Year 2009

VCE

Specialist Mathematics Solutions Trial Examination 1

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Question 1

$\left(x^2+y^2\right)^2 = x^3+y^3$ expanding gives $x^4+2x^2y^2+y^4 = x^3+y^3$

taking $\frac{d}{dx}$ of each term (implicit differentiation)

$$\frac{d}{dx}\left(x^4\right)+\frac{d}{dx}\left(2x^2y^2\right)+\frac{d}{dx}\left(y^4\right)=\frac{d}{dx}\left(x^3\right)+\frac{d}{dx}\left(y^3\right)$$

product rule in the second term

$$4x^3+4xy^2+4x^2y\frac{dy}{dx}+4y^3\frac{dy}{dx}=3x^2+3y^2\frac{dy}{dx}$$ M1

$$4x^3+4xy^2-3x^2=\left(3y^2-4x^2y-4y^3\right)\frac{dy}{dx}$$

$$\frac{dy}{dx}=\frac{4x^3+4xy^2-3x^2}{3y^2-4x^2y-4y^3}$$ A1

Question 2

$y=4\sin\left(3x\right) \qquad V=\pi\int_a^b y^2\,dx$

$$V=\pi\int_0^{\frac{\pi}{6}} 16\sin^2\left(3x\right)dx$$ A1

$$V=8\pi\int_0^{\frac{\pi}{6}}\left(1-\cos\left(6x\right)\right)dx$$ M1

$$V=8\pi\left[x-\frac{1}{6}\sin\left(6x\right)\right]_0^{\frac{\pi}{6}}$$

$$V=8\pi\left[\left(\frac{\pi}{6}-\frac{1}{6}\sin\left(\pi\right)\right)-\left(0-\frac{1}{6}\sin\left(0\right)\right)\right]$$

$$V=\frac{4\pi^2}{3}$$ A1

Question 3

Let $\alpha = 2+\sqrt{3}\,i$, then by the conjugate root theorem, since *a* and *b* are real,

$\beta = 2-\sqrt{3}\,i$ is also a root. Now $\alpha+\beta = 4$ and $\alpha\beta = 4-3i^2 = 7$, so that

$z^2 - 4z + 7$ is a factor. A1

$P(z) = z^3 + az^2 + bz - 21 = 0$

$P(z) = (z^2 - 4z + 7)(z-3) = 0$ expanding gives

$z^2: \quad a = -3-4 = -7$

$z: \quad b = 7+12 = 19$ A1

all the roots are $z = 2 \pm \sqrt{3}\,i$ and $z = 3$. A1

Question 4

$\underset{\sim}{c} = \alpha \underset{\sim}{a} + \beta \underset{\sim}{b}$

$\underset{\sim}{c} = 3\underset{\sim}{i} + y\underset{\sim}{j} + 7\underset{\sim}{k} = \alpha\left(2\underset{\sim}{i} - 3\underset{\sim}{j} + 4\underset{\sim}{k}\right) + \beta\left(\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}\right)$ M1

$\underset{\sim}{i} \quad (1) \quad 3 = 2\alpha + \beta$

$\underset{\sim}{j} \quad (2) \quad y = -3\alpha - \beta$ A1

$\underset{\sim}{k} \quad (3) \quad 7 = 4\alpha + \beta$

$(3)-(1) \quad \Rightarrow 2\alpha = 4$

$\alpha = 2$ and $\beta = -1$, substituting gives $y = -5$ A1

Question 5

a. $v = \sqrt{9-4x^2} \quad \Rightarrow \quad \dfrac{dv}{dx} = -8x \times \dfrac{1}{2} \times \left(9-4x^2\right)^{-\frac{1}{2}} = \dfrac{-4x}{\sqrt{9-4x^2}}$

$a = v\dfrac{dv}{dx} = -4x$ A1

b. $v = \dfrac{dx}{dt} = \sqrt{9-4x^2}$

$t = \displaystyle\int \frac{1}{\sqrt{9-4x^2}}\,dx$ A1

$t = \dfrac{1}{2}\sin^{-1}\left(\dfrac{2x}{3}\right) + C$ now when $x = 0 \quad t = 0 \quad \Rightarrow C = 0$ A1

$2t = \sin^{-1}\left(\dfrac{2x}{3}\right) \quad \Rightarrow \quad \sin(2t) = \dfrac{2x}{3}$

$x = \dfrac{3}{2}\sin(2t)$ A1

Question 6

a. $\underset{\sim}{r}(t)=\left(t+\frac{1}{t}\right)\underset{\sim}{i}+\left(t^2+\frac{1}{t^2}\right)\underset{\sim}{j}$ for $t>0$ vector equation,

the parametric equations are $(1)\quad x=t+\frac{1}{t}$ $\qquad (2)\quad y=t^2+\frac{1}{t^2}$ M1

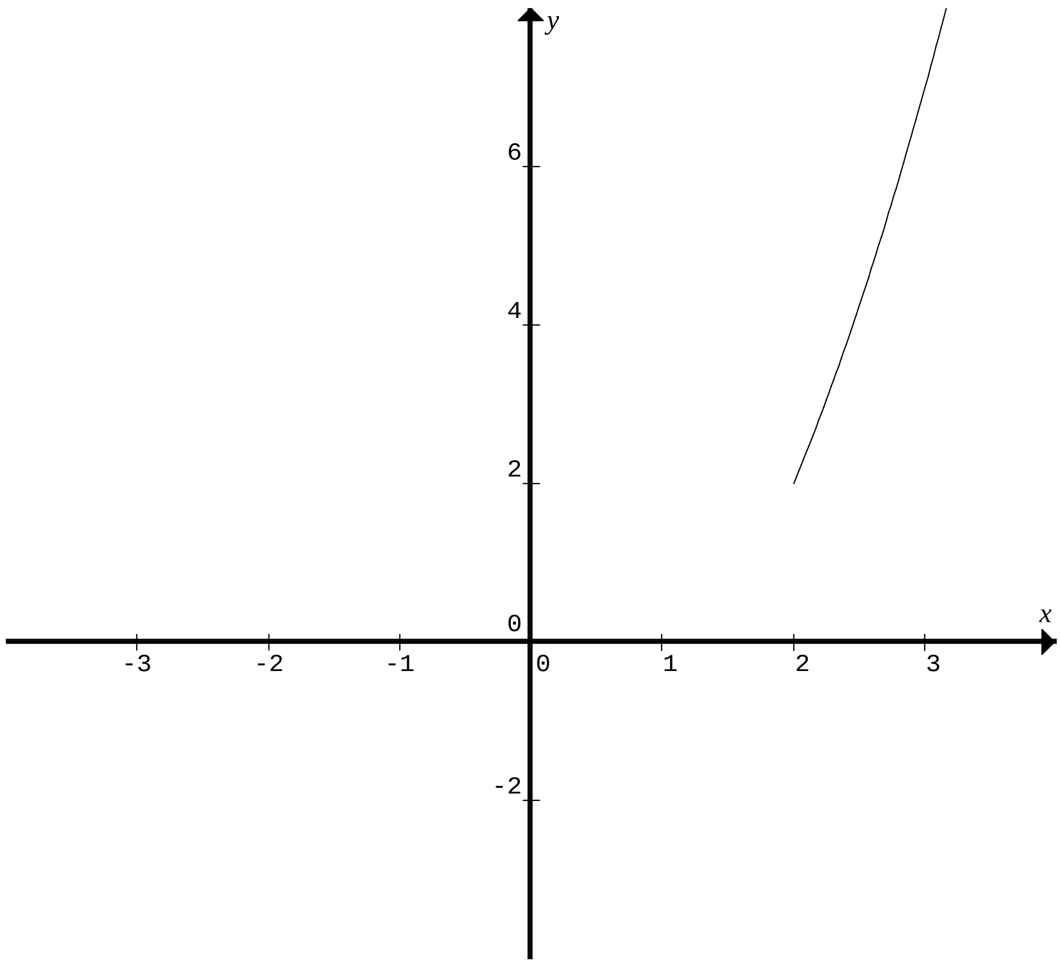
squaring (1) $x^2=t^2+2+\frac{1}{t^2}=\left(t^2+\frac{1}{t^2}\right)+2=y+2$

$y=x^2-2$ is the Cartesian equation of the path. A1

b. since $t>0$, the minimum value of *x*, occurs when $\frac{dx}{dt}=1-\frac{1}{t^2}=0 \Rightarrow t=1$

$\Rightarrow \quad x\geq 2$ and $y\geq 2$ A1

graph starts from the point $(2,2)$ G1



Question 7

$y=\dfrac{x^4-16}{2x^2}=\dfrac{x^2}{2}-\dfrac{8}{x^2}$

$y=\dfrac{x^2}{2}$ is an asymptote, and $x=0$ is a vertical asymptote A1

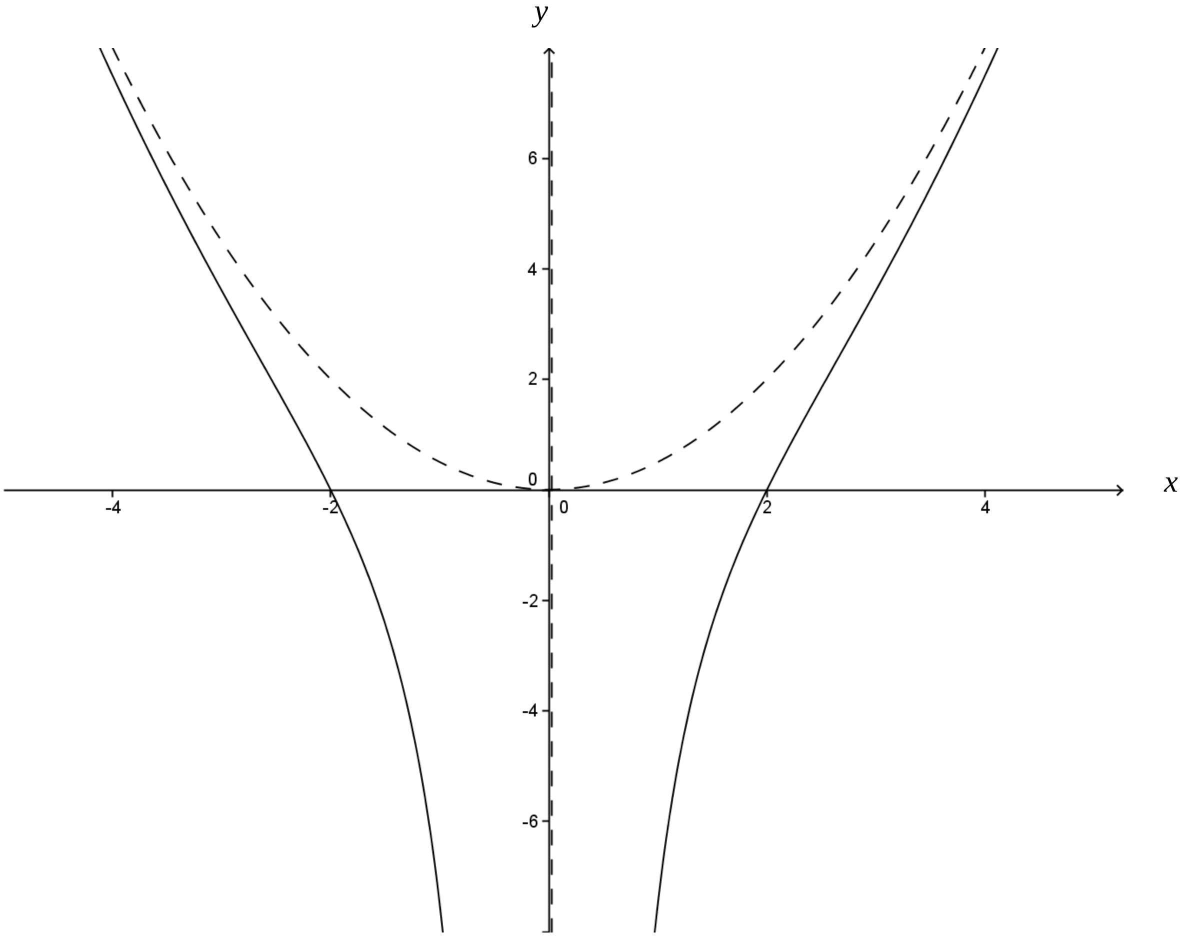
the graph does not cross the *y*-axis,

crosses the *x*-axis when $y=0 \Rightarrow x^4=16 \Rightarrow x=\pm 2$ at $(2,0)$ $(-2,0)$ A1

for turning points, $\dfrac{dy}{dx}=x+\dfrac{16}{x^3}=0 \Rightarrow x^4=-16$ this has no real solution,

so there are no turning points. A1

correct graph G1



Question 8

$$\frac{dy}{dx}=\frac{x}{\sqrt{2x+3}}$$

$$y=\int\frac{x}{\sqrt{2x+3}}\,dx \qquad \text{let} \quad u=2x+3 \quad \frac{du}{dx}=2 \qquad x=\frac{1}{2}(u-3)$$

$$y=\frac{1}{4}\int\frac{u-3}{\sqrt{u}}\,du$$

$$y=\frac{1}{4}\int\left(u^{\frac{1}{2}}-3u^{-\frac{1}{2}}\right)du$$ M1

$$y=\frac{1}{4}\left(\frac{2}{3}u^{\frac{3}{2}}-6u^{\frac{1}{2}}\right)+C$$

$$y=\frac{u^{\frac{1}{2}}}{2}\left(\frac{u}{3}-3\right)+C=\frac{\sqrt{u}}{2}\left(\frac{u-9}{3}\right)+C$$ A1

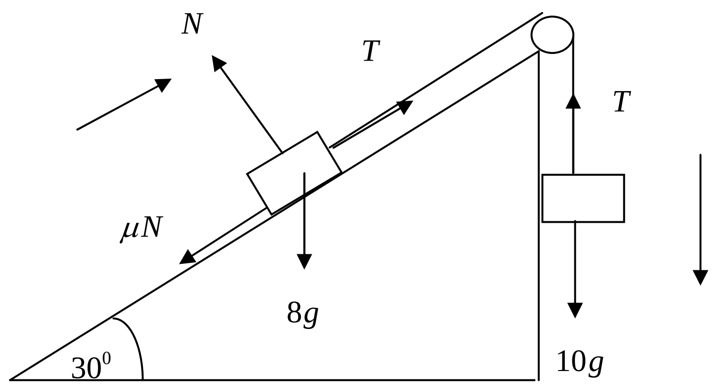
$$y=\frac{1}{6}(2x-6)\sqrt{2x+3}+C=\frac{1}{3}(x-3)\sqrt{2x+3}+C$$

now when $x=3 \quad y=0 \quad \Rightarrow \quad C=0$

$$y=\left(\frac{x}{3}-1\right)\sqrt{2x+3} \qquad a=\frac{1}{3} \qquad b=-1$$ A1

Question 9

i.



correct forces on the diagram A1

ii. resolving downwards for the 10 kg weight hanging vertically

$(1) \quad 10g - T = 10a$

for the crate on the incline plane, resolving upwards parallel to the plane

$(2) \quad T - \mu N - 8g\sin\left(30^0\right) = 8a$ A1

resolving perpendicular to the plane

$(3) \quad N - 8g\cos\left(30^0\right) = 0 \qquad N = 8g\cos\left(30^0\right)$

(2) becomes $T - 8\mu g\cos\left(30^0\right) - 8g\sin\left(30^0\right) = 8a$ A1

adding this to equation (1) to eliminate *T*,

$10g - 8g\left(\sin\left(30^0\right) + \mu\cos\left(30^0\right)\right) = 18a$ substituting M1

$\mu = \frac{\sqrt{3}}{4} \quad \cos\left(30^0\right) = \frac{\sqrt{3}}{2} \quad \sin\left(30^0\right) = \frac{1}{2}$

$g\left[10 - 8\left(\frac{1}{2} + \frac{3}{8}\right)\right] = 18a$

$a = \frac{g}{6}$ m/s^2 A1

Question 10

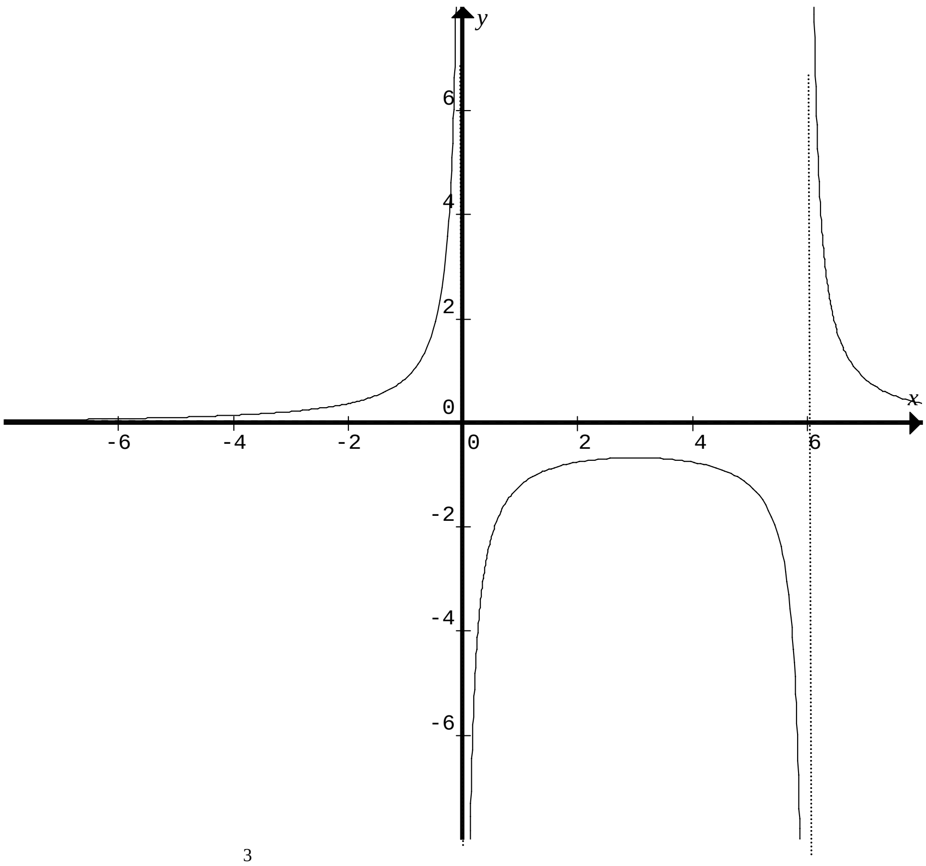
a. $y = \frac{6}{x^2 - 6x} = \frac{6}{x(x-6)}$

vertical asymptotes at $x = 0$ and $x = 6$

horizontal asymptotes at $y = 0$ (the *x*-axis) A1

the turning point occurs when $2x - 6 = 0 \quad \Rightarrow x = 3$

the maximum turning point is $\left(3, -\frac{2}{3}\right)$ and correct graph A1



b. the area is $\int_1^3 \frac{6}{x^2-6x}\,dx$ but this is below the *x*-axis and negative,

so the area is $A=\int_1^3 \frac{6}{6x-x^2}\,dx$ A1

by partial fractions $\frac{6}{6x-x^2}=\frac{B}{x}+\frac{C}{6-x}$ adding the partial fractions

$=\frac{B(6-x)+Cx}{x(6-x)}=\frac{x(C-B)+6B}{6x-x^2}$

(1) $6B=6$ and (2) $C-B=0$ so that $B=C=1$

$A=\int_1^3 \frac{6}{6x-x^2}\,dx=\int_1^3\left(\frac{1}{x}+\frac{1}{6-x}\right)dx$ M1

$A=\left[\log_e(x)-\log_e(6-x)\right]_1^3=\left[\log_e\left(\frac{x}{6-x}\right)\right]_1^3$

$A=\left[\log_e(1)-\log_e\left(\frac{1}{5}\right)\right]=\log_e(5)$ $\quad a=5$ A1

Question 11

$\left(\sqrt{3}+i\right)^m-\left(\sqrt{3}-i\right)^m=0$

now $\sqrt{3}+i=2\text{cis}\left(\frac{\pi}{6}\right)$ and $\sqrt{3}-i=2\text{cis}\left(-\frac{\pi}{6}\right)$ A1

$\left(2\text{cis}\left(\frac{\pi}{6}\right)\right)^m-\left(2\text{cis}\left(-\frac{\pi}{6}\right)\right)^m=0$, using DeMoivre's theorem

$2^m\text{cis}\left(\frac{m\pi}{6}\right)-2^m\text{cis}\left(-\frac{m\pi}{6}\right)=0$ M1

$2^m\left(\left(\cos\left(\frac{m\pi}{6}\right)+i\sin\left(\frac{m\pi}{6}\right)\right)-\left(\cos\left(-\frac{m\pi}{6}\right)+i\sin\left(-\frac{m\pi}{6}\right)\right)\right)=0$

but $\cos(-\theta)=\cos(\theta)$ and $\sin(-\theta)=-\sin(\theta)$, so that

$2^{m+1}i\sin\left(\frac{m\pi}{6}\right)=0$ A1

$\sin\left(\frac{m\pi}{6}\right)=0$

$\frac{m\pi}{6}=0,\pi,2\pi,3\pi,\ ....\ =k\pi$

$m=6k$ where $k\in Z$ A1

END OF SUGGESTED SOLUTIONS