**The Mathematical Association of Victoria**
**Trial Exam 2009**

# SPECIALIST MATHEMATICS

**STUDENT NAME** ______________________________________________

## Written Examination 1

**Reading time: 15 minutes**
**Writing time: 2 hours**

## QUESTION AND ANSWER BOOK

**Structure of Book**

| *Number of questions* | *Number of questions to be answered* | *Number of marks* |
|---|---|---|
| **11** | **11** | **40** |

Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers.
• Students are not permitted to bring into the examination room: notes of any kind, a calculator of any type, blank sheets of paper and/or white out liquid/tape.

**Materials supplied**
• Question and answer book of 10 pages with a detachable sheet of miscellaneous formulas at the back
• Working space is provided throughout the book.

**Instructions**
• Detach the formula sheet from the back of this book during reading time.
• Write your **name** in the space provided above on this page.
• All written responses must be in English.

**Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.**

**Instructions**

Answer **all** questions in the spaces provided.

A decimal approximation will not be accepted if an **exact** answer is required to a question.

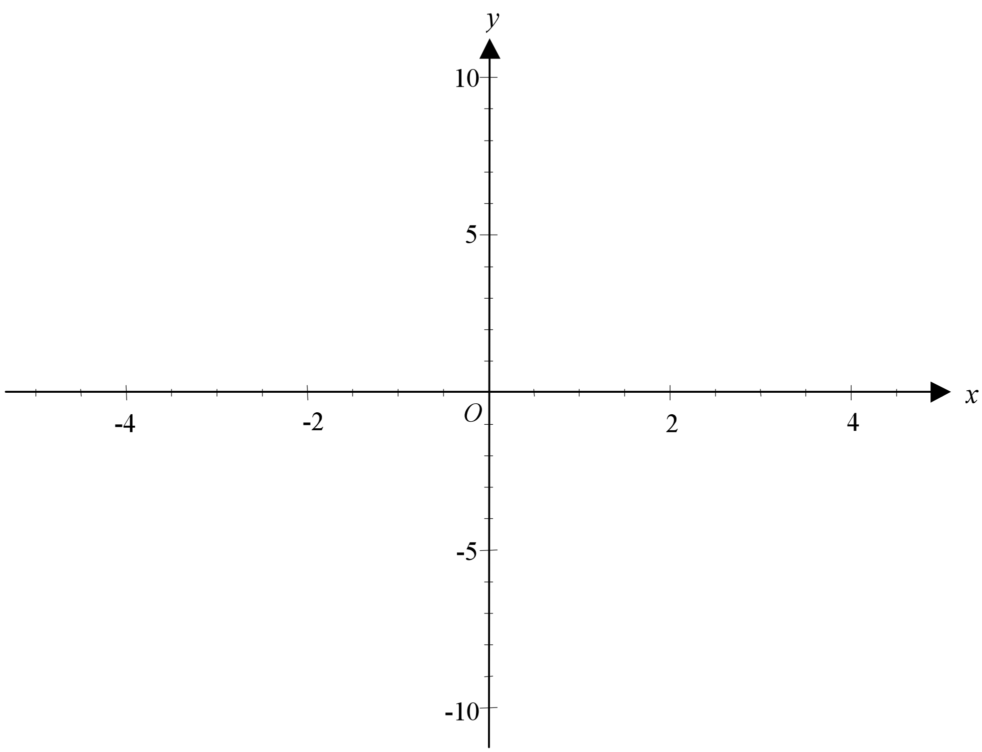
In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Take the **acceleration due to gravity** to have magnitude $g$ m/s$^2$, where $g = 9.8$

**Question 1**

Sketch the graph of $y = \frac{4}{x} - 2x^2$ on the axes below. Give co-ordinates of any intercepts and stationary points. State the equations of all asymptotes.



5 marks

**Question 2**

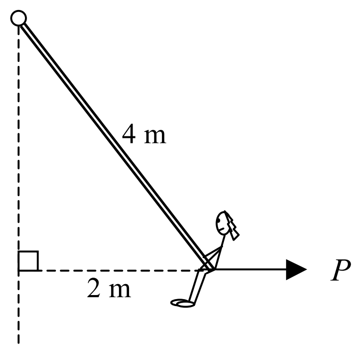
**a.** Show $z = -4$ is a solution to $z^2 + (1 + i\sqrt{2})z + 4i\sqrt{2} - 12 = 0$

1 mark

**b.** Hence or otherwise, find the other solution to $z^2 + (1 + i\sqrt{2})z + 4i\sqrt{2} - 12 = 0$

4 marks

**Question 3**
A girl of mass 24 kg holds onto a rope of length 4 m hanging from a fixed point vertically above. Her father pulls her back 2 m from the vertical as shown in the diagram below. He applies a horizontal force of $P$ newtons to momentarily hold her in this position before releasing her from rest to have a swing.




Find the exact horizontal force, $P$, in newtons, that the father applies.

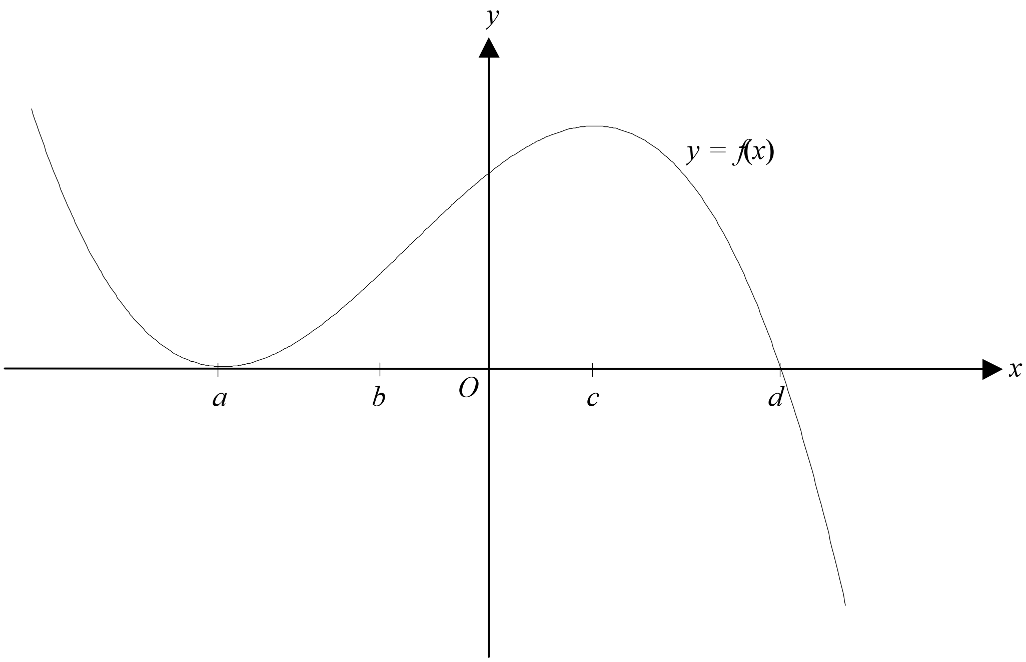
3 marks

**Question 4**

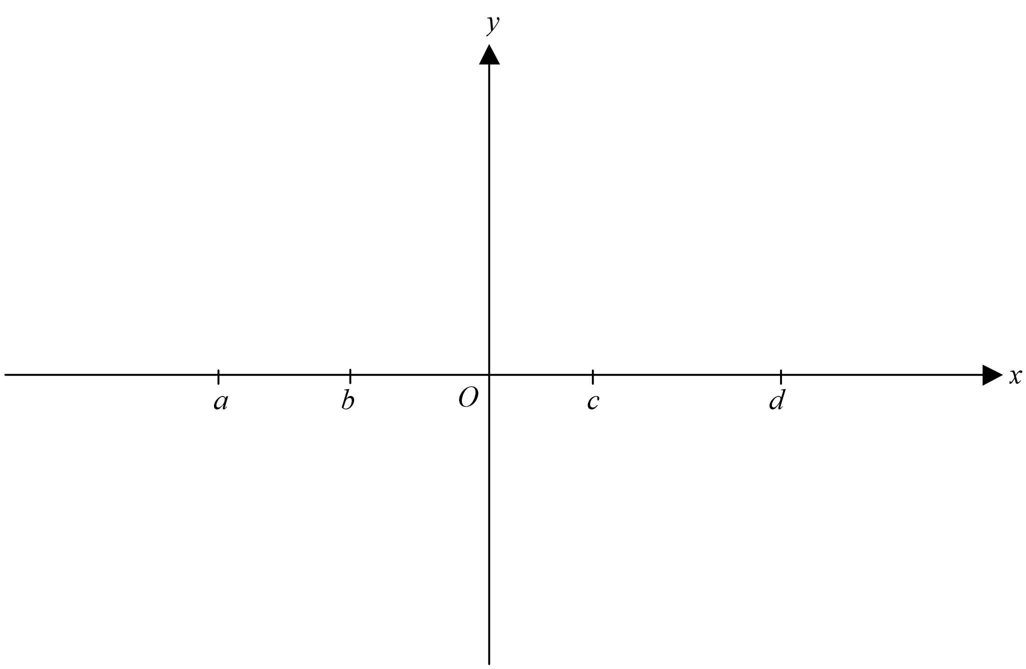
Solve the equation $\cot(2x) - \cot(x) = 2$ for $x \in [-\pi, \pi]$

4 marks

**Question 5**
The following is a graph of $y = f(x)$.




On the axes below, sketch the graph of $y = F(x)$, where $F(x)$ is the antiderivative of $f(x)$, given $F(0) = 0$.




3 marks

**Question 6**

Find the derivative of $\cot(x)$

2 marks

**Question 7**

Find an expression for $\frac{dy}{dx}$ in terms of $x$ and $y$ for the relation $5y = 2xy^2 - 3x$.

2 marks

**Question 8**

Using the substitution $u = \sqrt{x}$, find $\int \frac{\sqrt{x}}{x+1} dx$

4 marks

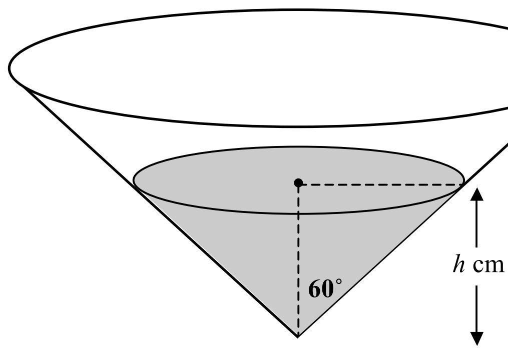
**Question 9**

Find $\int \frac{1}{1+\cos(x)} dx$

3 marks

**Question 10**

A solution is being filtered through filter paper that is folded in the shape of an inverted right circular cone with a semi-vertex angle of 60˚. The solution drips from the bottom of the filter at a rate of $8\sqrt{h}$ cm$^3$/min, where *h* is the height of the solution at time *t* minutes.




**a.** Show that a differential equation which models the rate of change of the height of the solution in the filter at time *t* minutes is given by $\frac{dh}{dt} = -\frac{8}{3\pi\sqrt{h^3}}$.

2 marks

**b.** Solve the differential equation $\frac{dh}{dt} = -\frac{8}{3\pi\sqrt{h^3}}$ to find the exact time, in minutes, for the solution to completely pass through the filter, given that its initial height was 4 cm.

3 marks

**Question 11**

A particle moves in a straight line so that at time $t$ seconds, it is situated $x$ m from a fixed origin, $O$, moving with a velocity $v$ m/s and an acceleration $a$ m/s$^2$, where $a = -\frac{3\sqrt{x}}{8}$, $x \geq 0$.
Initially the particle is at the origin moving with a velocity of $-2$ m/s.

**a.** Find an expression for the velocity of the particle in terms of its position from $O$.

3 marks

**b.** Determine the values of $x$ for which this acceleration model is valid.

1 mark

**END OF QUESTION AND ANSWER BOOK**

# Specialist Mathematics Formulas

## Mensuration

| | |
|---|---|
| area of a trapezium: | $\frac{1}{2}(a+b)h$ |
| curved surface area of a cylinder: | $2\pi rh$ |
| volume of a cylinder: | $\pi r^2 h$ |
| volume of a cone: | $\frac{1}{3}\pi r^2 h$ |
| volume of a pyramid: | $\frac{1}{3}Ah$ |
| volume of a sphere: | $\frac{4}{3}\pi r^3$ |
| area of a triangle: | $\frac{1}{2}bc\sin A$ |
| sine rule: | $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ |
| cosine rule: | $c^2 = a^2 + b^2 - 2ab\cos C$ |

## Coordinate geometry

ellipse: $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

hyperbola: $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

## Circular (trigonometric) functions

$\cos^2(x) + \sin^2(x) = 1$

$1 + \tan^2(x) = \sec^2(x)$

$\cot^2(x) + 1 = \operatorname{cosec}^2(x)$

$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$

$\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$

$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$

$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$

$\tan(x+y) = \frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$

$\tan(x-y) = \frac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}$

$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$

$\sin(2x) = 2\sin(x)\cos(x)$

$\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)}$

| function | $\sin^{-1}$ | $\cos^{-1}$ | $\tan^{-1}$ |
|---|---|---|---|
| domain | $[-1, 1]$ | $[-1, 1]$ | $R$ |
| range | $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ | $[0, \pi]$ | $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ |

## Algebra (complex numbers)

$z = x + yi = r(\cos\theta + i\sin\theta) = r\,\mathrm{cis}\,\theta$

$|z| = \sqrt{x^2 + y^2} = r$ $\qquad -\pi < \mathrm{Arg}\,z \le \pi$

$z_1 z_2 = r_1 r_2\,\mathrm{cis}(\theta_1 + \theta_2)$ $\qquad \dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\mathrm{cis}(\theta_1 - \theta_2)$

$z^n = r^n\,\mathrm{cis}(n\theta)$ (de Moivre's theorem)

## Calculus

$\dfrac{d}{dx}\left(x^n\right) = nx^{n-1}$ $\qquad \int x^n\,dx = \dfrac{1}{n+1}x^{n+1} + c, n \ne -1$

$\dfrac{d}{dx}\left(e^{ax}\right) = ae^{ax}$ $\qquad \int e^{ax}\,dx = \dfrac{1}{a}e^{ax} + c$

$\dfrac{d}{dx}\left(\log_e(x)\right) = \dfrac{1}{x}$ $\qquad \int \dfrac{1}{x}\,dx = \log_e|x| + c$

$\dfrac{d}{dx}\left(\sin(ax)\right) = a\cos(ax)$ $\qquad \int \sin(ax)\,dx = -\dfrac{1}{a}\cos(ax) + c$

$\dfrac{d}{dx}\left(\cos(ax)\right) = -a\sin(ax)$ $\qquad \int \cos(ax)\,dx = \dfrac{1}{a}\sin(ax) + c$

$\dfrac{d}{dx}\left(\tan(ax)\right) = a\sec^2(ax)$ $\qquad \int \sec^2(ax)\,dx = \dfrac{1}{a}\tan(ax) + c$

$\dfrac{d}{dx}\left(\sin^{-1}(x)\right) = \dfrac{1}{\sqrt{1-x^2}}$ $\qquad \int \dfrac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\dfrac{x}{a}\right) + c, a > 0$

$\dfrac{d}{dx}\left(\cos^{-1}(x)\right) = \dfrac{-1}{\sqrt{1-x^2}}$ $\qquad \int \dfrac{-1}{\sqrt{a^2-x^2}}\,dx = \cos^{-1}\left(\dfrac{x}{a}\right) + c, a > 0$

$\dfrac{d}{dx}\left(\tan^{-1}(x)\right) = \dfrac{1}{1+x^2}$ $\qquad \int \dfrac{a}{a^2+x^2}\,dx = \tan^{-1}\left(\dfrac{x}{a}\right) + c$

product rule: $\dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

quotient rule: $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

chain rule: $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

Euler's method: If $\dfrac{dy}{dx} = f(x)$, $x_0 = a$ and $y_0 = b$, then $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + hf(x_n)$

acceleration: $a = \dfrac{d^2x}{dt^2} = \dfrac{dv}{dt} = v\dfrac{dv}{dx} = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$

constant (uniform) acceleration: $v = u + at$ $\qquad s = ut + \dfrac{1}{2}at^2$ $\qquad v^2 = u^2 + 2as$ $\qquad s = \dfrac{1}{2}(u + v)t$

## Vectors in two and three dimensions

$$\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$$

$$|\underset{\sim}{r}| = \sqrt{x^2 + y^2 + z^2} = r$$

$$\underset{\sim}{r}_1 \cdot \underset{\sim}{r}_2 = r_1 r_2 \cos\theta = x_1 x_2 + y_1 y_2 + z_1 z_2$$

$$\dot{\underset{\sim}{r}} = \frac{d\underset{\sim}{r}}{dt} = \frac{dx}{dt}\underset{\sim}{i} + \frac{dy}{dt}\underset{\sim}{j} + \frac{dz}{dt}\underset{\sim}{k}$$

## Mechanics

momentum: $\underset{\sim}{p} = m\underset{\sim}{v}$

equation of motion: $\underset{\sim}{R} = m\underset{\sim}{a}$

friction: $F \leq \mu N$

**END OF FORMULA SHEET**