > 0$, .: $-1 < x < 1$ and $x \neq 0$.

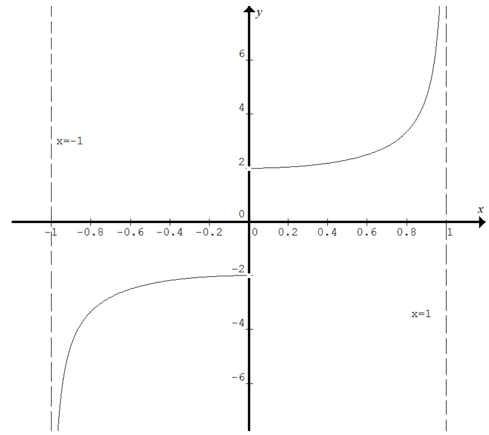
For $x \in (0, a)$ where $a = 1$, $x > 0$ and .: $f'(x) = \dfrac{2}{\sqrt{1 - x^2}}$.

For $x \in (-1, 0)$, $x < 0$, $\dfrac{x}{|x|} = -1$, .: $f'(x) = -\dfrac{2}{\sqrt{1 - x^2}}$.

Q4d

$$f'(x) = \begin{cases} -\dfrac{2}{\sqrt{1 - x^2}} & for \quad x \in (-1, 0) \\ \dfrac{2}{\sqrt{1 - x^2}} & for \quad x \in (0, 1) \end{cases}$$

Q4e



Q5a $u = 6 - 2i$, $w = 1 + 3i$, $u + w = 7 + i$, $\bar{u} = 6 + 2i$

$$z_1 = \frac{(u + w)\bar{u}}{iw} = \frac{(7 + i)(6 + 2i)}{-3 + i} = \frac{(7 + i)(6 + 2i)(-3 - i)}{(-3 + i)(-3 - i)}$$

$= -10(1 + i)$

$|z_1| = 10\sqrt{2}$

Q5b $z^3 = z_1 = 200^{\frac{1}{2}} cis\left(-\dfrac{3\pi}{4} + 2k\pi\right)$ where $k = 0, \pm1, \pm2, ...$

.: $z = 200^{\frac{1}{6}} cis\left(\dfrac{-3\pi + 8k\pi}{12}\right)$

$k = 0$, $z = 200^{\frac{1}{6}} cis\left(\dfrac{-\pi}{4}\right)$

$k = 1$, $z = 200^{\frac{1}{6}} cis\left(\dfrac{5\pi}{12}\right)$

$k = -1$, $z = 200^{\frac{1}{6}} cis\left(\dfrac{-11\pi}{12}\right)$

Q5c $u = 2\sqrt{10} cis(-\sigma)$, .: $\bar{u} = 2\sqrt{10} cis(\sigma)$,

$iw = -\dfrac{1}{2}u = \sqrt{10} cis(\pi - \alpha)$

$\dfrac{\bar{u}}{iw} = \dfrac{2\sqrt{10} cis(\alpha)}{\sqrt{10} cis(\pi - \alpha)} = 2cis(2\alpha - \pi)$

Q5d $z_1 = \dfrac{(u + w)\bar{u}}{iw}$, .: $200^{\frac{1}{2}} cis\left(-\dfrac{3\pi}{4}\right) = (u + w) \times 2cis(2\alpha - \pi)$

.: $10\sqrt{2} cis\left(-\dfrac{3\pi}{4}\right) = (u + w) \times 2cis(2\alpha - \pi)$

.: $u + w = 5\sqrt{2} cis\left(-\dfrac{3\pi}{4} - 2\alpha + \pi\right) = 5\sqrt{2} cis\left(\dfrac{\pi}{4} - 2\alpha\right)$.

.: $Arg(u + w) = \dfrac{\pi}{4} - 2\alpha$. Note: $\alpha = \tan^{-1}\left(\dfrac{1}{3}\right)$, a 'small' angle.

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