INSIGHT

Trial Exam Paper

2009

SPECIALIST MATHEMATICS

Written examination 2

STUDENT NAME:

QUESTION AND ANSWER BOOK

Reading time: 15 minutes
Writing time: 2 hours

Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
1	22	22	22
2	5	5	58
		Total	80

- Students are permitted to bring the following items into the examination: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, once bound reference, one approved graphics calculator or CAS (memory DOES NOT need to be cleared) and, if desired, one scientific calculator.
- Students are NOT permitted to bring sheets of paper or white out liquid/tape into the examination.

Materials provided

- The question and answer book of 29 pages with a separate sheet of miscellaneous formulas.
- Answer sheet for multiple-choice questions.

Instructions

- Write your **name** in the box provided and on the multiple-choice answer sheet.
- Remove the formula sheet during reading time.
- You must answer the questions in English.

At the end of the exam

- Place the multiple-choice answer sheet inside the front cover of this book.

Students are NOT permitted to bring mobile phones or any other electronic devices into the examination.

SECTION 1

Instructions for Section 1

Answer **all** questions in pencil on the multiple-choice answer sheet provided.
Choose the response that is **correct** for the question.
One mark will be awarded for a correct answer; no marks will be awarded for an incorrect answer.
Marks **are not** deducted for incorrect answers.
No marks will be awarded if more than one answer is completed for any question.
Take the **acceleration due to gravity** to have magnitude g m/s^2, where $g = 9.8$

Question 1

The parametric equations $x = 2\sec(t + 4) - 2$ and $y = 3\tan(t + 4) + 1$ define a relation given by

A. $\dfrac{(x+2)^2}{4} - \dfrac{(y+1)^2}{9} = 1$

B. $\dfrac{(x+2)^2}{4} + \dfrac{(y-1)^2}{9} = 1$

C. $\dfrac{(x+2)^2}{3} - \dfrac{(y-1)^2}{2} = 1$

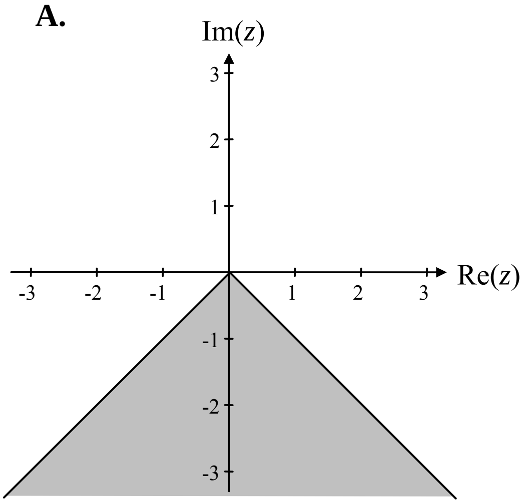
D. $\dfrac{(x+2)^2}{4} - \dfrac{(y-1)^2}{9} = 1$

E. $\dfrac{(x+2)^2}{2} - \dfrac{(y-1)^2}{3} = 1$

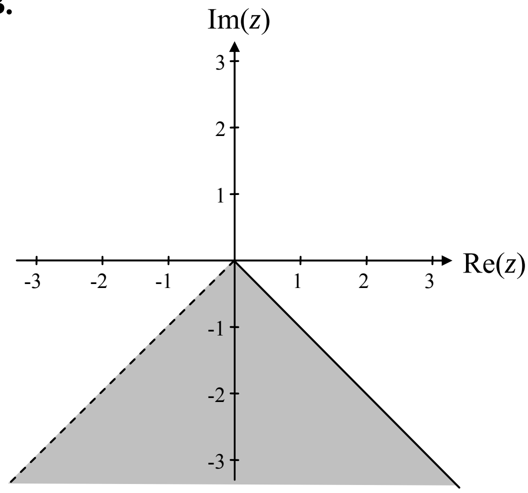
Question 2

The region of the complex plane defined by $\left\{z:\ -\frac{\pi}{4} \le Arg\left(i(z-1)\right) < \frac{\pi}{4}\right\}$ is

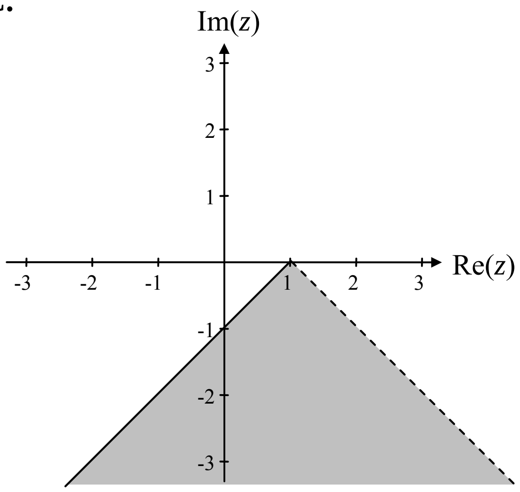
A.



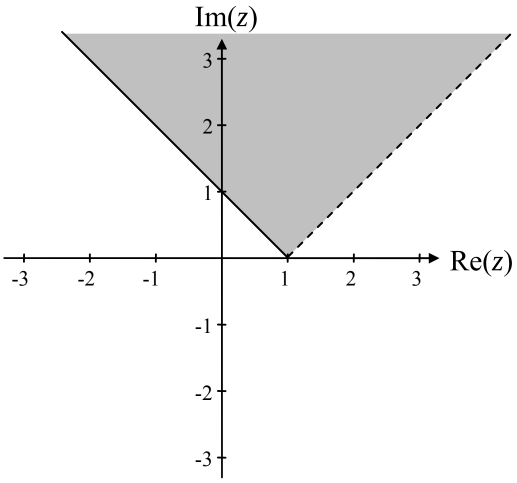
B.



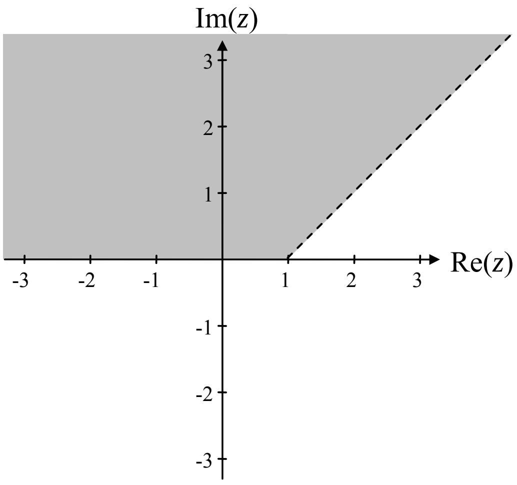
C.



D.



E.



Question 3

The maximal domain and range of the function $f(x) = 3\text{artan}(2x - \pi)$ are given by

A. $d_f = (\pi, 3\pi)$ and $r_f = R$

B. $d_f = R$ and $r_f = [-\frac{\pi}{2}, \frac{\pi}{2}]$

C. $d_f = R$ and $r_f = (-\frac{3\pi}{2}, \frac{3\pi}{2})$

D. $d_f = R$ and $r_f = (\frac{\pi}{4}, \frac{3\pi}{4})$

E. $d_f = (-\frac{\pi}{2}, \frac{\pi}{2})$ and $r_f = R$

Question 4

If $z^2 - z - 2$ is a factor of $P(z) = z^4 - 3z^3 + 10z^2 - 6z - 20,\ z \in C$, then all of the factors must be

A. $z - 2$, $z + 1$, $z - 1 + 3i$ and $z - 1 - 3i$

B. $z - 2$, $z + 1$, $z - 1 + 3i$ and $z + 1 - 3i$

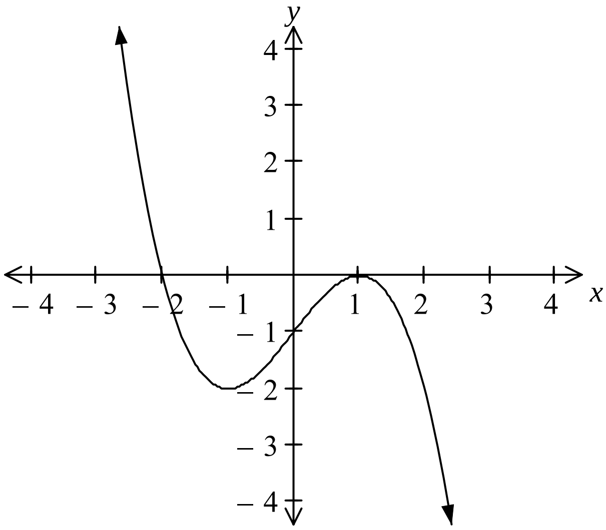
C. $z + 2$, $z - 1$, $z - 3 + i$ and $z - 3 - i$

D. $z - 2$, $z + 1$, $z + 3 + i$ and $z + 3 - i$

E. $z - 2$, $z + 1$, $z + 2$ and $z + 5$

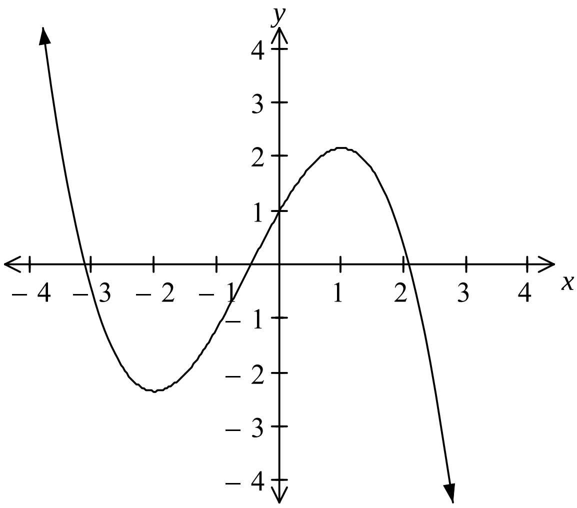
Question 5

The graph of $y = f'(x)$ is shown below.

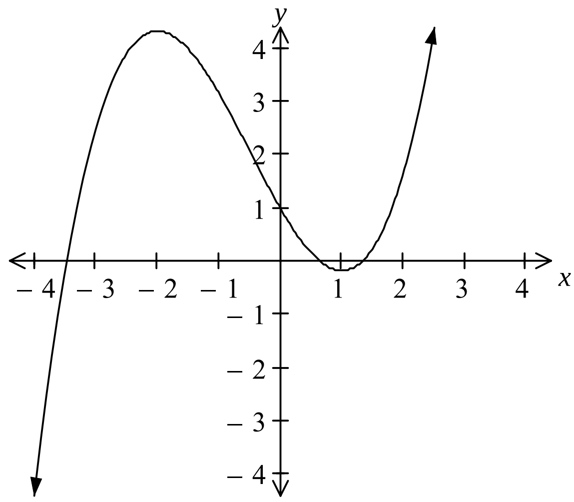


If $f(0) = 1$, then the graph of $y = f(x)$ could be

A.



B.



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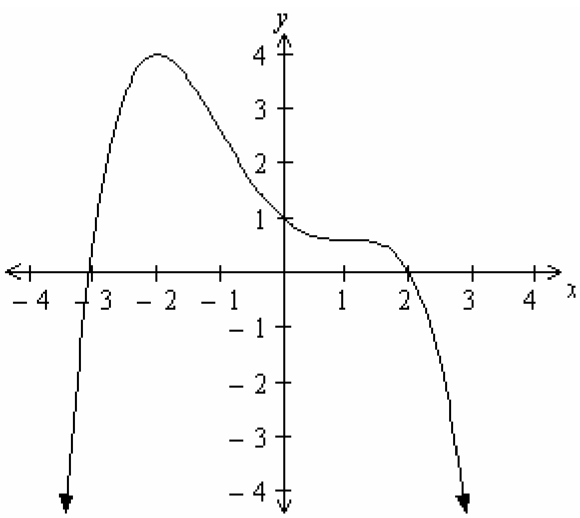

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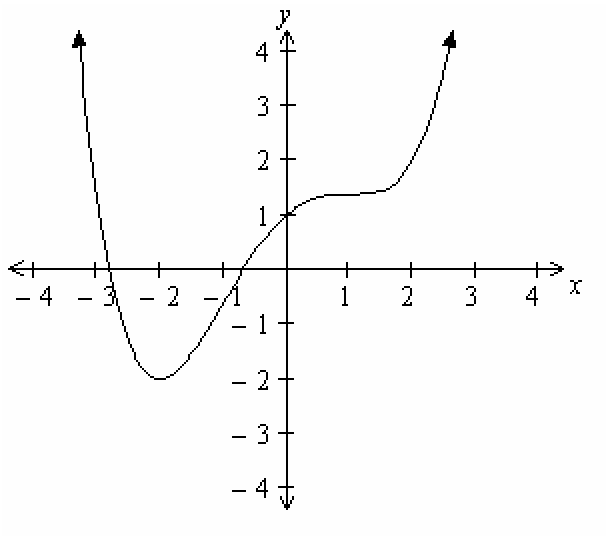
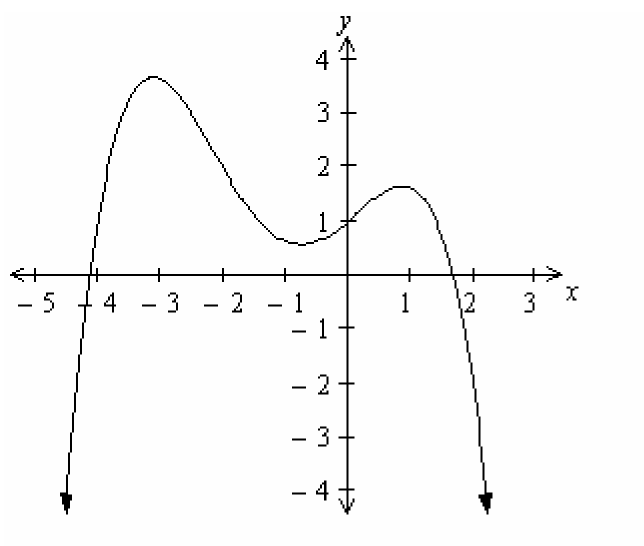
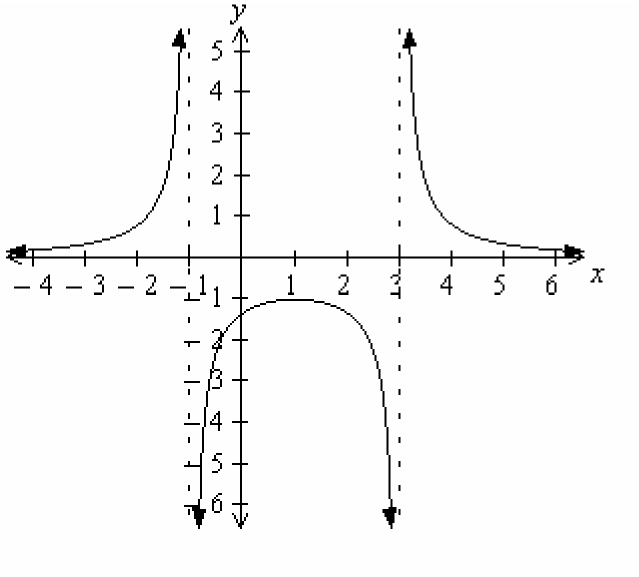


E.



Question 6

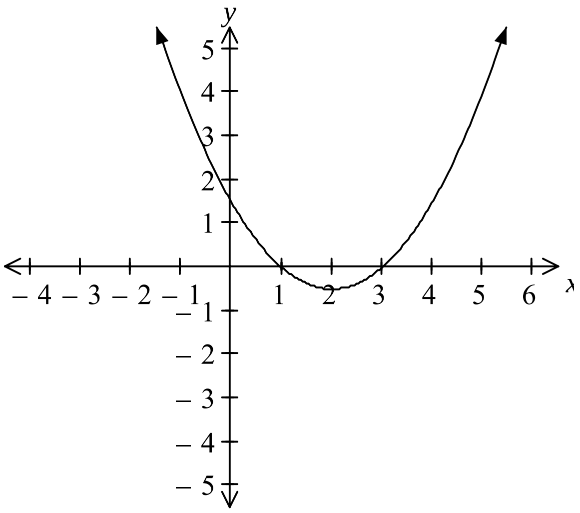
The graph of $y = \dfrac{1}{f(x)}$ is shown below.



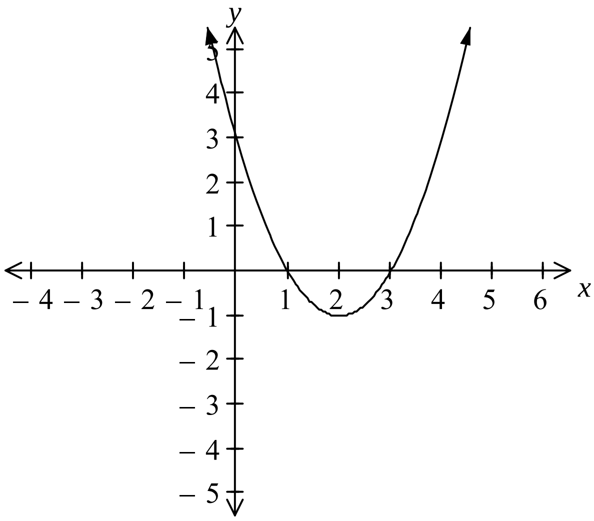
SECTION 1 – Question 6 – continued

The graph of $y = f(x)$ could be

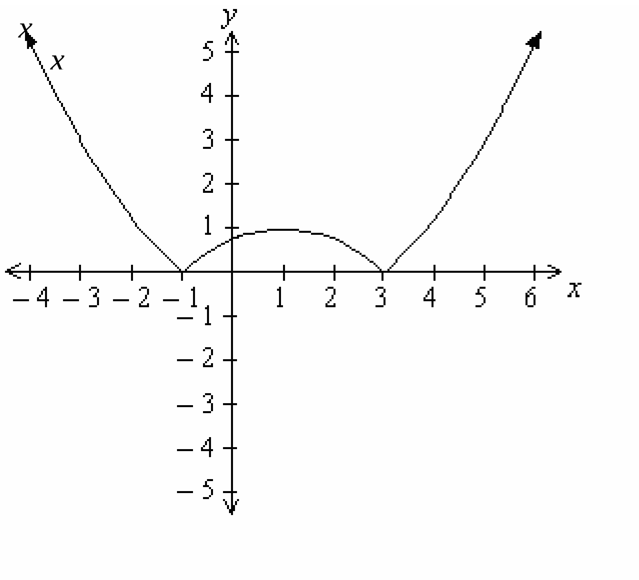
A.



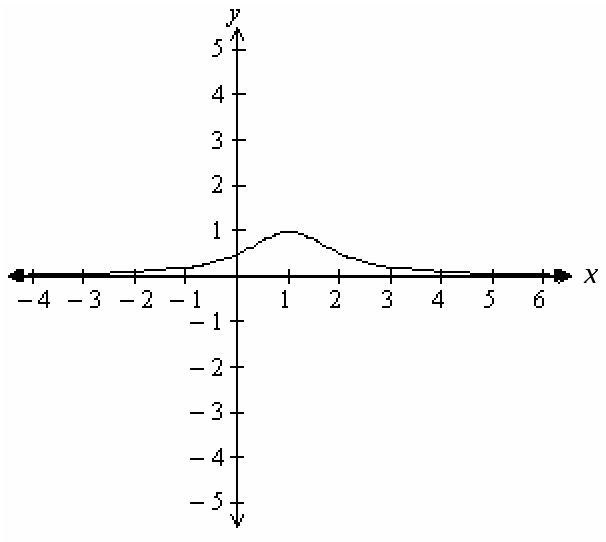
B.



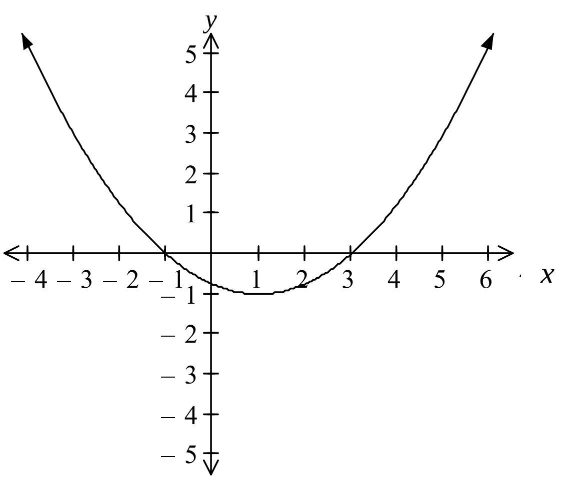
C.



D.



E.



Question 7

The solutions to $z^2 = a + \sqrt{3}ai$, where $z \in C$ and $a \in R^+$, are

A. $\pm\frac{\sqrt{2}}{2}(\sqrt{3} + i)$

B. $\pm\frac{\sqrt{2a}}{2}(\sqrt{3} + i)$

C. $\pm\frac{\sqrt{2}}{2}(1 + \sqrt{3}\,i)$

D. $\pm\frac{\sqrt{2a}}{2}(1 + \sqrt{3}\,i)$

E. $\pm\frac{\sqrt{2}}{2}(1 - \sqrt{3}\,i)$

Question 8

For the vectors $\underset{\sim}{a} = 3\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$, $\underset{\sim}{b} = \underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}$ and $\underset{\sim}{c} = x\underset{\sim}{i} - 7\underset{\sim}{j} + 10\underset{\sim}{k}$ to be linearly dependent, the value of x must be

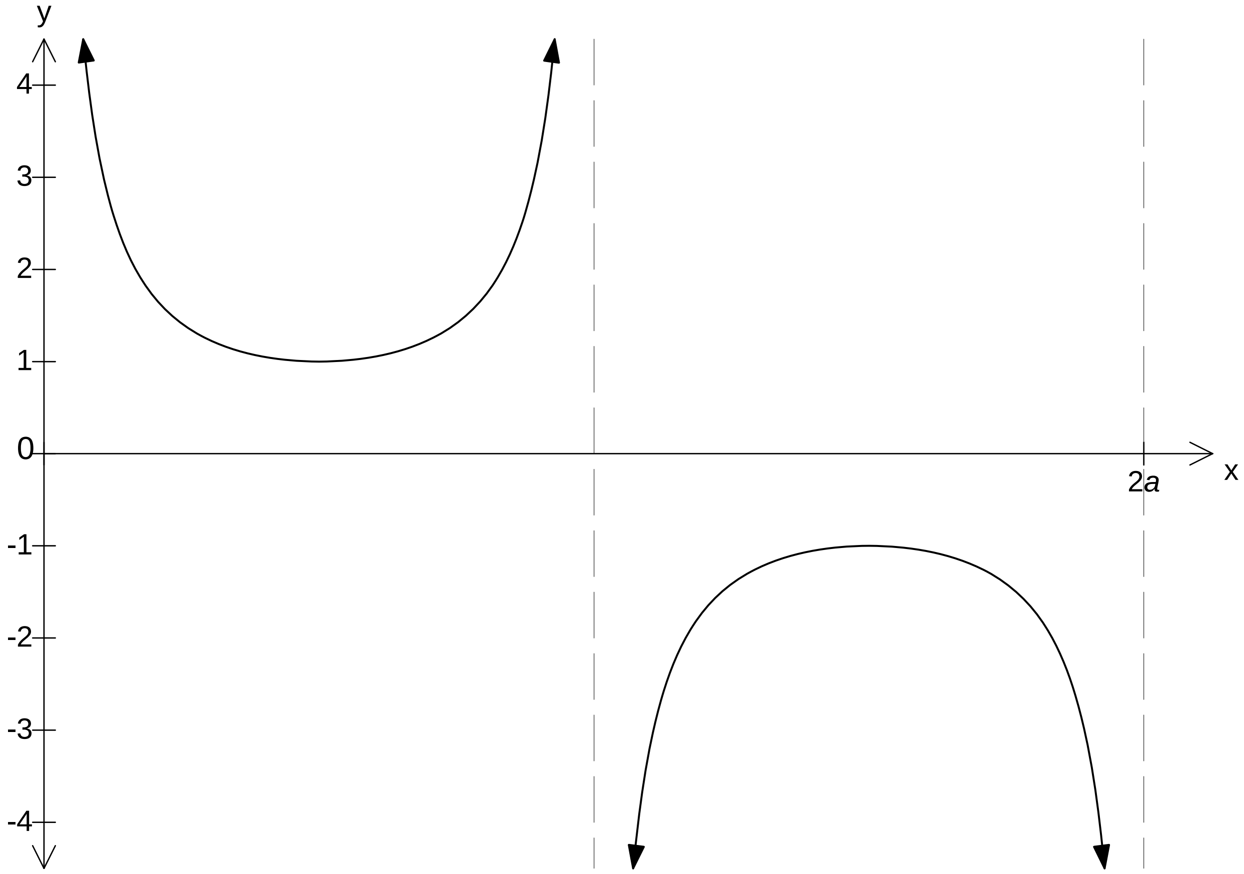
A. 4
B. 7
C. 2
D. – 7
E. –2

Question 9

The graph of the relation $\{z: \quad z\bar{z} - 2\,\mathrm{Re}(z) = 8, \quad z \in C\}$ would be

A. a circle with centre (0, 0) and radius $2\sqrt{2}$.
B. a circle with centre (–1, 0) and radius 3.
C. a straight line with gradient 2 and y-intercept of 8.
D. a straight line with gradient 1 and y-intercept of 8.
E. a circle with centre (1, 0) and radius 3.

Question 10



The rule for the function graphed above, where $a > 0$, could be

A. $y = \text{cosec}\left(\frac{\pi x}{a}\right)$

B. $y = \sec\left(\frac{\pi x}{a}\right)$

C. $y = \sec\left(\frac{\pi}{a}\left(x - \frac{a}{2}\right)\right)$

D. $y = -\text{cosec}\left(\frac{\pi x}{a}\right)$

E. $y = \text{cosec}\left(\frac{\pi}{a}\left(x + \frac{a}{2}\right)\right)$

Question 11

$\int_0^1 \left(\frac{2-3x}{4-x^2}\right)dx$ is equal to

A. $\log_e\left(\frac{9}{2}\right)$

B. $\log_e 18$

C. 0

D. $\log_e 72$

E. $\log_e\left(\frac{9}{8}\right)$

Question 12

The gradient of the tangent to the curve $2x\log_e(y) - x = y$ at the point where $y = e$ is

A. 3

B. –1

C. –3

D. 1

E. $\frac{1}{2}$

Question 13

Using a suitable substitution, $\int_0^1 \frac{\cos^{-1}\left(\frac{x}{2}\right)}{\sqrt{4-x^2}}\,dx$ is equal to

A. $2\int_2^0 u\,du$

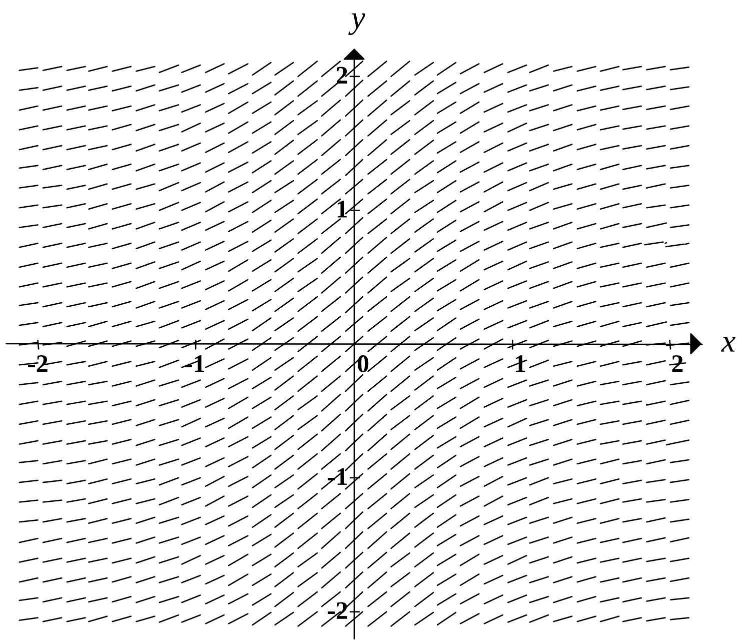
B. $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} u\,du$

C. $\frac{1}{2}\int_0^2 u\,du$

D. $2\int_0^{\frac{\pi}{4}} u\,du$

E. $\int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{u}\,du$

Question 14



The direction (slope) field for a certain first-order differential equation is shown above.

The differential equation could be

A. $\frac{dy}{dx} = \frac{1}{1+x^2}$

B. $\frac{dy}{dx} = \tan^{-1} x$

C. $\frac{dy}{dx} = 1 + x^2 + y^2$

D. $\frac{dy}{dx} = |x+1|$

E. $\frac{dy}{dx} = \frac{1}{|x+y+1|}$

Question 15

If $\frac{dy}{dx} = \log_e(x)$ and $y(1) = 2$, then the value of y when $x = 3$ can be found by evaluating

A. $1 + \int_2^3 \log_e(t)\, dt$

B. $2 + \int_1^3 \frac{1}{t}\, dt$

C. $2 + \int_1^3 \log_e(t)\, dt$

D. $1 - \int_2^3 \log_e(t)\, dt$

E. $3 + \int_1^2 \log_e(t)\, dt$

Question 16

The position vectors of two moving particles, R and S, at any time t seconds are given by

$\underset{\sim}{r} = at\,\underset{\sim}{i} - 4\,\underset{\sim}{j}$ and $\underset{\sim}{s} = t^2\,\underset{\sim}{i} + 2t\,\underset{\sim}{j},\ t \geq 0,\ a \in R$, respectively.

The angle between the directions of the two particles at $t = 1$ is

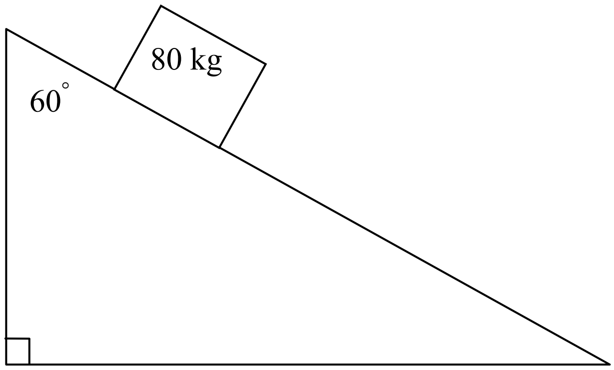
A. 69.3°
B. 45°
C. 35.3°
D. 19.5°
E. dependent on the value of a.

Question 17

The volume of a tank is given by $V = 0.4\pi h^{\frac{5}{2}}$, where h cm is the depth of water in the tank at time t minutes. Water leaks from the tank at a rate of 16 cm^3/minute. The depth of water in the tank when the height is decreasing at a rate of $\frac{2}{\pi}$ cm/minute is

A. 16 cm
B. 8 cm
C. 4π cm
D. 4 cm
E. 8π cm

Question 18



A skier of mass 80 kilograms slides from rest down a straight slope inclined at 60° to the vertical. Assuming it is a smooth slope, the speed of the skier after moving 100 metres down the slope is nearest to

A. 41.2 m/s
B. 22.1 m/s
C. 31.3 m/s
D. 44.3 m/s
E. 10 m/s

Question 19

A mass of 4 kilograms is at rest when two forces, $\underset{\sim}{F}_1 = (\underset{\sim}{i} - 3\underset{\sim}{j})$ newtons and $\underset{\sim}{F}_2 = (2\underset{\sim}{i} - \underset{\sim}{j})$ newtons, act on it. The time taken for the mass to travel 10 metres is

A. 1 s
B. 2 s
C. 4 s
D. 5 s
E. 8 s

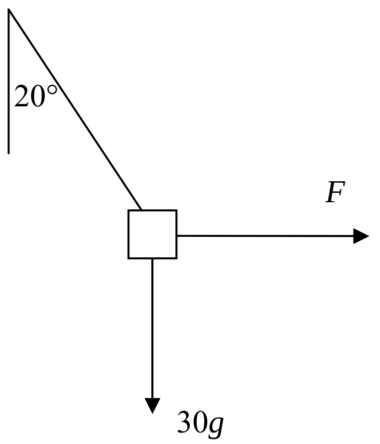
Question 20

The velocity of a particle moving in a straight line is given by $v(x) = \cos(x^2)$, where x is the displacement from the origin O.

The acceleration of the particle is

A. $a(x) = -2x\sin(x^2)$
B. $a(x) = \cos(2x)$
C. $a(x) = -2x\tan(x^2)$
D. $a(x) = -x\sin(2x^2)$
E. $a(x) = -2x\tan(x^2)\sec(x^2)$

Question 21



The magnitude of the horizontal force, F newtons, required to hold a 30 kilogram child in equilibrium on a swinging rope, as shown in the diagram above, is

A. $\dfrac{30g}{\tan 70^\circ}$

B. $\dfrac{30g \sin 70^\circ}{\sin 20^\circ}$

C. $30g \sin 20^\circ$

D. $\dfrac{30g}{\sin 70^\circ}$

E. $30g \tan 20^\circ$

Question 22

A lift travelling upwards accelerates at a m/s^2 ($a > 0$) with a person of mass 100 kilograms standing on a set of weight scales in the lift. It then decelerates at twice the magnitude of the acceleration. The magnitude of the change in the reading on the scales will be

A. $100a$ kg

B. $200g$ kg

C. $300a$ kg

D. $100(g + a)$ kg

E. $-100a$ kg

END OF SECTION 1

SECTION 2

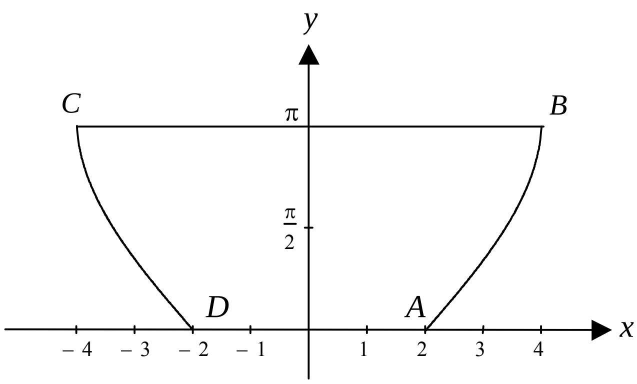
Instructions for Section 2

Answer all the questions in the spaces provided.
A decimal approximation will not be accepted if an exact answer is required to a question.
In questions where more than one mark is available, approximate working must be shown.
Unless otherwise indicated, the diagrams in this book have not been drawn to scale.
Take the **acceleration due to gravity** to have magnitude g m/s^2, where $g = 9.8$

Question 1

The diagram below shows the profile of a symmetrical small bowl *ABCD*. The bowl is generated by rotating the area between the curve *AB* and the *y*-axis about the *y*-axis. The top and base of the bowl have radii of 4 cm and 2 cm, respectively, and the height of the bowl is π cm.

The curve *AB* can be modelled by the function $y = a\sin^{-1}(bx - c),\ \ x \in [2, 4]$.



a. Show that $a = 2$, $b = \dfrac{1}{2}$ and $c = 1$.

3 marks

b. If h cm is the height of water in the bowl at any time, express the volume of water, V cm^3, in terms of h.

4 marks

c. **Hence**, find the exact volume of water in a full bowl.

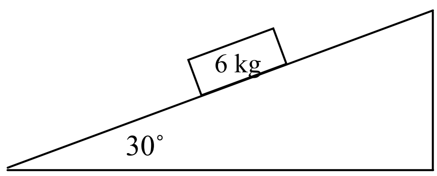
1 mark

d. To the nearest millimetre, what would be the height of the water when the bowl is filled to half its capacity?

2 marks
Total 3 + 4 + 1 + 2 = 10 marks

Question 2

A miniature racing car of mass 6 kilograms is propelled from rest up a rough ramp 19.6 metres long and inclined at an angle of 30° to the horizontal. The car is powered up the ramp by a constant force of 10g newtons. This causes the car to accelerate at 9.8 m/s^2.



a. Label the forces acting on the car as it moves up the ramp.

1 mark

b. Show that at the top of the ramp the car is g metres above the ground and its speed is 2g m/s when it leaves the ramp.

2 marks

c. Calculate the exact value of the coefficient of friction.

2 marks

When the car leaves the ramp it is only subject to the force of gravity.
Take $\underset{\sim}{i}$ as the unit vector in the horizontal direction and $\underset{\sim}{j}$ as the unit vector in the vertical direction from the point on the ground, directly below the top of the ramp.

d. Determine the velocity vector $\underset{\sim}{v}$ and the position vector $\underset{\sim}{r}$ of the car at any time t seconds.

2 marks

e. Find the exact Cartesian equation of the path of the car after it leaves the ramp.

2 marks

f. Find the exact magnitude of the momentum of the car when it hits the ground.

3 marks
Total 1 + 2 + 2 + 2 + 2 + 3 = 12 marks

Question 3

a. Given $w = a + bi$, where $a, b \in R$ and $b > 0$.

If $w + \overline{w} = 2$ and $w\,\overline{w} = 2$, show that $w = 1 + i$.

2 marks

b. If $v = 1 + \sqrt{3}\,i$,

i. Find $\dfrac{v}{w}$ in simplest exact Cartesian form.

1 mark

ii. Find $\dfrac{v}{w}$ in polar form.

2 marks

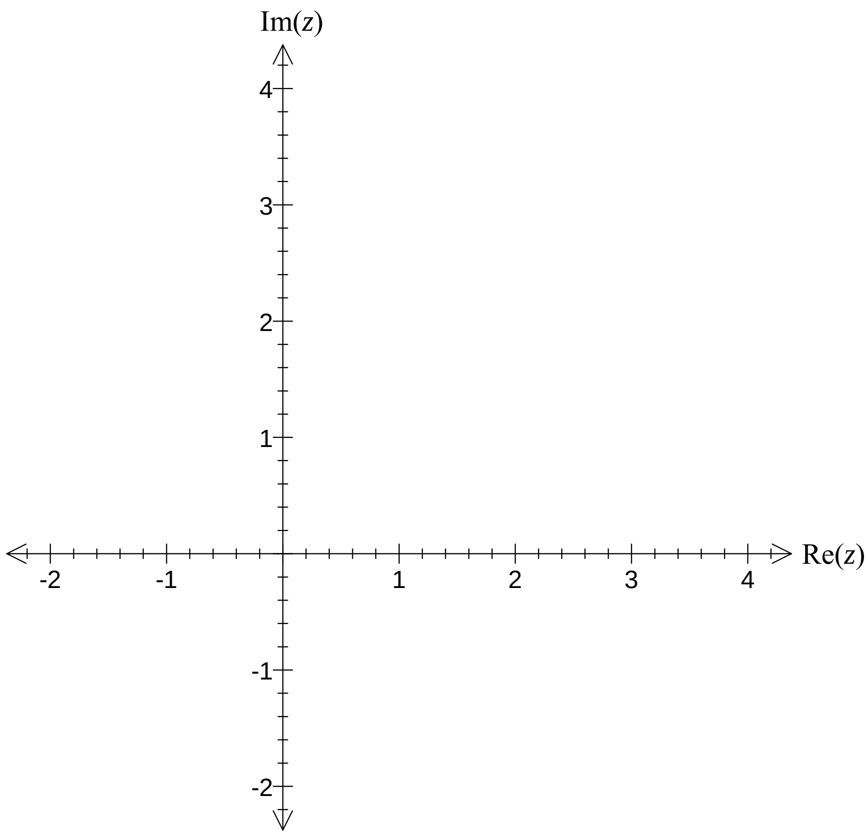
c. **Hence**, express $\tan\left(\frac{\pi}{12}\right)$ in the form $a-\sqrt{b}$, where a and b are positive integers.

3 marks

d. S is a subset of the complex plane, which is defined by

$$S = \{z : \ |z - w| = 1, \quad z \in C\}$$

Plot the points v and w and sketch the relation defined by S on the Argand diagram below.



2 marks

SECTION 2 – Question 3 – continued
TURN OVER

e. T is a subset of the complex plane defined by

$$T = \{z : \ |z - v| = |z - w|, \quad z \in C\}$$

i. Express the equation for the relation defined by T in Cartesian form.

1 mark

ii. Part of T is a chord to the relation $S = \{z : \ |z - w| = 1, \quad z \in C\}$

Find the exact length of this chord in the form $a^{\frac{b}{c}}$, where a, b and c are integers.

3 marks

Total 2 + 3 + 3 + 2 + 4 = 14 marks

Question 4

A tank contains 100 litres of sugar solution with a concentration of 0.05 kg/L. A sugar solution of concentration 0.1 kg/L flows into the tank at a rate of 2 L/min. The mixture, which is kept uniform by stirring, flows out of the tank at a rate of 2 L/min. After t minutes the tank contains x kilograms of sugar.

a. Show that the differential equation for x in terms of t is $\frac{dx}{dt} = \frac{10 - x}{50}$ kg/min.

1 mark

b. Solve this differential equation to give x as a function of t.

3 marks

c. Calculate the amount of sugar in the tank after 2 minutes. Give your answer correct to three decimal places.

1 mark

d. If this situation continued for a long period of time, how much sugar would be present in the tank?

1 mark

e. If the outflow from the tank was 1 L/min instead of 2 L/min, set up the new differential equation for x in terms of t.

1 mark

f. For the differential equation from part **e.** use Euler's method, with increments of 1 minute, to predict the amount of sugar in the tank after 2 minutes. Give your answer correct to three decimal places.

2 marks

Total 1 + 3 + 1 + 1 + 1 + 2 = 9 marks

Question 5

At 10 a.m. an aircraft is flying at an altitude of $(e^2 - e)$ km, 500 km north and 440 km east of a point $T(0, 0, 0)$, which is its touchdown point on a horizontal runway.

The position of the aircraft relative to the point T is given by the vector

$$\underset{\sim}{r}(t) = \left(a + \frac{2420}{t+5}\right)\underset{\sim}{i} + (500 - 24t + 0.28t^2)\underset{\sim}{j} + (e^{c-0.02t} - e)\underset{\sim}{k}, \text{ where } a,\ c \in R.$$

$\underset{\sim}{r}$ is in kilometres and t is the time in minutes after 10 a.m.

$\underset{\sim}{i}$ is the unit vector in an easterly direction, $\underset{\sim}{j}$ is the unit vector in a northerly direction and

$\underset{\sim}{k}$ is the unit vector representing the altitude of the aircraft.

(Treat the aircraft as a point in this problem.)

a. Show that $a = -44$ and $c = 2$.

1 mark

b. Show that the aircraft touches down at point T at 10.50 a.m.

1 mark

c. Show that the exact velocity of the aircraft at touchdown is $\underset{\sim}{r}' = -0.8\underset{\sim}{i} + 4\underset{\sim}{j} - 0.02e\underset{\sim}{k}$.

2 marks

d. Find the vertical angle to the runway at which the aircraft lands. Give your answer to the nearest hundredth of a degree.

3 marks

e. Relative to the point T, find the position vector $\underset{\sim}{p}(t)$ of the aircraft on the runway when the aircraft stops.

3 marks

f. A stationary fire engine is positioned 1.5 kilometres north and 200 metres west of *T*.

Determine, to the nearest metre, the minimum distance between the fire engine and the aircraft on its path after touchdown.

3 marks

Total 1 + 1 + 2 + 3 + 3 + 3 = 13 marks

END OF QUESTION AND ANSWER BOOK