

Year 2009
VCE
Specialist Mathematics
Solutions
Trial Examination 1



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Question 1

$$(x^2 + y^2)^2 = x^3 + y^3 \text{ expanding gives } x^4 + 2x^2y^2 + y^4 = x^3 + y^3$$

taking $\frac{d}{dx}$ of each term (implicit differentiation)

$$\frac{d}{dx}(x^4) + \frac{d}{dx}(2x^2y^2) + \frac{d}{dx}(y^4) = \frac{d}{dx}(x^3) + \frac{d}{dx}(y^3)$$

product rule in the second term

$$4x^3 + 4xy^2 + 4x^2y \frac{dy}{dx} + 4y^3 \frac{dy}{dx} = 3x^2 + 3y^2 \frac{dy}{dx} \quad \text{M1}$$

$$4x^3 + 4xy^2 - 3x^2 = (3y^2 - 4x^2y - 4y^3) \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{4x^3 + 4xy^2 - 3x^2}{3y^2 - 4x^2y - 4y^3} \quad \text{A1}$$

Question 2

$$y = 4 \sin(3x) \quad V = \pi \int_a^b y^2 dx$$

$$V = \pi \int_0^{\frac{\pi}{6}} 16 \sin^2(3x) dx \quad \text{A1}$$

$$V = 8\pi \int_0^{\frac{\pi}{6}} (1 - \cos(6x)) dx \quad \text{M1}$$

$$V = 8\pi \left[x - \frac{1}{6} \sin(6x) \right]_0^{\frac{\pi}{6}}$$

$$V = 8\pi \left[\left(\frac{\pi}{6} - \frac{1}{6} \sin(\pi) \right) - \left(0 - \frac{1}{6} \sin(0) \right) \right]$$

$$V = \frac{4\pi^2}{3} \quad \text{A1}$$

Question 3

Let $\alpha = 2 + \sqrt{3}i$, then by the conjugate root theorem, since a and b are real,

$\beta = 2 - \sqrt{3}i$ is also a root. Now $\alpha + \beta = 4$ and $\alpha\beta = 4 - 3i^2 = 7$, so that

$z^2 - 4z + 7$ is a factor. A1

$$P(z) = z^3 + az^2 + bz - 21 = 0$$

$$P(z) = (z^2 - 4z + 7)(z - 3) = 0 \quad \text{expanding gives}$$

$$z^2: \quad a = -3 - 4 = -7 \quad \text{A1}$$

$$z: \quad b = 7 + 12 = 19$$

all the roots are $z = 2 \pm \sqrt{3}i$ and $z = 3$. A1

Question 4

$$\underline{c} = \alpha \underline{a} + \beta \underline{b}$$

$$\underline{c} = 3\underline{i} + y\underline{j} + 7\underline{k} = \alpha(2\underline{i} - 3\underline{j} + 4\underline{k}) + \beta(\underline{i} - \underline{j} + \underline{k}) \quad \text{M1}$$

$$\underline{i} \quad (1) \quad 3 = 2\alpha + \beta$$

$$\underline{j} \quad (2) \quad y = -3\alpha - \beta \quad \text{A1}$$

$$\underline{k} \quad (3) \quad 7 = 4\alpha + \beta$$

$$(3) - (1) \Rightarrow 2\alpha = 4$$

$\alpha = 2$ and $\beta = -1$, substituting gives $y = -5$ A1

Question 5

$$\text{a.} \quad v = \sqrt{9 - 4x^2} \quad \Rightarrow \quad \frac{dv}{dx} = -8x \times \frac{1}{2} \times (9 - 4x^2)^{-\frac{1}{2}} = \frac{-4x}{\sqrt{9 - 4x^2}}$$

$$a = v \frac{dv}{dx} = -4x \quad \text{A1}$$

$$\text{b.} \quad v = \frac{dx}{dt} = \sqrt{9 - 4x^2}$$

$$t = \int \frac{1}{\sqrt{9 - 4x^2}} dx \quad \text{A1}$$

$$t = \frac{1}{2} \sin^{-1} \left(\frac{2x}{3} \right) + C \quad \text{now when } x = 0 \quad t = 0 \Rightarrow C = 0 \quad \text{A1}$$

$$2t = \sin^{-1} \left(\frac{2x}{3} \right) \quad \Rightarrow \quad \sin(2t) = \frac{2x}{3}$$

$$x = \frac{3}{2} \sin(2t) \quad \text{A1}$$

Question 6

a. $\underline{r}(t) = \left(t + \frac{1}{t}\right)\underline{i} + \left(t^2 + \frac{1}{t^2}\right)\underline{j}$ for $t > 0$ vector equation,

the parametric equations are (1) $x = t + \frac{1}{t}$ (2) $y = t^2 + \frac{1}{t^2}$ M1

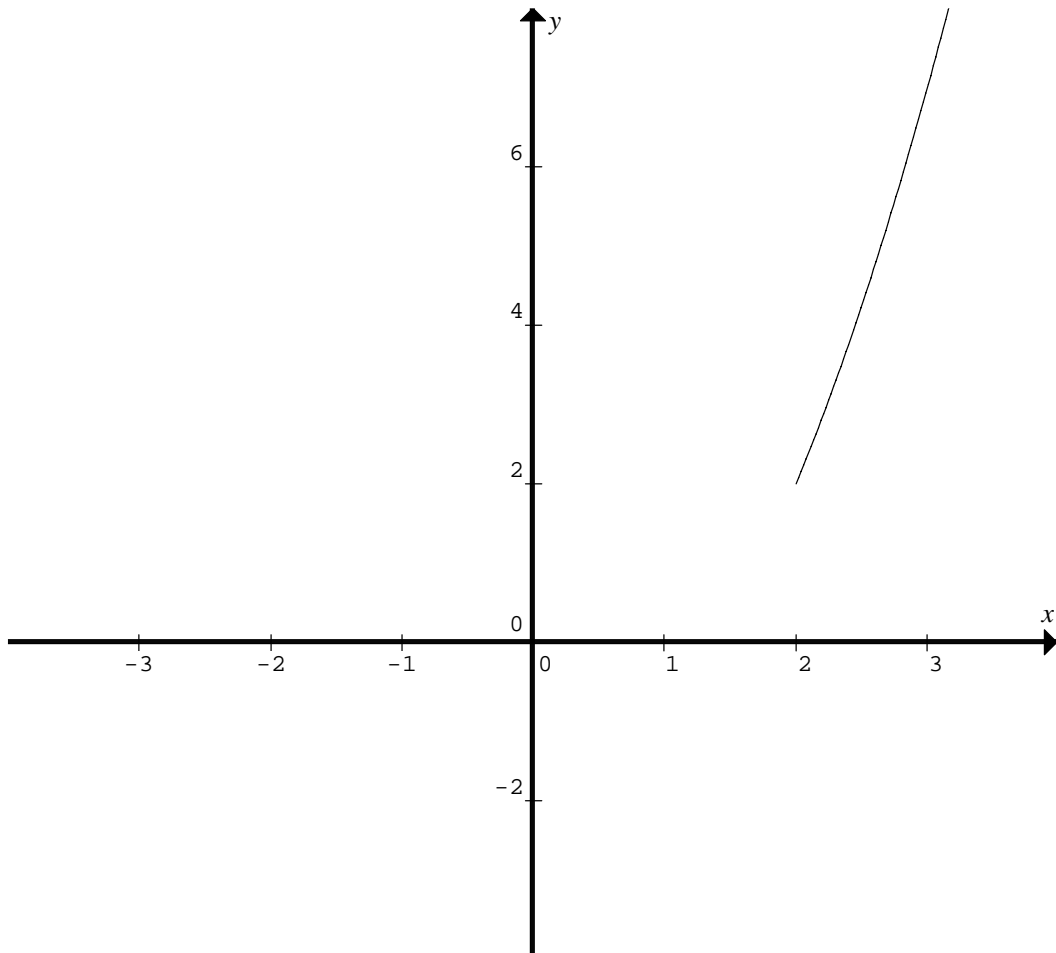
squaring (1) $x^2 = t^2 + 2 + \frac{1}{t^2} = \left(t^2 + \frac{1}{t^2}\right) + 2 = y + 2$

$y = x^2 - 2$ is the Cartesian equation of the path. A1

b. since $t > 0$, the minimum value of x , occurs when $\frac{dx}{dt} = 1 - \frac{1}{t^2} = 0 \Rightarrow t = 1$

$\Rightarrow x \geq 2$ and $y \geq 2$ A1

graph starts from the point (2, 2) G1



Question 7

$$y = \frac{x^4 - 16}{2x^2} = \frac{x^2}{2} - \frac{8}{x^2}$$

$y = \frac{x^2}{2}$ is an asymptote, and $x = 0$ is a vertical asymptote A1

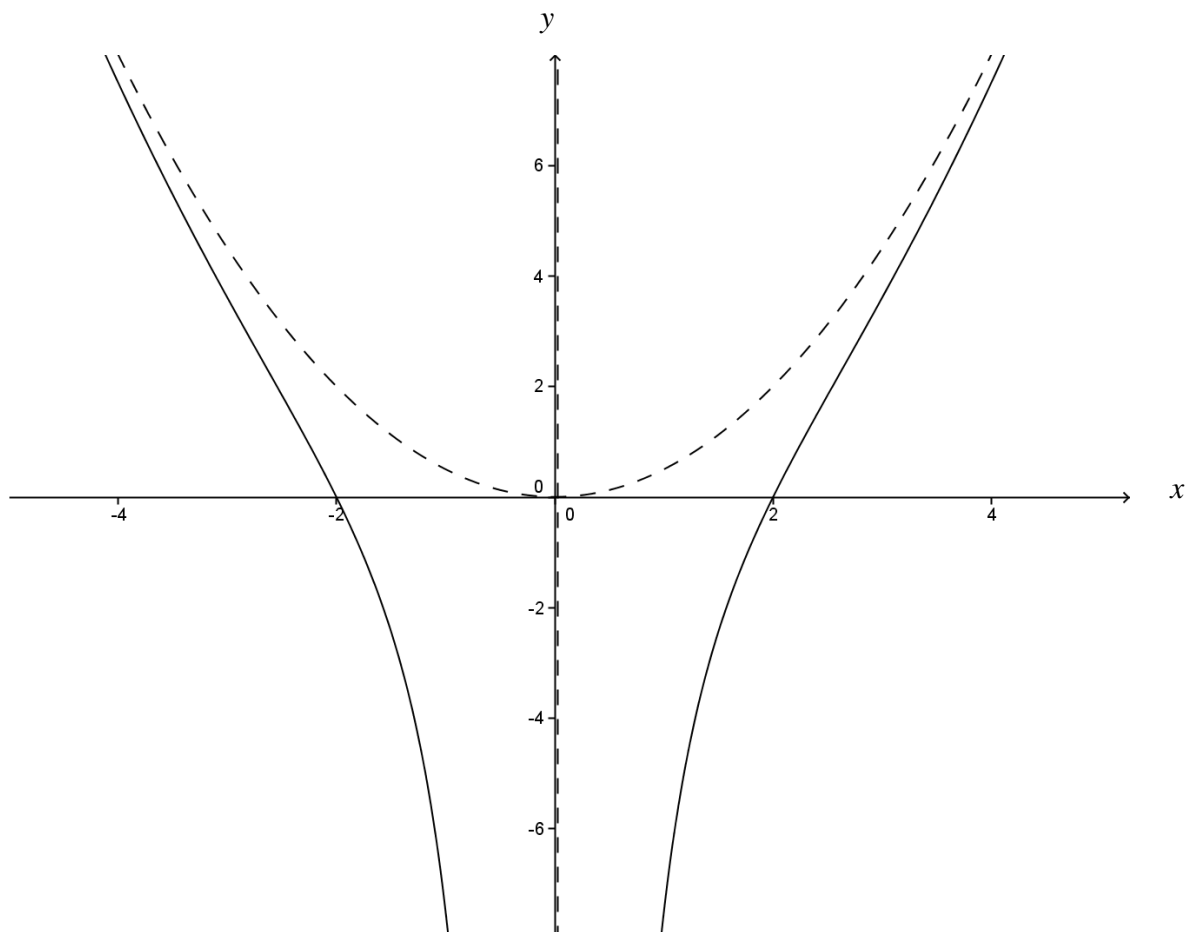
the graph does not cross the y-axis,

crosses the x-axis when $y = 0 \Rightarrow x^4 = 16 \Rightarrow x = \pm 2$ at $(2, 0)$ $(-2, 0)$ A1

for turning points, $\frac{dy}{dx} = x + \frac{16}{x^3} = 0 \Rightarrow x^4 = -16$ this has no real solution,

so there are no turning points. A1

correct graph G1



Question 8

$$\frac{dy}{dx} = \frac{x}{\sqrt{2x+3}}$$

$$y = \int \frac{x}{\sqrt{2x+3}} dx \quad \text{let } u = 2x+3 \quad \frac{du}{dx} = 2 \quad x = \frac{1}{2}(u-3)$$

$$y = \frac{1}{4} \int \frac{u-3}{\sqrt{u}} du$$

$$y = \frac{1}{4} \int \left(u^{\frac{1}{2}} - 3u^{-\frac{1}{2}} \right) du$$

M1

$$y = \frac{1}{4} \left(\frac{2}{3} u^{\frac{3}{2}} - 6u^{\frac{1}{2}} \right) + C$$

$$y = \frac{1}{2} \left(\frac{u}{3} - 3 \right) + C = \frac{\sqrt{u}}{2} \left(\frac{u-9}{3} \right) + C$$

A1

$$y = \frac{1}{6} (2x-6) \sqrt{2x+3} + C = \frac{1}{3} (x-3) \sqrt{2x+3} + C$$

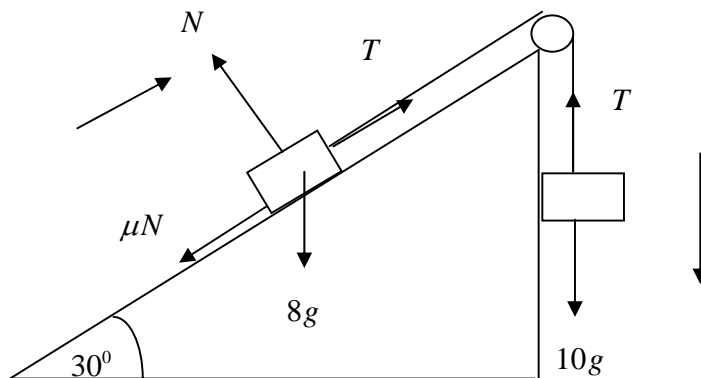
$$\text{now when } x=3 \quad y=0 \Rightarrow C=0$$

$$y = \left(\frac{x}{3} - 1 \right) \sqrt{2x+3} \quad a = \frac{1}{3} \quad b = -1$$

A1

Question 9

i.



correct forces on the diagram

A1

ii. resolving downwards for the 10 kg weight hanging vertically

$$(1) \quad 10g - T = 10a$$

for the crate on the incline plane, resolving upwards parallel to the plane

$$(2) \quad T - \mu N - 8g \sin(30^\circ) = 8a \quad \text{A1}$$

resolving perpendicular to the plane

$$(3) \quad N - 8g \cos(30^\circ) = 0 \quad N = 8g \cos(30^\circ)$$

$$(2) \text{ becomes } T - 8\mu g \cos(30^\circ) - 8g \sin(30^\circ) = 8a \quad \text{A1}$$

adding this to equation (1) to eliminate T ,

$$10g - 8g (\sin(30^\circ) + \mu \cos(30^\circ)) = 18a \text{ substituting} \quad \text{M1}$$

$$\mu = \frac{\sqrt{3}}{4} \quad \cos(30^\circ) = \frac{\sqrt{3}}{2} \quad \sin(30^\circ) = \frac{1}{2}$$

$$g \left[10 - 8 \left(\frac{1}{2} + \frac{3}{8} \right) \right] = 18a$$

$$a = \frac{g}{6} \text{ m/s}^2 \quad \text{A1}$$

Question 10

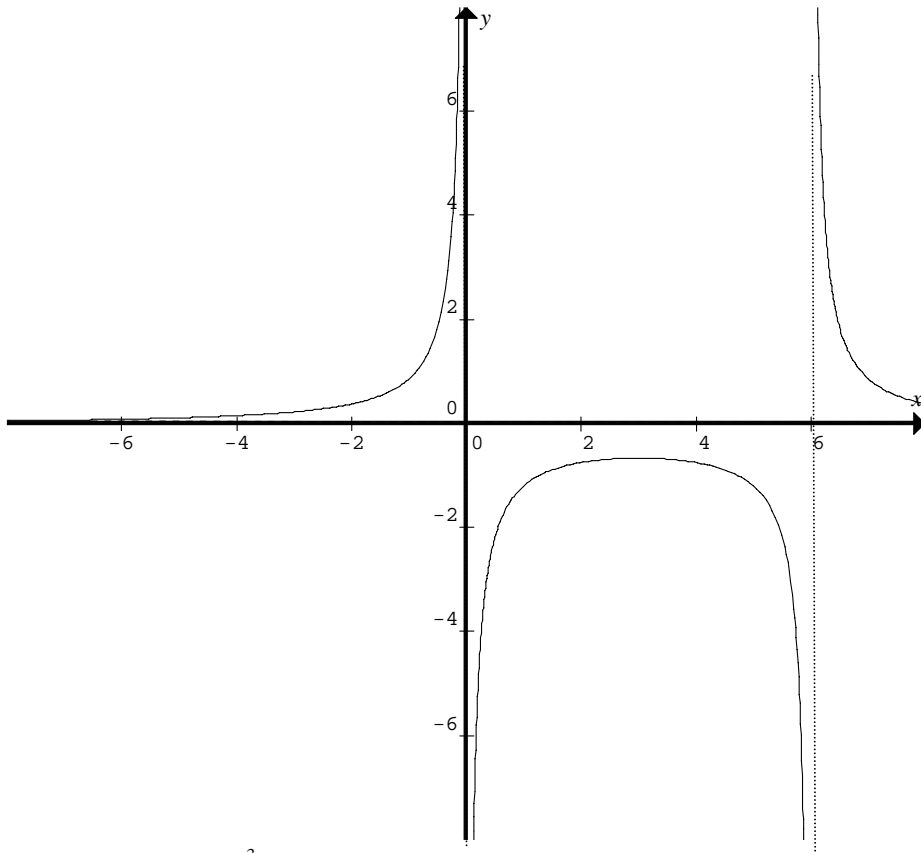
a.
$$y = \frac{6}{x^2 - 6x} = \frac{6}{x(x-6)}$$

vertical asymptotes at $x = 0$ and $x = 6$

horizontal asymptotes at $y = 0$ (the x -axis) A1

the turning point occurs when $2x - 6 = 0 \Rightarrow x = 3$

the maximum turning point is $\left(3, -\frac{2}{3} \right)$ and correct graph A1



b. the area is $\int_1^3 \frac{6}{x^2 - 6x} dx$ but this is below the x -axis and negative,

so the area is $A = \int_1^3 \frac{6}{6x - x^2} dx$ A1

by partial fractions $\frac{6}{6x - x^2} = \frac{B}{x} + \frac{C}{6-x}$ adding the partial fractions
 $= \frac{B(6-x) + Cx}{x(6-x)} = \frac{x(C-B) + 6B}{6x - x^2}$

(1) $6B = 6$ and (2) $C - B = 0$ so that $B = C = 1$

$A = \int_1^3 \frac{6}{6x - x^2} dx = \int_1^3 \left(\frac{1}{x} + \frac{1}{6-x} \right) dx$ M1

$A = \left[\log_e(x) - \log_e(6-x) \right]_1^3 = \left[\log_e \left(\frac{x}{6-x} \right) \right]_1^3$

$A = \left[\log_e(1) - \log_e \left(\frac{1}{5} \right) \right] = \log_e(5) \quad a = 5$ A1

Question 11

$$(\sqrt{3} + i)^m - (\sqrt{3} - i)^m = 0$$

$$\text{now } \sqrt{3} + i = 2\text{cis}\left(\frac{\pi}{6}\right) \quad \text{and} \quad \sqrt{3} - i = 2\text{cis}\left(-\frac{\pi}{6}\right) \quad \text{A1}$$

$$\left(2\text{cis}\left(\frac{\pi}{6}\right)\right)^m - \left(2\text{cis}\left(-\frac{\pi}{6}\right)\right)^m = 0, \text{ using DeMoivre's theorem}$$

$$2^m \text{cis}\left(\frac{m\pi}{6}\right) - 2^m \text{cis}\left(-\frac{m\pi}{6}\right) = 0 \quad \text{M1}$$

$$2^m \left(\left(\cos\left(\frac{m\pi}{6}\right) + i \sin\left(\frac{m\pi}{6}\right) \right) - \left(\cos\left(-\frac{m\pi}{6}\right) + i \sin\left(-\frac{m\pi}{6}\right) \right) \right) = 0$$

but $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$, so that

$$2^{m+1} i \sin\left(\frac{m\pi}{6}\right) = 0 \quad \text{A1}$$

$$\sin\left(\frac{m\pi}{6}\right) = 0$$

$$\frac{m\pi}{6} = 0, \pi, 2\pi, 3\pi, \dots = k\pi$$

$$m = 6k \quad \text{where} \quad k \in \mathbb{Z} \quad \text{A1}$$

END OF SUGGESTED SOLUTIONS