2009 VCAA Specialist Mathematics Exam 1 Solutions
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Q1 $z^4 - z^2 - 6 = 0$, $\left(z^2\right)^2 - z^2 - 6 = 0$, $\left(z^2 - 3\right)\left(z^2 + 2\right) = 0$,
$\left(z - \sqrt{3}\right)\left(z + \sqrt{3}\right)\left(z - i\sqrt{2}\right)\left(z + i\sqrt{2}\right) = 0$.
$\therefore z = \pm\sqrt{3}$, $\pm i\sqrt{2}$.

Q2 Take upward as the positive direction. Use $\sum \vec{F} = m\vec{a}$.
Q2a $\vec{R} + 50\times{}^{-}9.8 = 50\times{}^{-}2$, $\vec{R} = {}^{+}390$ newtons.

Q2b $\vec{R} + 50\times{}^{-}9.8 = 50\times{}^{+}2$, $\vec{R} = {}^{+}590$ newtons.

Q3 Unit vector parallel to $-2\tilde{i} - 2\tilde{j} + \tilde{k}$ is
$\dfrac{-2\tilde{i} - 2\tilde{j} + \tilde{k}}{\sqrt{(-2)^2 + (-2)^2 + 1^2}} = \dfrac{1}{3}\left(-2\tilde{i} - 2\tilde{j} + \tilde{k}\right)$.
Scalar resolute of $5\tilde{i} + \tilde{j} + 3\tilde{k}$ parallel to $-2\tilde{i} - 2\tilde{j} + \tilde{k}$ is
$\left(5\tilde{i} + \tilde{j} + 3\tilde{k}\right).\dfrac{1}{3}\left(-2\tilde{i} - 2\tilde{j} + \tilde{k}\right) = -3$.
$\therefore$ vector resolute of $5\tilde{i} + \tilde{j} + 3\tilde{k}$ parallel to $-2\tilde{i} - 2\tilde{j} + \tilde{k}$ is
$-3\times\dfrac{1}{3}\left(-2\tilde{i} - 2\tilde{j} + \tilde{k}\right) = 2\tilde{i} + 2\tilde{j} - \tilde{k}$, and
vector resolute of $5\tilde{i} + \tilde{j} + 3\tilde{k}$ perpendicular to $-2\tilde{i} - 2\tilde{j} + \tilde{k}$ is
$\left(5\tilde{i} + \tilde{j} + 3\tilde{k}\right) - \left(2\tilde{i} + 2\tilde{j} - \tilde{k}\right) = 3\tilde{i} - \tilde{j} + 4\tilde{k}$.

Q4 $\cos(2\theta) = \dfrac{3}{4}$, $2\cos^2(\theta) - 1 = \dfrac{3}{4}$, $\cos^2(\theta) = \dfrac{7}{8}$,
$\sin^2(\theta) = 1 - \cos^2(\theta) = \dfrac{1}{8}$.
Since $\theta \in \left(\dfrac{3\pi}{4}, \pi\right)$,
$\therefore \cos(\theta) = -\dfrac{\sqrt{7}}{2\sqrt{2}} = -\dfrac{\sqrt{14}}{4}$ and $\sin(\theta) = \dfrac{\sqrt{2}}{4}$.
$\therefore cis(\theta) = \cos(\theta) + i\sin(\theta) = -\dfrac{\sqrt{14}}{4} + i\dfrac{\sqrt{2}}{4}$.

Q5a $3(0)^3 - 4^2 + k(0) + 5(4) - 2(0)(4) = 4$
$\therefore (0,4)$ satisfies the equation, hence every curve in the family passes through it.
Let $x = 0$ to find the y-intercepts, $y^2 - 5y + 4 = 0$,
$(y-4)(y-1) = 0$.
$\therefore (0,1)$ is the other y-intercept.

Q5b Implicit differentiation:
$9x^2 - 2y\dfrac{dy}{dx} + k + 5\dfrac{dy}{dx} - 2y - 2x\dfrac{dy}{dx} = 0$,
$\therefore (2x + 2y - 5)\dfrac{dy}{dx} = 9x^2 + k - 2y$, $\therefore \dfrac{dy}{dx} = \dfrac{9x^2 + k - 2y}{2x + 2y - 5}$.

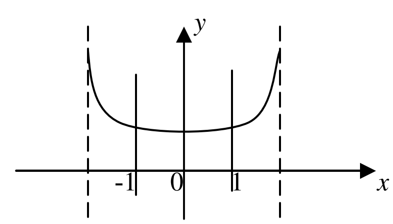
Q5c If $(1,1)$ is a point on the curve,
$3(1)^3 - 1^2 + k(1) + 5(1) - 2(1)(1) = 4$, $k = -1$.
At $(1,1)$, $\dfrac{dy}{dx} = \dfrac{9(1)^2 - 1 - 2(1)}{2(1) + 2(1) - 5} = -6$.

Q6 $y = e^{mx}$, $\dfrac{dy}{dx} = me^{mx}$, $\dfrac{d^2y}{dx^2} = m^2e^{mx}$.
$\therefore \dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} - 10y = m^2e^{mx} - 3me^{mx} - 10e^{mx} = 0$.
$\therefore \left(m^2 - 3m - 10\right)e^{mx} = (m-5)(m+2)e^{mx} = 0$.
Since $e^{mx} \neq 0$, $\therefore m = -2$ or 5.

Q7 $a = v^2 - 3$, $\therefore \dfrac{1}{2}\dfrac{d\left(v^2\right)}{dx} = v^2 - 3$, $\dfrac{dx}{d\left(v^2\right)} = \dfrac{1}{2}\times\dfrac{1}{v^2 - 3}$,
$x = \dfrac{1}{2}\int \dfrac{1}{v^2 - 3}\, d\left(v^2\right)$.
$\therefore 2x = \log_e\left|v^2 - 3\right| + c$.
At $x = 1$, $v = -2$, $\therefore c = 2$ and $2x = \log_e\left|v^2 - 3\right| + 2$.
$\therefore \left|v^2 - 3\right| = e^{2(x-1)}$, $\therefore v^2 - 3 = e^{2(x-1)}$ or $v^2 - 3 = -e^{2(x-1)}$.
Hence $v^2 = 3 + e^{2(x-1)}$ or $v^2 = 3 - e^{2(x-1)}$.
Only $v^2 = 3 + e^{2(x-1)}$ satisfies the condition that at $x = 1$, $v = -2$.
$\therefore v = -\sqrt{3 + e^{2(x-1)}}$.

Q8a $f(x) = \dfrac{2 + x^2}{4 - x^2} = \dfrac{-\left(4 - x^2\right) + 6}{4 - x^2} = -1 + \dfrac{6}{4 - x^2}$.

Q8b



Area $= \displaystyle\int_{-1}^{1}\left(-1 + \frac{6}{4 - x^2}\right)dx = 2\times\int_{0}^{1}\left(-1 + \frac{6}{4 - x^2}\right)dx$

$= 2\times\displaystyle\int_{0}^{1}\left(-1 + \frac{\frac{3}{2}}{2 - x} + \frac{\frac{3}{2}}{2 + x}\right)dx$ [Partial fractions]

$= 2\left[-x - \dfrac{3}{2}\log_e|2 - x| + \dfrac{3}{2}\log_e|2 + x|\right]_0^1 = \log_e 27 - 2$.

Q9a $\frac{dy}{dx}=4+(y+2)^2$, $\frac{dx}{dy}=\frac{1}{4+(y+2)^2}$,

$x=\int\frac{1}{4+(y+2)^2}dy=\frac{1}{2}\int\frac{2}{4+(y+2)^2}dy=\frac{1}{2}\times\tan^{-1}\left(\frac{y+2}{2}\right)+c$.

Given $y(0)=0$, $\therefore c=-\frac{\pi}{8}$.

$x=\frac{1}{2}\times\tan^{-1}\left(\frac{y+2}{2}\right)-\frac{\pi}{8}$, $\tan^{-1}\left(\frac{y+2}{2}\right)=2x+\frac{\pi}{4}$,

$\frac{y+2}{2}=\tan\left(2x+\frac{\pi}{4}\right)$, $\therefore y=2\tan\left(2x+\frac{\pi}{4}\right)-2$.

Q9b $x_0=0$, $y_0=0$, $\frac{dy}{dx}=8$.

$x_1=0.1$, $y_1\approx 0+0.1\times 8=0.8$.

Q10a $f(x)=\frac{2}{\pi}\arcsin\left(\frac{1}{2}x+1\right)-3=\frac{2}{\pi}\arcsin\left(\frac{1}{2}(x+2)\right)-3$.

Comparing with $\arcsin(x)$, which has a domain of $[-1,1]$ and a range of $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, the domain of $f(x)$ is dilated by a factor of 2 and translated to the left by 2, i.e. $[-4,0]$, the range of $f(x)$ is dilated by a factor of $\frac{2}{\pi}$ and translated downwards by 3, i.e. $[-4,-2]$.

Q10b $f(x)=\frac{2}{\pi}\arcsin\left(\frac{1}{2}x+1\right)-3$,

$f'(x)=\frac{2}{\pi}\times\frac{1}{\sqrt{1-\left(\frac{1}{2}x+1\right)^2}}\times\frac{1}{2}=\frac{1}{\pi\sqrt{1-\left(\frac{1}{2}x+1\right)^2}}$

$=\frac{1}{\pi\sqrt{-\frac{1}{4}x(x+4)}}=\frac{2}{\pi\sqrt{-x(x+4)}}$.

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