Insight Publications

INSIGHT

Trial Exam Paper

2008

SPECIALIST MATHEMATICS

Written examination 2

Worked solutions

This book presents:

- worked solutions, giving you a series of points to show you how to work through the questions.
- mark allocations
- tips on how to approach the questions.

SECTION 1

Question 1

The vectors $4\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$ and $-2\underset{\sim}{i} - \underset{\sim}{j} + a\underset{\sim}{k}$ are linearly dependent when a is equal to

A. -10

B. $-\frac{1}{2}$

C. $\mathbf{\frac{1}{2}}$

D. 4

E. 10

Answer is C.

Tip

- *Two vectors will be linearly dependent when they are parallel.*

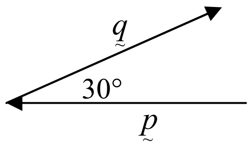
Worked solution

$-2\underset{\sim}{i} - \underset{\sim}{j} + a\underset{\sim}{k} \equiv -\frac{1}{2}\left(4\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}\right)$

$\therefore\ a = \frac{1}{2}$

Question 2

Vectors $\underset{\sim}{p}$ and $\underset{\sim}{q}$ are shown in the diagram below.



Given $\left|\underset{\sim}{p}\right| = 2\sqrt{3}$ and $\left|\underset{\sim}{q}\right| = 3$, it follows that $\underset{\sim}{p} \cdot \underset{\sim}{q}$ will be equal to

A. **–9**

B. $-3\sqrt{3}$

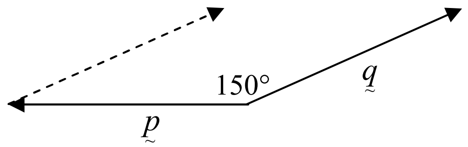
C. $3\sqrt{3}$

D. $6\sqrt{3}$

E. 9

Answer is A.

Worked solution



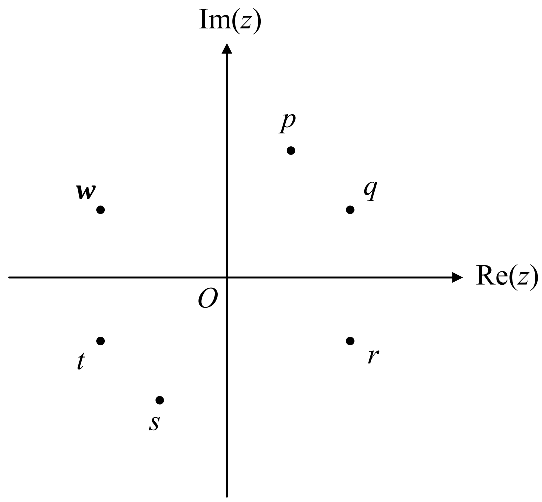
$\underset{\sim}{p} \cdot \underset{\sim}{q} = \left|\underset{\sim}{p}\right|\left|\underset{\sim}{q}\right|\cos\theta$

$\underset{\sim}{p} \cdot \underset{\sim}{q} = 2\sqrt{3} \times 3\cos(150°)$

$\underset{\sim}{p} \cdot \underset{\sim}{q} = 2\sqrt{3} \times 3 \times \left(-\frac{\sqrt{3}}{2}\right)$

$\underset{\sim}{p} \cdot \underset{\sim}{q} = -9$

Question 3



The complex number ***w*** is plotted in the Argand diagram above.

The point that represents the complex number –*iw* would be

A. ***p***
B. *q*
C. *r*
D. *s*
E. *t*

Answer is A.

Tip

- *Multiplying any complex number by –i rotates it by 90° in a clockwise direction.*

Worked solution

Let $w = -a + bi$, where a and b are positive real numbers.

$-iw = -i(-a + bi) = ai - bi^2 = ai + b = b + ai$

$b + ai$ is represented in the first quadrant by p.

Question 4

Let $z = x + yi$, where x and y are non-zero real numbers.

Which one of the following is not a real number?

A. $z.\bar{z}$

B. $z + \bar{z}$

C. $\frac{1}{z} + \frac{1}{\bar{z}}$

D. $(z - \bar{z})^2$

E. $\mathbf{z^2 - \bar{z}^2}$

Answer is E.

Worked solution

$z.\bar{z} = (x + yi)(x - yi) = x^2 - (yi)^2 = x^2 + y^2$ real

$z + \bar{z} = (x + yi) + (x - yi) = 2x$ real

$\frac{1}{z} + \frac{1}{\bar{z}} = \frac{\bar{z} + z}{z.\bar{z}} = \frac{2x}{x^2 + y^2}$ real

$(z - \bar{z})^2 = (x + yi - (x - yi))^2 = (2yi)^2 = -4y^2$ real

$z^2 - \bar{z}^2 = (x + yi)^2 - (x - yi)^2 = x^2 - y^2 + 2xyi - (x^2 - y^2 - 2xyi) = 4xyi$ imaginary

Question 5

Complex numbers u and v are such that $\frac{u}{v} = 10\,\text{cis}\left(\frac{\pi}{3}\right)$.

If $v = 2\,\text{cis}\left(\frac{\pi}{5}\right)$, then u is equal to

A. $5\,\text{cis}\left(\frac{\pi}{15}\right)$

B. $5\,\text{cis}\left(\frac{2\pi}{15}\right)$

C. $20\,\text{cis}\left(\frac{\pi}{15}\right)$

D. $20\,\text{cis}\left(\frac{\pi}{8}\right)$

E. $\mathbf{20\,\text{cis}\left(\frac{8\pi}{15}\right)}$

Answer is E.

Worked solution

$$\frac{u}{v} = 10\,\text{cis}\left(\frac{\pi}{3}\right)$$

$$u = v \times 10\,\text{cis}\left(\frac{\pi}{3}\right) \qquad \text{(Given that } v = 2\,\text{cis}\left(\frac{\pi}{5}\right)\text{).}$$

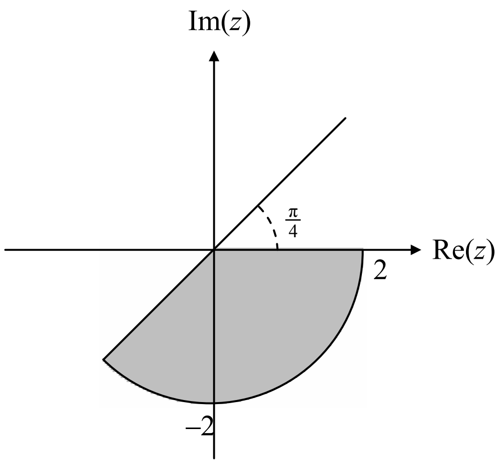
$$u = 2\,\text{cis}\left(\frac{\pi}{5}\right) \times 10\,\text{cis}\left(\frac{\pi}{3}\right)$$

$$u = 20\,\text{cis}\left(\frac{\pi}{5} + \frac{\pi}{3}\right)$$

$$u = 20\,\text{cis}\left(\frac{3\pi}{15} + \frac{5\pi}{15}\right)$$

$$u = 20\,\text{cis}\left(\frac{8\pi}{15}\right)$$

Question 6



The shaded region shown in the diagram above with boundaries included represents

A. $\{z : |z| \le 2\} \cap \left\{z : \text{Arg}(z) \ge -\frac{3\pi}{4}\right\}$

B. $\mathbf{\{z : |z| \le 2\} \cap \left\{z : -\frac{3\pi}{4} \le \text{Arg}(z) \le 0\right\}}$

C. $\{z : |z| \le 2\} \cap \left\{z : \text{Arg}(z) \ge -\frac{\pi}{4}\right\}$

D. $\{z : |z| \le 4\} \cap \left\{z : -\frac{\pi}{4} \le \text{Arg}(z) \le 0\right\}$

E. $\{z : |z| \le 4\} \cap \left\{z : -\frac{3\pi}{4} \le \text{Arg}(z) \le 0\right\}$

Answer is B.

Worked solution

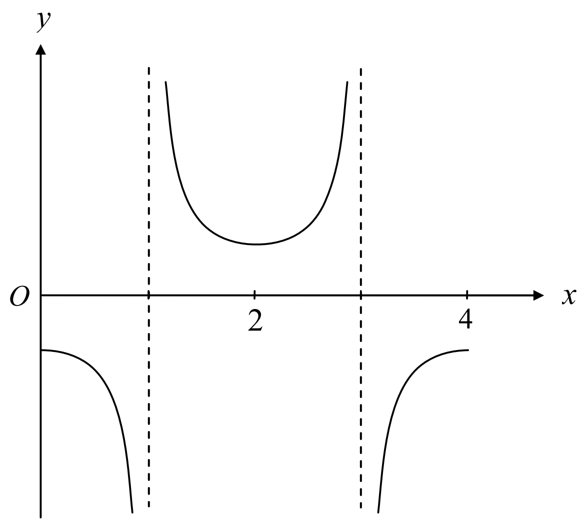
Interior of circle centre (0, 0) radius 2 units with boundaries included is given by $\{z : |z| \le 2\}$.

The shading is between the rays $\text{Arg}(z) = -\frac{3\pi}{4}$ and $\text{Arg}(z) = 0$ with boundaries included.

Therefore, the shaded region is represented by $\{z : |z| \le 2\} \cap \left\{z : -\frac{3\pi}{4} \le \text{Arg}(z) \le 0\right\}$.

Question 7

The graph of $f:[0, 4] \to R,\ \ f(x) = \sec(nx - p)$ is shown on the axes below.



The values of n and p, respectively, would be

A. $\frac{1}{2},\ \ \pi$

B. $\frac{1}{2},\ \ 2$

C. $\frac{\pi}{2},\ \ 2$

D. $\boldsymbol{\frac{\pi}{2},\ \ \pi}$

E. $\pi,\ \ \frac{\pi}{2}$

Answer is D.

Worked solution

The period of the graph is 4.

$$\therefore\ \frac{2\pi}{n} = 4$$

$$n = \frac{\pi}{2}$$

The graph of $y = \sec\left(\frac{\pi}{2}x\right)$ has been translated 2 units right.

The equation is $y = \sec\left(\frac{\pi}{2}(x-2)\right)$.

Putting equation in the form $y = \sec(nx - p)$ gives $y = \sec\left(\frac{\pi}{2}x - \pi\right)$.

$$n = \frac{\pi}{2},\ \ p = \pi$$

Question 8

$y = \dfrac{1}{x^2 + bx + 1}$ has domain $x \in R$ when b is an element of

A. R
B. $(0, \infty)$
C. $(-2, \infty)$
D. $\mathbf{(-2, 2)}$
E. $[-2, 2]$

Answer is D.

Worked solution

For the domain of $y = \dfrac{1}{x^2 + bx + 1}$ to be R, $x^2 + bx + 1 \neq 0$.

This occurs when the discriminant of the quadratic $x^2 + bx + 1$ is negative.

$\Delta = b^2 - 4ac < 0$

Here, $a = 1$, $b = b$, $c = 1$.

$\therefore \; b^2 - 4 \times 1 \times 1 < 0$

$b^2 - 4 < 0$

$b \in (-2, 2)$

Question 9

The following information is known about the function $y = f(x)$.

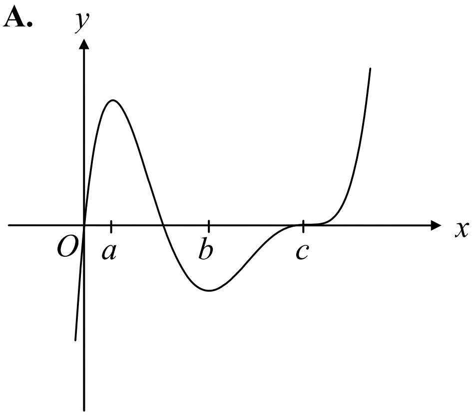
$f'(a) = 0, \quad f''(a) < 0$

$f'(b) < 0, \quad f''(b) = 0$

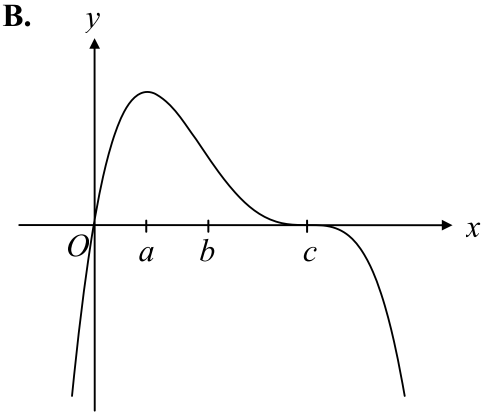
$f'(c) = 0, \quad f''(c) = 0$

A graph of $y = f(x)$ could be

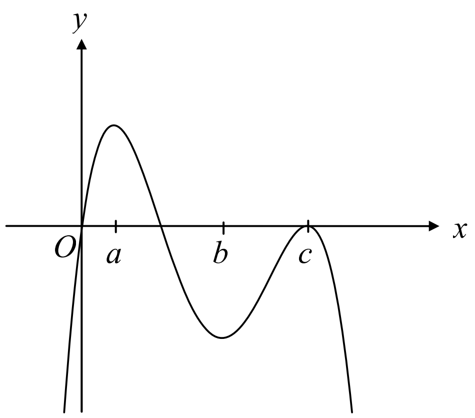
A.



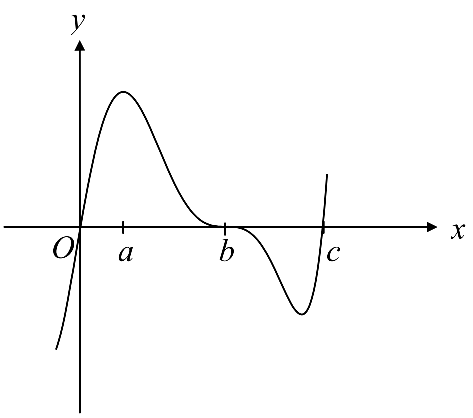
B.



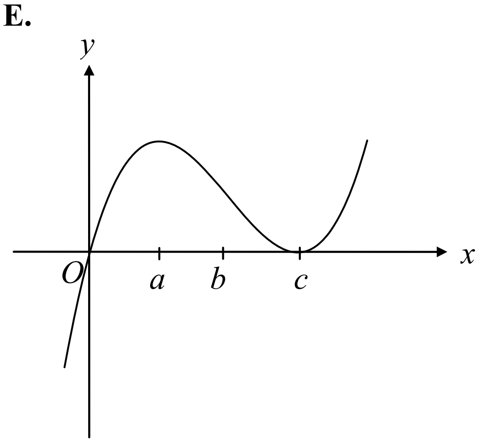
C.



D.



E.



Answer is B.

Worked solution

Since $f'(a)=0$ we know that there is a stationary point at $x = a$.

All alternatives have a maximum turning point at $x = a$.

Since $f'(b)<0$ and $f''(b)=0$, there is a non-stationary point of inflexion (i.e. point of greatest slope) at $x = b$.

Since $f'(c)=0$ and $f''(c)=0$, there is a stationary point of inflexion at $x = c$.

The only graph that satisfies all of these conditions is graph **B**.

Question 10

Using a suitable substitution $\int \frac{\log_e(2x)}{2x}\,dx$, $x \neq 0$ can be expressed as

A. $\mathbf{\frac{1}{2}\int u\,du}$

B. $2\int u\,du$

C. $\int u\,du$

D. $\int \frac{1}{u}\,du$

E. $\frac{1}{2}\int \frac{1}{u}\,du$

Answer is A.

Worked solution

Let $u = \log_e(2x)$

$\Rightarrow \quad \frac{du}{dx} = \frac{1}{x}$

$\Rightarrow \quad du = \frac{1}{x}dx$

$\int \frac{\log_e(2x)}{2x}\,dx$

$= \frac{1}{2}\int \log_e(2x)\left(\frac{1}{x}\,dx\right)$

$= \frac{1}{2}\int u\,du$

Question 11

If $f(x)=\int_{1}^{x^2}\frac{1}{\sqrt{4-t^2}}\,dt$, then $f(\sqrt{2})$ is equal to

A. $\frac{\pi}{12}$

B. $\frac{\pi}{6}$

C. $\frac{\pi}{4}$

D. $\boldsymbol{\frac{\pi}{3}}$

E. $\frac{\pi}{2}$

Answer is D.

Worked solution

$$f(x)=\int_{1}^{x^2}\frac{1}{\sqrt{4-t^2}}\,dt$$

$$f(x)=\left[\sin^{-1}\left(\frac{t}{2}\right)\right]_{1}^{x^2}$$

$$f(x)=\sin^{-1}\left(\frac{x^2}{2}\right)-\sin^{-1}\left(\frac{1}{2}\right)$$

$$f(x)=\sin^{-1}\left(\frac{x^2}{2}\right)-\frac{\pi}{6}$$

$$f(\sqrt{2})=\sin^{-1}\left(\frac{(\sqrt{2})^2}{2}\right)-\frac{\pi}{6}$$

$$f(\sqrt{2})=\sin^{-1}(1)-\frac{\pi}{6}$$

$$f(\sqrt{2})=\frac{\pi}{2}-\frac{\pi}{6}$$

$$f(\sqrt{2})=\frac{\pi}{3}$$

Question 12

Euler's method with an increment of 0.2 is used to find an approximate solution to the differential equation $\frac{dy}{dx} = \frac{x}{y}$ at the point where $x = 3.6$.

If $y = 5$ when $x = 3$, the approximate solution, correct to 3 decimal places is

A. 5.245
B. **5.375**
C. 5.509
D. 5.600
E. 5.625

Answer is B.

Worked solution

Let $y = f(x)$ and $\frac{dy}{dx} = f'(x, y) = \frac{x}{y}$.

Using $f(x+h) \approx hf'(x, y) + f(x)$ with $h = 0.2$, given $y = 5$ when $x = 3$.

$f(3.2) = f(3+0.2) \approx 0.2f'(3, 5) + f(3)$ $f'(3, 5) = \frac{3}{5} = 0.6$

$f(3.2) \approx 0.2 \times 0.6 + 5 = 5.12$

$f(3.4) = f(3.2+0.2) \approx 0.2f'(3.2, 5.12) + f(3.2)$ $f'(3.2, 5.12) = \frac{3.2}{5.12} = 0.625$

$f(3.4) \approx 0.2 \times 0.625 + 5.12 = 5.245$

$f(3.6) = f(3.4+0.2) \approx 0.2f'(3.4, 5.245) + f(3.4)$ $f'(3.4, 5.245) = \frac{3.4}{5.245} = 0.648$

$f(3.6) \approx 0.2 \times 0.648 + 5.245$

$\therefore f(3.6) \approx 5.375$

Question 13

Initially, a vat contains 500 litres of water with 200 kg of sugar dissolved. A solution containing 0.1 kg of sugar per litre runs into the vat at a rate of 5 L/min. The mixture is stirred in the vat and it flows out the same rate. A differential equation to model the amount of sugar, Q kg, in the vat after t minutes is given by

A. $\boldsymbol{\frac{dQ}{dt} = 0.5 - 0.01Q}$

B. $\frac{dQ}{dt} = 0.4 - 0.01Q$

C. $\frac{dQ}{dt} = 0.5 - 0.002Q$

D. $\frac{dQ}{dt} = 0.5 - 0.4Q$

E. $\frac{dQ}{dt} = 0.5 - 2Q$

Answer is A.

Worked solution

There are Q kg of sugar in the vat at time t.

The concentration of sugar in the vat at time t is $\frac{Q}{500}$ kg/L.

As the mixed solution flows out at a rate of 5 L/min, the amount of sugar leaving the vat each minute is $\frac{Q}{500} \times 5 = \frac{Q}{100} = 0.01Q$ kg/min.

The amount of sugar flowing into the vat each minute is $0.1 \times 5 = 0.5$ kg/min.

$$\frac{dQ}{dt} = \text{Rate of inflow} - \text{Rate of outflow}$$

$$\therefore \frac{dQ}{dt} = 0.5 - 0.01Q$$

Question 14

$y = 4\sin(2.5x)$ is a solution of which one of the following differential equations?

A. $\dfrac{d^2y}{dx^2} + y = 0$

B. $\dfrac{d^2y}{dx^2} + 21y = 0$

C. $\mathbf{4\dfrac{d^2y}{dx^2} + 25y = 0}$

D. $25\dfrac{d^2y}{dx^2} + 4y = 0$

E. $-25\dfrac{d^2y}{dx^2} + 4y = 0$

Answer is C.

Tip

- *Find the first and second derivatives of* $y = 4\sin(2.5x)$*, then use* $\dfrac{d^2y}{dx^2}$ *and y to derive the differential equation.*

Worked solution

$$\frac{dy}{dx} = 10\cos(2.5x)$$

$$\frac{d^2y}{dx^2} = -25\sin(2.5x)$$

$$\sin(2.5x) = \frac{1}{4}y = -\frac{1}{25}\left(\frac{d^2y}{dx^2}\right)$$

$$\therefore\ 25y = -4\frac{d^2y}{dx^2}$$

$$4\frac{d^2y}{dx^2} + 25y = 0$$

*The following information relates to questions **15** and **16**.*

The position of a particle measured, in metres, from the origin O at time t seconds is given by $\underset{\sim}{r}(t) = \left(\frac{1}{t+1}\right)\underset{\sim}{i} + (t-1)\underset{\sim}{j}$, $t \geq 0$.

Question 15

Correct to 2 decimal places, the speed of the particle in m/s at $t = 1$ is

A. 0.25
B. 0.75
C. **1.03**
D. 1.06
E. 1.12

Answer is C.

Worked solution

Velocity $\dot{\underset{\sim}{r}}(t) = -\frac{1}{(t+1)^2}\underset{\sim}{i} + 1\underset{\sim}{j}$

At $t = 1$, $\dot{\underset{\sim}{r}}(1) = -\frac{1}{(1+1)^2}\underset{\sim}{i} + 1\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(1) = -0.25\underset{\sim}{i} + 1\underset{\sim}{j}$

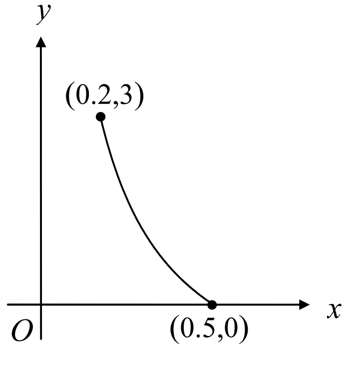
Speed $= \left|\dot{\underset{\sim}{r}}(1)\right| = \sqrt{(-0.25)^2 + (1)^2}$

$\left|\dot{\underset{\sim}{r}}(1)\right| = \sqrt{1.0625} = 1.03$

The speed of the particle at $t = 1$ is 1.03 m/s (correct to 2 decimal places).

Question 16

The path of the particle is graphed in the Cartesian plane over the interval $t \in [a, b]$.



SECTION 1 – Question 16 – continued
TURN OVER

The value of b would be

A. 0.5
B. 1
C. 1.2
D. 3
E. **4**

Answer is E.

Worked solution

Parametric equations of the curve are $x = \frac{1}{1+t}$ and $y = 1 - t$.

At the point (0.5, 0):

$0.5 = \frac{1}{1+t}$ and $0 = 1 - t$

$0.5(1+t) = 1$

$0.5t = 0.5$

$\therefore \; t = 1$

At the point (0.2, 3):

$0.2 = \frac{1}{1+t}$ and $3 = 1 - t$

$0.2(1+t) = 1$

$0.2t = 0.8$

$\therefore \; t = 4$

Hence, $[a, b] = [1, 4]$

i.e. $b = 4$

Question 17

A particle is moving in a straight line. Its velocity, v m/s, when it is x metres from the origin is given by $v = \sqrt{x^2 - 3x + 6}$. The acceleration of the particle, in m/s^2, when it is 2 metres from the origin is

A. $\frac{1}{4}$
B. $\mathbf{\frac{1}{2}}$
C. 1
D. 2
E. 4

Answer is B.

Worked solution

$$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$a = \frac{d}{dx}\left(\frac{1}{2}\left(\sqrt{x^2 - 3x + 6}\right)^2\right)$$

$$a = \frac{d}{dx}\left(\frac{1}{2}\left(x^2 - 3x + 6\right)\right)$$

$$a = \frac{1}{2}(2x - 3)$$

At $x = 2$, $a = \frac{1}{2}(2 \times 2 - 3) = \frac{1}{2}$ m/s^2

Question 18

A car travels 1 km on a straight road. It starts from rest and accelerates at 1.5 m/s^2 until it reaches a velocity of 18 m/s. The car travels at this velocity until it approaches the destination, when it decelerates at 1.2 m/s^2 until it comes to rest.

The time, in seconds, the car takes to travel the 1 km is closest to

A. 42
B. 55
C. **69**
D. 76
E. 108

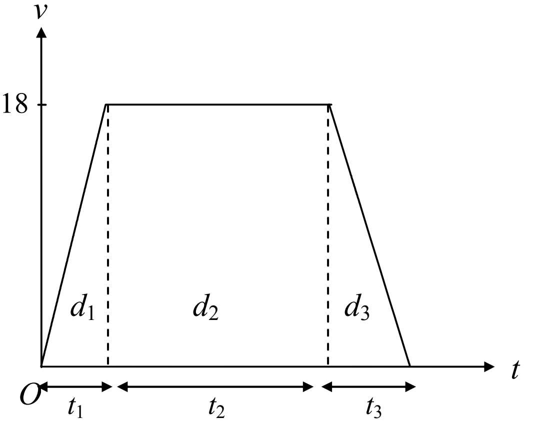
Answer is C.

Tip

- *Use the information given to draw a velocity–time graph.*

Worked solution

Let t_1 be the time spent accelerating.



$$a = \frac{v}{t}$$

$$1.5 = \frac{18}{t_1}$$

$$t_1 = \frac{18}{1.5} = 12 \text{ s}$$

Let d_1 be the distance travelled whilst accelerating.

Area under velocity–time graph:

$$d_1 = \frac{1}{2} \times 18 \times 12 = 108 \text{ m}$$

Let t_3 be the time spent decelerating.

$$a = \frac{v}{t}$$

$$1.2 = \frac{18}{t_3}$$

$$t_3 = \frac{18}{1.2} = 15 \text{ s}$$

Let d_3 be the distance travelled whilst accelerating.

Area under velocity–time graph:

$$d_3 = \frac{1}{2} \times 18 \times 15 = 135 \text{ m}$$

Let t_2 be the time travelling with a constant velocity.

$$d_2 = 1000 - 108 - 135 = 757 \text{ m}$$

$$v = \frac{d_2}{t_2}$$

$$18 = \frac{757}{t_2}$$

$$t_2 = \frac{757}{18} = 42.06 \text{ s}$$

$\therefore$ Total time taken is 12 + 42 + 15 = 69 s.

Question 19

A 70 kg cyclist travelling at a speed of 8 m/s accelerates at 0.2 m/s^2 for 30 seconds.

The cyclist's change in momentum, measured in kg m/s, is

A. 140
B. **420**
C. 560
D. 980
E. 1540

Answer is B.

Tip

- *Use the acceleration to determine the change in velocity over the 30 seconds.*

Worked solution

$\Delta v = a \times t$

$\Delta v = 0.2 \times 30 = 6$ m/s

Change in momentum = Mass × Change in velocity
= 70 × 6
= 420 kg m/s

Question 20

Forces $2\underset{\sim}{i} - 3\underset{\sim}{j}$, $5\underset{\sim}{i} + 2\underset{\sim}{j}$ and $\underset{\sim}{i} - 5\underset{\sim}{j}$ newtons act simultaneously on a particle of mass 0.5 kg, which is initially at rest. The magnitude of the acceleration of the particle, in m/s^2, is closest to

A. 5
B. 10
C. 13
D. **20**
E. 26

Answer is D.

SECTION 1 – Question 20 – continued
TURN OVER

Tip

- *First, find the magnitude of the resultant force.*

Worked solution

$$\underset{\sim}{F} = \left(2\underset{\sim}{i} - 3\underset{\sim}{j}\right) + \left(5\underset{\sim}{i} + 2\underset{\sim}{j}\right) + \left(\underset{\sim}{i} - 5\underset{\sim}{j}\right)$$

$$\underset{\sim}{F} = 8\underset{\sim}{i} - 6\underset{\sim}{j}$$

$$\left|\underset{\sim}{F}\right| = \sqrt{8^2 + (-6)^2} = 10 \text{ newtons}$$

Equation of motion:

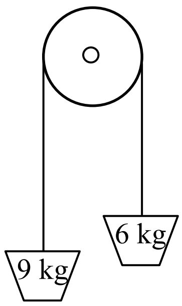
$F = ma$

$10 = 0.5a$

$a = 20 \text{ m/s}^2$

Question 21

A 9 kg mass and a 6 kg mass are connected by a light string passing over a smooth pulley, as shown in the diagram below. The connected system is moving under the force of gravity.



The tension in the string, in newtons, will be

A. $0.2g$
B. $3g$
C. $5g$
D. $6g$
E. $\mathbf{7.2g}$

Answer is E.

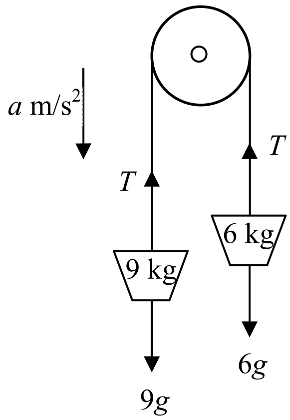
Tip

- *First, draw forces acting on the diagram before attempting the calculations.*

Worked solution

T Tension in the string

a Acceleration of the connected system



The 9 kg mass is accelerating downwards.

Resolving forces around the 9 kg mass:

$$9g - T = 9a$$

$$T = 9g - 9a \ldots(1)$$

Resolving forces around the 6 kg mass:

$$T - 6g = 6a$$

$$a = \frac{T - 6g}{6} \ldots(2)$$

Substituting (2) into (1):

$$T = 9g - 9\left(\frac{T - 6g}{6}\right)$$

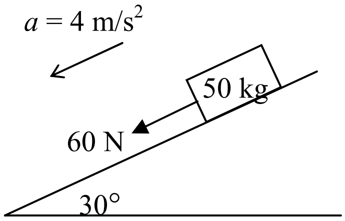
$$T = 9g - 1.5T + 9g$$

$$2.5T = 18g$$

$$T = 7.2g$$

Question 22

A force of 60 newtons is applied to a 50 kg mass sitting on a plane inclined at an angle of 30° to the horizontal level.

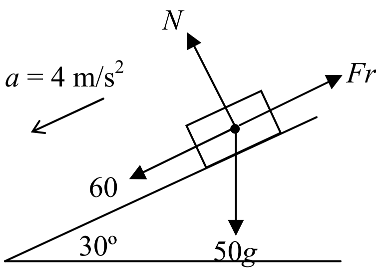


If the mass accelerates down the plane at 4 m/s^2, the coefficient of friction between the two surfaces will be closest to

A. 0.10
B. 0.15
C. 0.20
D. **0.25**
E. 0.35

Answer is D.

Worked solution



Fr	Friction
N	Normal reaction
$50g$	Weight force
μ	Coefficient of friction

Resolving forces parallel to the plane:

$$60 + 50g\sin(30°) - Fr = 50 \times 4$$

$$Fr = 60 + 25g - 200$$

$Fr = 105$ newtons … (1)

$Fr = \mu N$

$Fr = \mu \times 25\sqrt{3}g$ newtons …(2)

Equating (1) and (2):

$$\mu \times 25\sqrt{3}g = 105$$

$$\mu = \frac{105}{25\sqrt{3}g}$$

$\mu = 0.25$ (to 2 decimal places)

Resolving forces perpendicular to plane:

$$N = 50g\cos(30°)$$

$$N = 50g \times \frac{\sqrt{3}}{2} = 25\sqrt{3} \text{ newtons}$$

Tip

- *First, draw all forces acting on diagram before attempting the calculations.*

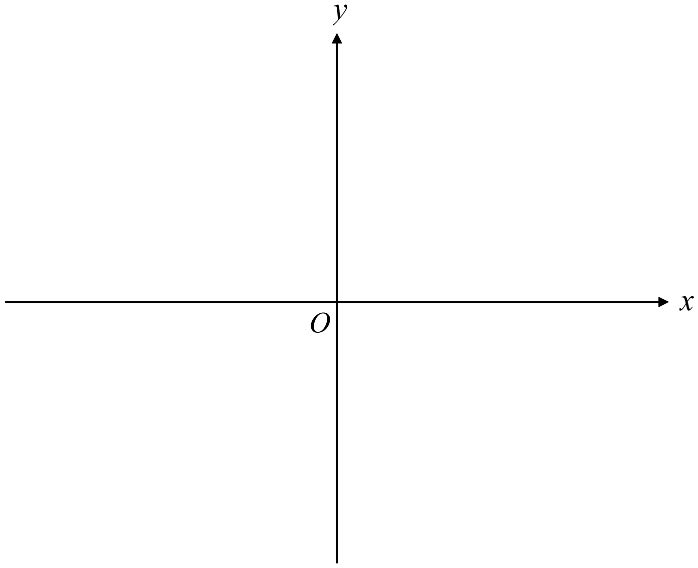
END OF SECTION 1

SECTION 2

Question 1

Given the curve with equation $\frac{x^2}{2} - y^2 = 1$.

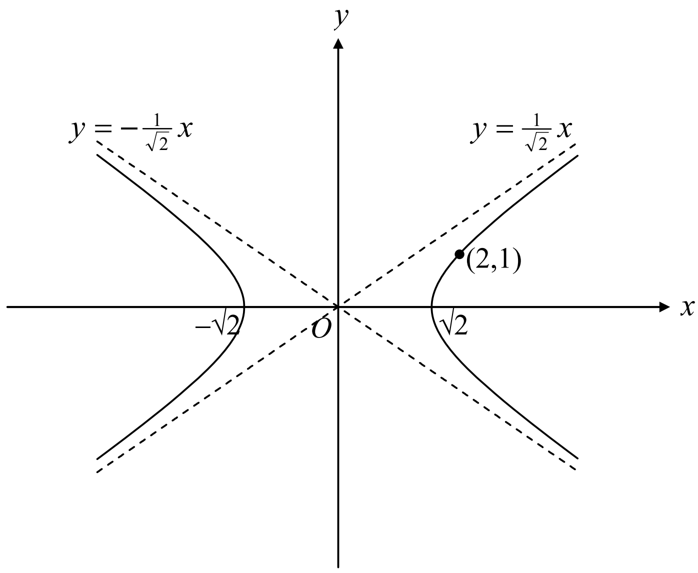
a. Sketch the graph of the curve on the axes below, showing all features clearly.



Worked solution

Asymptotes occur at $y = \pm\frac{b}{a}x$, where $a = \sqrt{2}$, $b = 1$. $\therefore\ y = \pm\frac{1}{\sqrt{2}}x$

x-intercepts occur where $y = 0$. $\frac{x^2}{2} = 1$ $\therefore\ x = -\sqrt{2}, \sqrt{2}$



3 marks

Mark allocation

- 1 mark for asymptotes.
- 1 mark for intercepts.
- 1 mark for shape.

b. **i.** Show that $\frac{dy}{dx} = \frac{x}{2y}$.

Tip

- *Use implicit differentiation.*

Worked solution

$$\frac{x^2}{2} - y^2 = 1$$

$$\frac{2x}{2} - 2y\frac{dy}{dx} = 0 \qquad \text{M1, A1}$$

$$\frac{dy}{dx} = \frac{x}{2y}$$

ii. Determine the gradient of the curve at the points where $x = 2$.

Find y when $x = 2$:

$$y = \pm\sqrt{\frac{2^2}{2} - 1} = \pm 1$$

At (2, 1) $\quad \frac{dy}{dx} = \frac{2}{2\times 1} = 1$ A1

At (2, –1) $\quad \frac{dy}{dx} = \frac{2}{2\times(-1)} = -1$ A1

2 + 2 = 4 marks

Mark allocation

- 1 mark for using a correct method to differentiate.
- 1 mark for working leading to the correct derivative.
- 1 mark for finding the gradient at (2, 1).
- 1 mark for finding the gradient at (-2, 1).

c. Determine the equation of the tangent to the curve at the points where $x = 2$.

Worked solution

At (2, 1), $m = 1$ $\qquad$ At (2, –1), $m = -1$

Substitute into $y = mx + c$ to find c:

$1 = 1\times 2 + c$ and $-1 = -1\times 2 + c$

$c = -1$ $\qquad$ $c = 1$

Equations of the tangents are $y = x - 1$ and $y = -x + 1$. A1, A1

2 marks

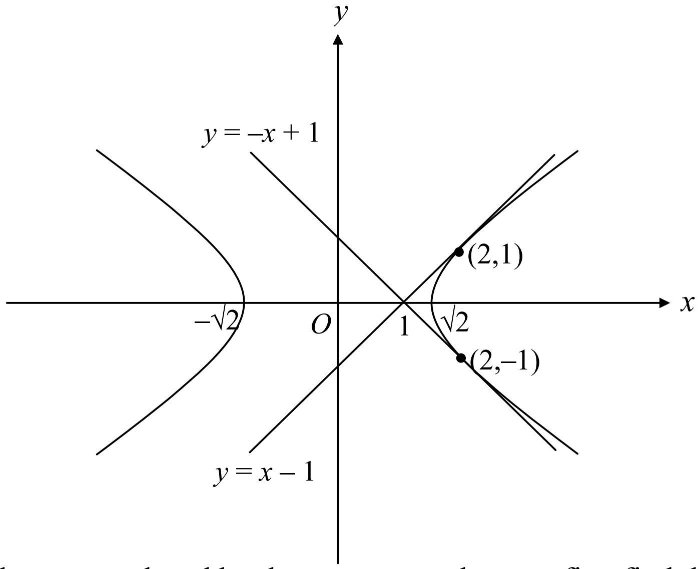
Mark allocation

- 1 mark for a correct tangent equation.
- 1 mark for the other tangent equation.

d. **i.** Write down a definite integral that will give the area enclosed by the curve $\frac{x^2}{2} - y^2 = 1$ and its tangents at $x = 2$.

Worked solution

The graph below shows the curve with the tangents drawn at $x = 2$.



To find the area enclosed by the tangents and curve, first find the area between the line $y = x - 1$, the x-axis, and the ordinates $x = 1$ and $x = 2$.

This is given by $\int_1^2 (x-1)\,dx$.

Then subtract the area between the curve $y = \sqrt{\frac{x^2}{2} - 1}$, the x-axis, and the ordinates $x = \sqrt{2}$ and $x = 2$. This is given by $\int_{\sqrt{2}}^{2} \sqrt{\left(\frac{x^2}{2} - 1\right)}\,dx$.

Doubling this result gives the required area. M1

$$\text{Area} = 2 \times \left[\int_1^2 (x-1)\,dx - \int_{\sqrt{2}}^{2} \sqrt{\left(\frac{x^2}{2} - 1\right)}\,dx\right]$$ A1

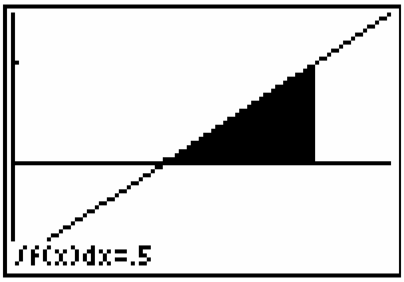
Mark allocation

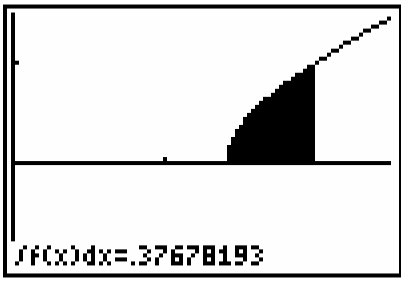
- 1 mark for correct working.
- 1 mark for correct integral.

ii. Find this area, correct to 3 decimal places.

Worked solution

Answer can be determined using your calculator by finding the area between the graph and the *x*-axis.





Area $= 2\times[0.5 - 0.37678193]$

Area = 0.246 square units A1

2 + 1 = 3 marks

Mark allocation

- 1 mark for using a correct method to find the required area.
- 1 mark for the correct integral.
- 1 mark for the answer.

Total 3 + 4 + 2 + 3 = 12 marks

Question 2

a. **i.** Show that the numbers $\sqrt{2}+\sqrt{2}\,i$ and $\sqrt{2}-\sqrt{2}\,i$ may be written in polar form as $2\,\text{cis}\left(\frac{\pi}{4}\right)$ and $2\,\text{cis}\left(-\frac{\pi}{4}\right)$.

Worked solution

The numbers are complex conjugates.

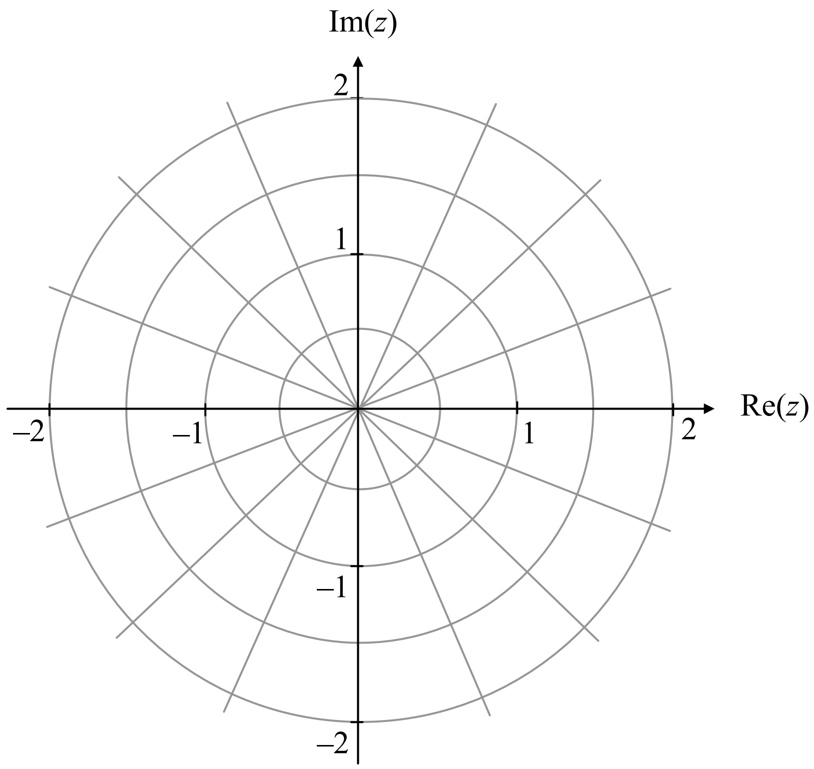
$r=\sqrt{\left(\sqrt{2}\right)^2+\left(\pm\sqrt{2}\right)^2}=\sqrt{4}=2$

$\theta=\tan^{-1}\left(\frac{\pm\sqrt{2}}{\sqrt{2}}\right)=\tan^{-1}\left(\pm 1\right)=\pm\frac{\pi}{4}$ A1

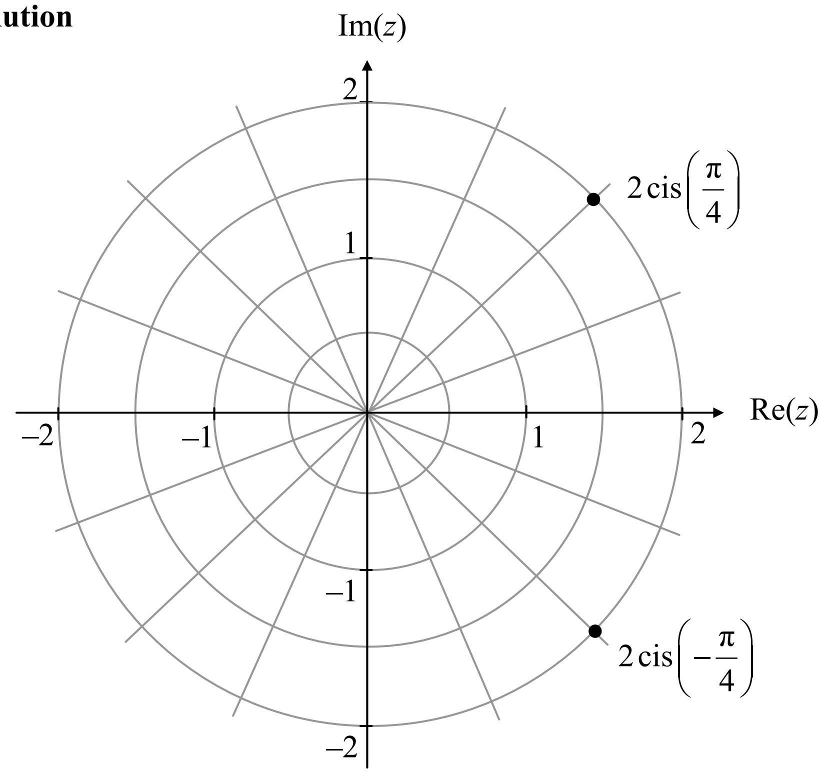
$\sqrt{2}+\sqrt{2}\,i=2\text{cis}\left(\frac{\pi}{4}\right)$

$\sqrt{2}-\sqrt{2}\,i=2\text{cis}\left(-\frac{\pi}{4}\right)$

ii. Plot and label these numbers in polar form on the Argand diagram below.



Worked solution



1 + 1 = 2 marks

Mark allocation

- 1 mark for showing calculations for r and θ.
- 1 mark for both numbers plotted correctly.

b. **i.** $\sqrt{2}+\sqrt{2}\,i$ and $\sqrt{2}-\sqrt{2}\,i$ are solutions of the equation $z^2+mz+n=0$.

Use algebra to show that $m=-2\sqrt{2}$ and $n = 4$.

Worked solution

If $\sqrt{2}+\sqrt{2}\,i$ and $\sqrt{2}-\sqrt{2}\,i$ are solutions of $z^2+mz+n=0$,

then $\left(z-\left(\sqrt{2}+\sqrt{2}\,i\right)\right)\left(z-\left(\sqrt{2}-\sqrt{2}\,i\right)\right)=0$ M1

$$\left(z-\sqrt{2}-\sqrt{2}\,i\right)\left(z-\sqrt{2}+\sqrt{2}\,i\right)=0$$

$$\left(\left(z-\sqrt{2}\right)-\sqrt{2}\,i\right)\left(\left(z-\sqrt{2}\right)+\sqrt{2}\,i\right)=0$$

$$\left(z-\sqrt{2}\right)^2-\left(\sqrt{2}\,i\right)^2=0$$

$$\left(z-\sqrt{2}\right)^2+2=0$$ A1

$$z^2-2\sqrt{2}z+2+2=0$$

$$z^2-2\sqrt{2}z+4=0$$

$$z^2+mz+n=0$$

$$\therefore\quad m=-2\sqrt{2},\quad n=4$$

ii. Hence, find all Cartesian solutions of the equation $z^4 - 2\sqrt{2}\,z^2 + 4 = 0$.

Worked solution

$$z^4 - 2\sqrt{2}\,z^2 + 4 = 0$$

$$\left(z^2\right)^2 - 2\sqrt{2}\left(z^2\right) + 4 = 0$$

$$\left(z^2 - \left(\sqrt{2} + \sqrt{2}\,i\right)\right)\left(z^2 - \left(\sqrt{2} - \sqrt{2}\,i\right)\right) = 0$$

$$z^2 - \left(\sqrt{2} + \sqrt{2}\,i\right) = 0 \text{ and } z^2 - \left(\sqrt{2} - \sqrt{2}\,i\right) = 0 \qquad \text{M1}$$

$$z^2 = \sqrt{2} + \sqrt{2}\,i \text{ and } z^2 = \sqrt{2} - \sqrt{2}\,i$$

$$z = \pm\sqrt{\sqrt{2} + \sqrt{2}\,i} \text{ and } z = \pm\sqrt{\sqrt{2} - \sqrt{2}\,i}$$

The four Cartesian solutions are:

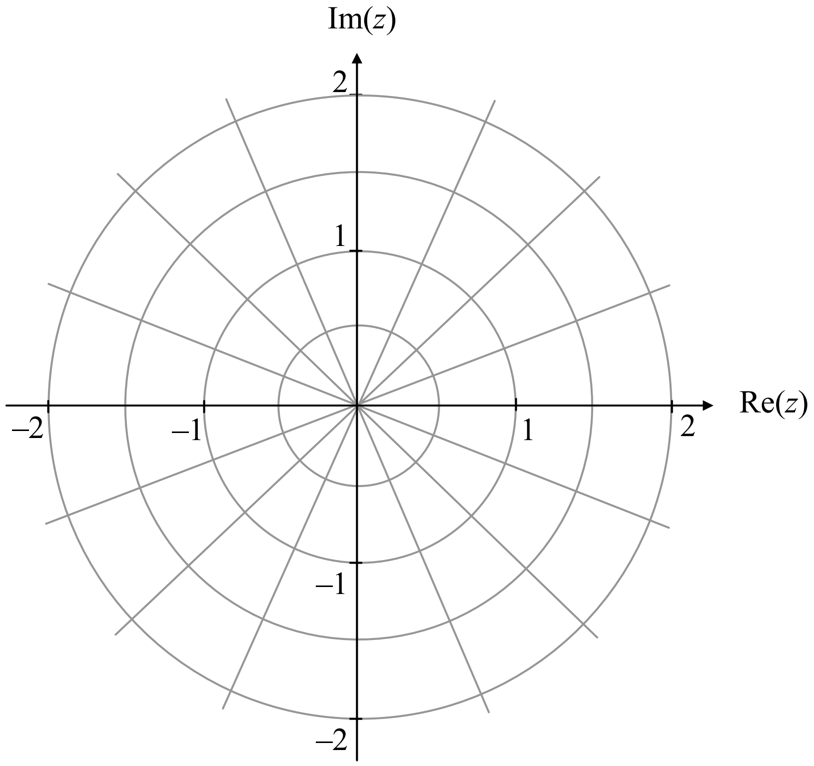
$\sqrt{\sqrt{2} + \sqrt{2}\,i}$, $-\sqrt{\sqrt{2} + \sqrt{2}\,i}$, $\sqrt{\sqrt{2} - \sqrt{2}\,i}$, $-\sqrt{\sqrt{2} - \sqrt{2}\,i}$ A1

2 + 2 = 4 marks

Mark allocation

- 1 mark for using the given solutions to find the equation.
- 1 mark for finding *m* and *n*.
- 1 mark for a correct method used to find all solutions.
- 1 mark for four correct solutions.

c. Determine the polar solutions of $z^4 - 2\sqrt{2}\,z^2 + 4 = 0$, then plot and label these solutions on the Argand diagram below.



Tip

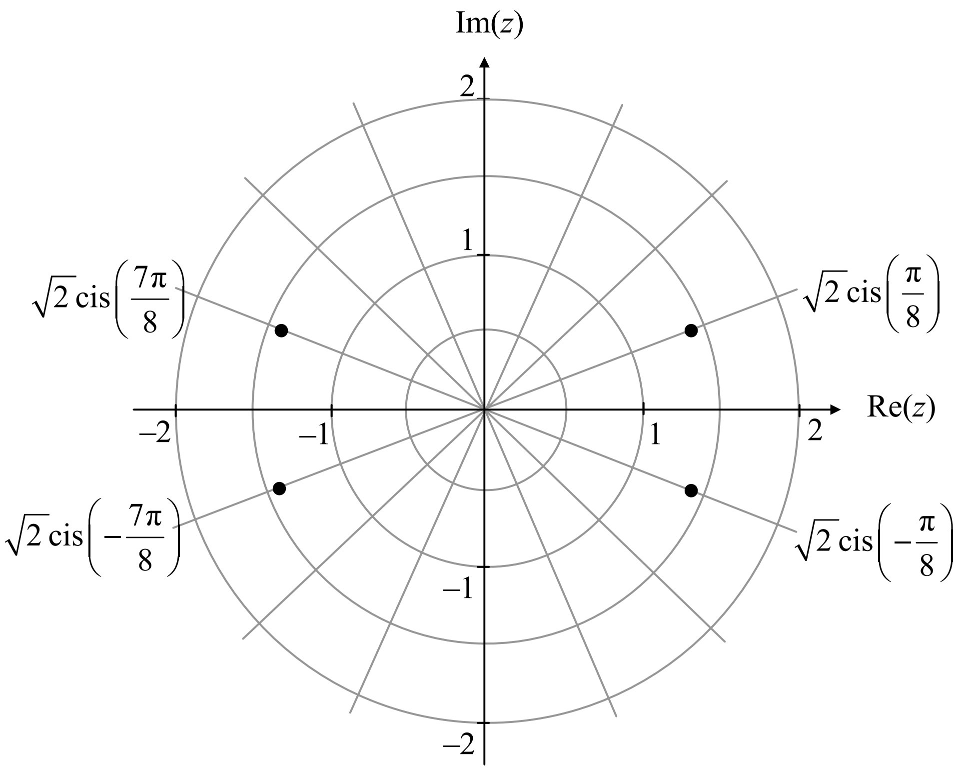
- *Use the polar form found in part **a**, then take the square root of these numbers.*

Worked solution

$z^2 = \sqrt{2} + \sqrt{2}\,i = 2\,\text{cis}\left(\frac{\pi}{4}\right)$ and $z^2 = \sqrt{2} - \sqrt{2}\,i = 2\,\text{cis}\left(-\frac{\pi}{4}\right)$

$z = \left(2\,\text{cis}\left(\frac{\pi}{4}\right)\right)^{\frac{1}{2}}$	$z = \left(2\,\text{cis}\left(-\frac{\pi}{4}\right)\right)^{\frac{1}{2}}$
$z = 2^{\frac{1}{2}}\text{cis}\left(\frac{1}{2}\left(\frac{\pi}{4} + 2k\pi\right)\right)$	$z = 2^{\frac{1}{2}}\text{cis}\left(\frac{1}{2}\left(-\frac{\pi}{4} + 2k\pi\right)\right)$
$z = \sqrt{2}\,\text{cis}\left(\frac{\pi}{8} + k\pi\right)$	$z = \sqrt{2}\,\text{cis}\left(-\frac{\pi}{8} + k\pi\right)$
$k = 0, -1$	$k = 0, 1$
$z = \sqrt{2}\,\text{cis}\left(\frac{\pi}{8}\right), \sqrt{2}\,\text{cis}\left(-\frac{7\pi}{8}\right)$	$z = \sqrt{2}\,\text{cis}\left(-\frac{\pi}{8}\right), \sqrt{2}\,\text{cis}\left(\frac{7\pi}{8}\right)$

The four polar solutions are $\sqrt{2}\,\text{cis}\left(-\frac{7\pi}{8}\right)$, $\sqrt{2}\,\text{cis}\left(-\frac{\pi}{8}\right)$, $\sqrt{2}\,\text{cis}\left(\frac{\pi}{8}\right)$ and $\sqrt{2}\,\text{cis}\left(\frac{7\pi}{8}\right)$.



4 marks

Mark allocation

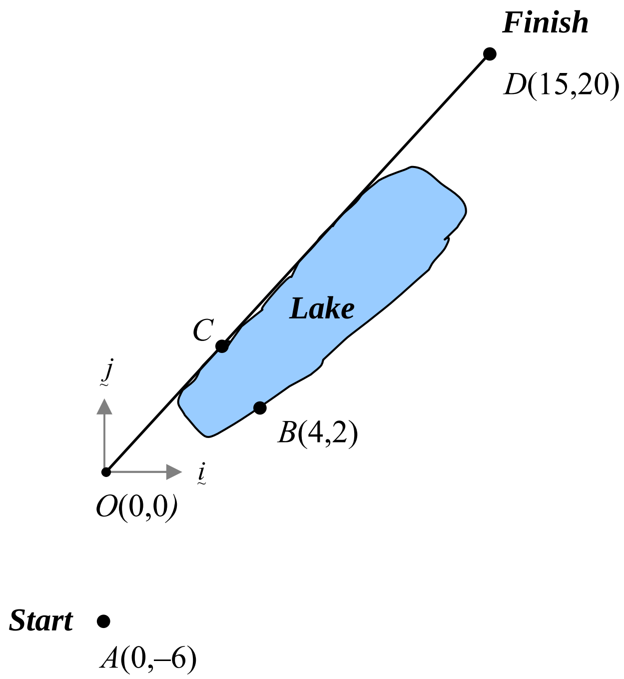
- 1 mark for each solution plotted correctly and labelled in polar form.

Total 2 + 4 + 4 = 10 marks

Question 3

The diagram below shows the course for an event involving running, swimming and cycling. The race starts at *A* and finishes at *D*. Point *O* is the origin of the coordinate system. Distances are measured in kilometres.

The first stage of the race involves running from *A*(0, –6) to *B*(4, 2). The second stage involves swimming across the lake from *B* to some point *C* on *OD*. The third and final stage of the race involves cycling along a straight road from *C* to the finishing point at *D*(15, 20).



a. Jack starts running from *A* with a velocity of $\underset{\sim}{v} = 0.1\underset{\sim}{i} + 0.01t\underset{\sim}{j}$ km/min, $t \in [0, 40]$.

Show that Jack's position at any time *t* on the run is given by the vector $\underset{\sim}{r} = 0.1t\underset{\sim}{i} + \left(0.005t^2 - 6\right)\underset{\sim}{j}$

Worked solution

Integrate the velocity vector to find the position vector.

$$\underset{\sim}{v} = \int \left(0.1\underset{\sim}{i} + 0.01t\underset{\sim}{j}\right) dt$$

$$\underset{\sim}{r} = 0.1t\underset{\sim}{i} + 0.005t^2\underset{\sim}{j} + \underset{\sim}{c}$$ M1

When $t = 0$, $\underset{\sim}{r} = 0\underset{\sim}{i} - 6\underset{\sim}{j}$

$$0\underset{\sim}{i} - 6\underset{\sim}{j} = 0.1 \times 0\underset{\sim}{i} + 0.005 \times 0^2\underset{\sim}{j} + \underset{\sim}{c}$$

$$\underset{\sim}{c} = -6\underset{\sim}{j}$$ A1

$$\underset{\sim}{r} = 0.1t\underset{\sim}{i} + 0.005t^2\underset{\sim}{j} - 6\underset{\sim}{j}$$

$$\underset{\sim}{r} = 0.1t\underset{\sim}{i} + \left(0.005t^2 - 6\right)\underset{\sim}{j}$$

2 marks

Mark allocation

- 1 mark for using integration to find position vector.
- 1 mark for finding the constant of integration correctly.

b. Show that Jack reaches point *B*, the starting position for the swimming, after 40 minutes.

Tip

- *Substitute t = 40 into the position vector.*

Worked solution

$\underset{\sim}{r} = 0.1 \times 40\underset{\sim}{i} + \left(0.005 \times 40^2 - 6\right)\underset{\sim}{j}$ A1

$\underset{\sim}{r} = 4\underset{\sim}{i} + (8 - 6)\underset{\sim}{j}$

$\underset{\sim}{r} = 4\underset{\sim}{i} + 2\underset{\sim}{j}$

After 40 minutes Jack is at point *B*, which has coordinates (4, 2).

1 mark

Mark allocation

- 1 mark for substituting $t = 40$ into the position vector.

c. Find the Cartesian equation of the path Jack took whilst running.

Worked solution

From position vector $\underset{\sim}{r} = 0.1t\underset{\sim}{i} + \left(0.005t^2 - 6\right)\underset{\sim}{j}$, resolve into *x* and *y* components.

$x = 0.1t \ \ldots (1)$ and $y = 0.005t^2 - 6 \ \ldots (2)$

Eliminate variable *t*:

From (1): $t^2 = \left(\dfrac{x}{0.1}\right)^2 = 100x^2$, substitute into (2), giving: M1

$y = 0.005 \times 100x^2 - 6$

$y = 0.5x^2 - 6$ A1

2 marks

Mark allocation

- 1 mark for attempting to eliminate the parameter *t*.
- 1 mark for finding the Cartesian equation.

d. At B, Jack proceeds to swim across the lake to some point C situated on $\vec{OD}$.

i. Write a vector expression for $\vec{OC}$ given $\vec{OC} = c\,\vec{OD}$, where $c > 0$.

Worked solution

$\vec{OC} = c\left(15\underset{\sim}{i} + 20\underset{\sim}{j}\right) = 15c\,\underset{\sim}{i} + 20c\,\underset{\sim}{j}$ A1

ii. Hence, find a vector expression for $\vec{BC}$.

Worked solution

$\vec{BC} = \vec{BO} + \vec{OC}$

$\vec{BC} = \left(-4\underset{\sim}{i} - 2\underset{\sim}{j}\right) + \left(15c\,\underset{\sim}{i} + 20c\,\underset{\sim}{j}\right)$

$\vec{BC} = (15c - 4)\underset{\sim}{i} + (20c - 2)\underset{\sim}{j}$ A1

iii. Given that Jack wishes to swim the shortest distance across the lake, use vectors to find the coordinates of point C.

Tip

- *The shortest distance will be when vectors $\vec{BC}$ and $\vec{OD}$ meet at right angles. Therefore, must find the dot product of these vectors.*

Worked solution

$\vec{BC}\,.\,\vec{OD} = 0$

$\left((15c - 4)\underset{\sim}{i} + (20c - 2)\underset{\sim}{j}\right).\left(15\underset{\sim}{i} + 20\underset{\sim}{j}\right) = 0$ A1

$225c - 60 + 400c - 40 = 0$

$625c = 100$

$c = \frac{100}{625} = 0.16$

$\vec{OC} = 15 \times 0.16\,\underset{\sim}{i} + 20 \times 0.16\,\underset{\sim}{j}$

$\vec{OC} = 2.4\,\underset{\sim}{i} + 3.2\,\underset{\sim}{j}$

Point C is (2.4, 3.2). A1

1 + 1 + 2 = 4 marks

Mark allocation

- 1 mark for writing $\vec{OC}$ correctly.
- 1 mark for using $\vec{OC}$ to find $\vec{BC}$ in terms of c.
- 1 mark for finding dot product.
- 1 mark for finding coordinates of point C.

e. Determine how far Jack swam.

Worked solution

$\overrightarrow{BC} = (15\times 0.16-4)\underset{\sim}{i} + (20\times 0.16-2)\underset{\sim}{j}$

$\overrightarrow{BC} = -1.6\underset{\sim}{i} + 1.2\underset{\sim}{j}$

Jack swam $\left|\overrightarrow{BC}\right| = \sqrt{(-1.6)^2+(1.2)^2} = \sqrt{4} = 2\text{ km}$ A1

1 mark

Mark allocation

- 1 mark for the answer.

f. Jack then cycles from *C* to *D*. If he rode at an average speed of 28 km/h and swam at an average speed of 2.5 km/h, how many minutes did he take to complete the race?

Worked solution

Distance from *C* to *D*:

$\left|\overrightarrow{CD}\right| = \sqrt{(15-2.4)^2+(20-3.2)^2} = \sqrt{441} = 21\ \text{km}$ A1

Cycling time $= \frac{21}{28}\times 60 = 45\ \text{min}$

Swimming time $= \frac{2}{2.5}\times 60 = 48\ \text{min}$

Running time = 40 min (Given in part **c**.)

Time taken to complete race = 45 + 48 + 40 + 133 min. A1

2 marks

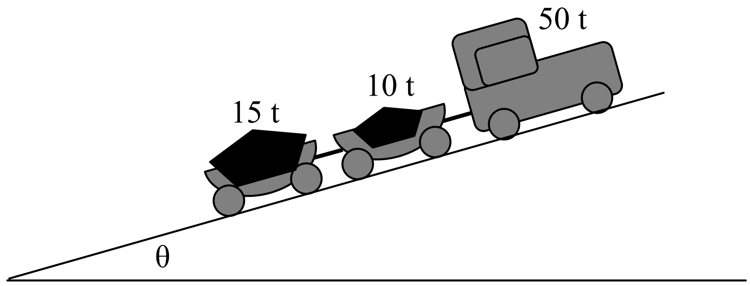
Mark allocation

- 1 mark for finding the distance *CD*.
- 1 mark for the answer.

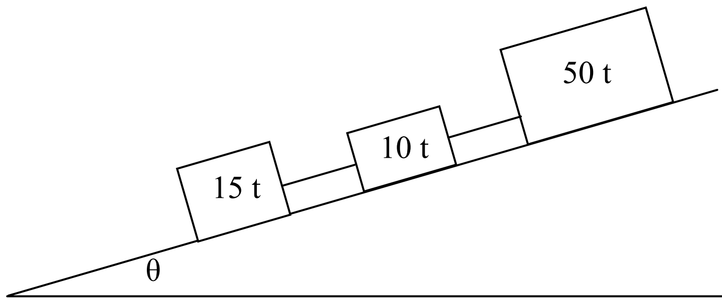
Total 2 + 1 + 2 + 4 + 1 + 2 = 12 marks

Question 4

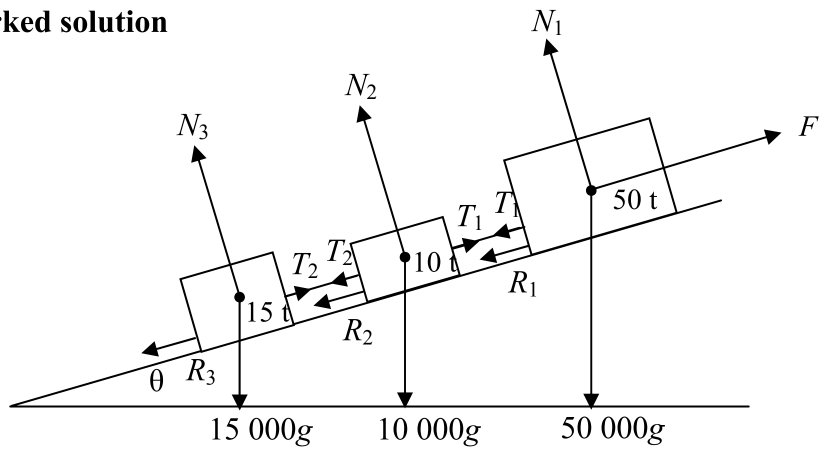
A 50 tonne engine is pulling two carriages containing loads of 10 tonnes and 15 tonnes, respectively, along a straight track at a constant speed. The track is inclined at an angle of θ to the horizontal level, where $\sin(\theta) = \frac{1}{10}$. Resistance forces of 98 newtons per tonne act on the engine and the carriages.



a. In the diagram below, show all forces acting.



Worked solution



F	Force exerted by engine.
T_1	Tension in the coupling joining the 50 t engine and the 10 t carriage.
T_2	Tension in the coupling joining the 10 t carriage and the 15 t carriage.
R_1	Resistance acting against the motion of the 50 t engine: 50 × 98 = 4900 newtons.
R_2	Resistance acting against the motion of the 10 t carriage: 10 × 98 = 980 newtons.
R_3	Resistance acting against the motion of 15 t carriage: 15 × 98 = 1470 newtons.
N_1, N_2, N_3	Normal reaction of the track on each mass.

1 mark

Mark allocation

- 1 mark for all forces shown correctly.

SECTION 2 – Question 4 – continued
TURN OVER

Tip

- *Change tonne to kilograms.*

b. Show that the engine exerts a tractive force of 80 850 newtons parallel to the track.

Worked solution

Resolving forces acting on the 50 tonne engine parallel to the track:

$F - 50\,000g\sin(\theta) - R_1 - T_1 = 50\,000a$ M1

$F - 50\,000g \times \frac{1}{10} - 4900 - T_1 = 0$ $a = 0$ (i.e. engine is moving at constant speed)

$F = 53\,900 + T_1$ A1

Resolving forces acting on the 10 tonne carriage parallel to the track:

$T_1 - 10\,000g\sin(\theta) - R_2 - T_2 = 0$

$T_1 = 10\,000g \times \frac{1}{10} + 980 + T_2$

$T_1 = 10\,780 + T_2$

Resolving forces acting on the 15 tonne carriage parallel to the track:

$T_2 - 15\,000g\sin(\theta) - R_3 = 0$

$T_2 = 15\,000g \times \frac{1}{10} + 1470$

$T_2 = 16\,170$ newtons A1

$\Rightarrow \quad T_1 = 10\,780 + 16\,170 = 26\,950$ newtons A1

$\Rightarrow \quad F = 53\,900 + 26950 = 80\,850$ newtons

$F = 80\,850$ newtons

4 marks

Mark allocation

- 1 mark for resolving forces parallel to the truck.
- 1 mark for expressing F in terms of T_1.
- 1 mark for finding T_2.
- 1 mark for work leading to the answer.

c. Some time later, the coupling between the 10 tonne carriage and the 15 tonne carriage fails. The 15 tonne carriage becomes disconnected from the system. The engine continues to exert the same tractive force, which results in the engine and 10 tonne carriage accelerating along the track.

Show that the engine and 10 tonne carriage start to accelerate at 0.2695 m/s^2.

Worked solution

$F = 80\,850$ newtons

Resolving forces acting on the 50 tonne engine parallel to the track:

$$F - 50000g\sin(\theta) - R_1 - T_1 = 50000a$$

$$80\,850 - 49\,000 - 4900 - T_1 = 50\,000a$$

$$26\,950 - T_1 = 50\,000a \text{ .... (1)} \qquad \text{A1}$$

Resolving forces acting on the 10 tonne carriage parallel to the track:

$$T_1 - 10\,000g\sin(\theta) - R_2 = 10\,000a$$

$$T_1 - 9800 - 980 = 10\,000a$$

$$T_1 - 10\,780 = 10\,000a$$

$$T_1 = 10\,000a + 10\,780 \text{ .... (2)} \qquad \text{A1}$$

Substituting (2) into (1):

$$26\,950 - (10\,000a + 10\,780) = 50\,000a$$

$$16\,170 = 60\,000a$$

$$a = 0.2695 \text{ m/s}^2 \qquad \text{A1}$$

3 marks

Mark allocation

- 1 mark for one correct equation involving T_1 and *a*.
- 1 mark for another correct equation involving T_1 and *a*.
- 1 mark for the answer.

d. The engine is moving at a speed of 20 m/s when the 15 tonne carriage disconnects. The driver realises this has occurred after 10 seconds.

How far does the engine and 10 tonne carriage travel in this time?

Write your answer, in metres, correct to 1 decimal place.

Worked solution

System is moving under constant acceleration.

$u = 20,\ a = 0.2695,\ t = 10$

$s = ut + \frac{1}{2}at^2$

$s = 20 \times 10 + \frac{1}{2} \times 0.2695 \times 100$

$s = 213.5\text{ m}$ A1

1 mark

Mark allocation

- 1 mark for the answer.

e. When the engine driver realises the 15 tonne carriage has disconnected, he applies the brakes and brings the engine and 10 tonne carriage to rest in 100 m.

Find the deceleration of the engine in m/s^2, correct to 3 decimal places.

Tip

- *First, find the velocity of the engine after 10 seconds.*

Worked solution

Find the velocity of the engine after 10 seconds:

$v = u + at$

$v = 20 + 0.2695 \times 10$

$v = 22.695\ \text{m/s}$ A1

$u = 22.695,\ v = 0,\ s = 100$

$v^2 = u^2 + 2as$

$0 = 22.695^2 + 2 \times a \times 100$

$a = -\frac{22.695^2}{200}$

$a = -2.575\ \text{m/s}^2$

Engine decelerates at 2.575 m/s^2. A1

2 marks

Mark allocation

- 1 mark for finding the velocity of the engine after 10 seconds.
- 1 mark for the answer.

Total 1 + 4 + 3 + 1 + 2 = 11 marks

Question 5

A 100 kg mass falls from rest from a stationary pontoon floating on the surface of a lake. As it travels vertically downwards through the water, the mass is subject to a force of 980 newtons due to gravity and a retarding force of $10v^2$ newtons due to the resistance of the water, where v is the velocity of the mass measured in m/s.

a. **i.** Write down the equation of motion of the mass as it travels through the water.

Worked solution

Equation of motion:

$$F = ma$$

$$980 - 10v^2 = 100a \qquad \text{A1}$$

ii. Hence, show that the rate of change in the velocity of the mass with respect to its position, x metres from the pontoon, is modelled by the differential equation

$$\frac{dv}{dx} = \frac{98 - v^2}{10v}.$$

Worked solution

From equation of motion in part **i**:

$$a = \frac{980 - 10v^2}{100}$$

$$v\frac{dv}{dx} = \frac{98 - v^2}{10}$$

$$\frac{dv}{dx} = \frac{98 - v^2}{10v} \qquad \text{A1}$$

1 + 1 = 2 marks

Mark allocation

- 1 mark for the equation of motion.
- 1 mark correct work leading to the differential equation.

b. Use calculus to show that the distance travelled by the mass is given by

$$x = 5\log_e\left(\frac{98}{98 - v^2}\right).$$

Worked solution

$$\frac{dx}{dv} = \frac{10v}{98 - v^2} \qquad \text{M1}$$

$$x = \int\left(\frac{10v}{98 - v^2}\right)dv$$

$$x = -5\int\left(\frac{-2v}{98 - v^2}\right)dv$$

$$x = -5\log_e\left(98 - v^2\right) + c$$

When $x = 0$, $v = 0$:

$0 = -5\log_e(98) + c$ M1

$c = 5\log_e(98)$

$x = -5\log_e(98 - v^2) + 5\log_e(98)$

$x = 5\left(\log_e(98) - \log_e(98 - v^2)\right)$ A1

$x = 5\log_e\left(\dfrac{98}{98 - v^2}\right)$

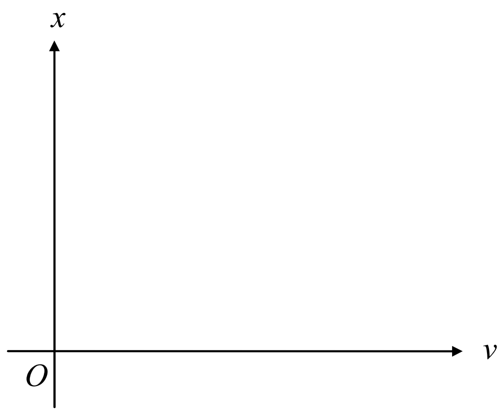
As the mass travels vertically downwards, its position, x metres, represents the distance travelled.

3 marks

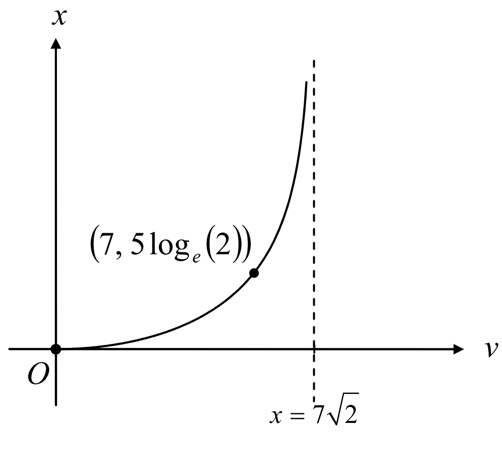
Mark allocation

- 1 mark for using $\dfrac{dx}{dv}$.
- 1 mark for integrating and substituting $x = 0$ and $v = 0$ into the equation.
- 1 mark for correct working leading to the given equation.

c. Sketch a graph of $x = 5\log_e\left(\dfrac{98}{98 - v^2}\right)$, $v \geq 0$, on the axes below, showing all relevant features.



Worked solution



The coordinates of the point on the graph need to be shown.

When $v = 7$, $x = 5\log_e\left(\frac{98}{98-7^2}\right) = 5\log_e(2)$.

2 marks

Mark allocation

- 1 mark for equation of asymptote.
- 1 mark for shape and position.

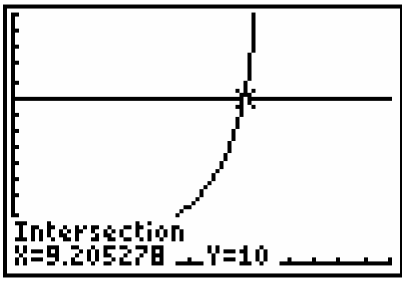
d. **i.** Determine the velocity of the mass after it has travelled 10 metres.

Give your answer, correct to 1 decimal place.

Worked solution

Solve $10 = 5\log_e\left(\frac{98}{98-v^2}\right)$.

Answer obtained from calculator:



When mass has travelled 10 m, its velocity will be 9.2 m/s. A1

ii. Give the exact value of the limiting velocity of the mass.

Tip

- *The limiting velocity of the mass is found from the vertical asymptote of the graph, which occurs where* $98 - v^2 = 0, v \geq 0$.

Worked solution

$v = \sqrt{98} = 7\sqrt{2}$ m/s A1

1 + 1 = 2 marks

Mark allocation

- 1 mark for correct answer for part i.
- 1 mark for correct answer for part ii.

e. Determine the how long it will take for the mass to reach the bottom of the lake, 40 metres below the surface of the water.

Give your answer in seconds, correct to 2 decimal places.

Tip

- *First find an expression relating velocity and time.*

Worked solution

$$a = \frac{dv}{dt} = \frac{980 - 10v^2}{100}$$

$$\frac{dt}{dv} = \frac{100}{980 - 10v^2}$$

$$t = \int \left(\frac{10}{98 - v^2} \right) dv \qquad \text{M1}$$

$$t = 10\int \left(\frac{1}{98 - v^2} \right) dv$$

Resolving into partial fractions:

$$\frac{1}{98 - v^2} = \frac{A}{\sqrt{98} + v} + \frac{B}{\sqrt{98} - v}$$

$$1 = A\left(\sqrt{98} - v\right) + B\left(\sqrt{98} + v\right)$$

When $v = -\sqrt{98}$, $A = \frac{1}{2\sqrt{98}}$ and when $v = \sqrt{98}$, $B = \frac{1}{2\sqrt{98}}$.

$$t = 10\int \left(\frac{\frac{1}{2\sqrt{98}}}{\sqrt{98} + v} + \frac{\frac{1}{2\sqrt{98}}}{\sqrt{98} - v} \right) dx$$

$$t = \frac{5}{\sqrt{98}} \int \left(\frac{1}{\sqrt{98} + v} - \frac{-1}{\sqrt{98} - v} \right) dx$$

$$t = \frac{5}{\sqrt{98}} \left(\log_e\left(\sqrt{98} + v\right) - \log_e\left(\sqrt{98} - v\right) \right) + c$$

$$t = \frac{5}{\sqrt{98}} \log_e \left(\frac{\sqrt{98} + v}{\sqrt{98} - v} \right) + c$$

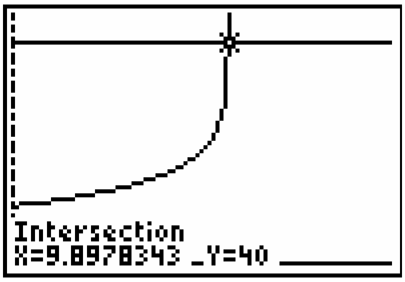
When $t = 0$, $v = 0$:

$$0 = \frac{5}{\sqrt{98}} \log_e \left(\frac{\sqrt{98} + 0}{\sqrt{98} - 0} \right) + c$$

$$c = \frac{5}{\sqrt{98}} \log_e(1) = 0$$

$$t = \frac{5}{\sqrt{98}} \log_e \left(\frac{\sqrt{98} + v}{\sqrt{98} - v} \right) \quad \ldots (1) \qquad \text{A1}$$

Solve $40 = 5\log_e\left(\frac{98}{98-v^2}\right)$.



The mass has travelled 40 metres when v = 9.8978343 m/s. A1

Substituting v = 9.8978343 m/s into (1) to find t:

$$t = \frac{5}{\sqrt{98}}\log_e\left(\frac{\sqrt{98}+9.8978343}{\sqrt{98}-9.8978343}\right)$$

t = 4.74 s

The mass takes 4.74 seconds to reach the bottom of the lake. A1

4 marks

Mark allocation

- 1 mark for attempting to express t as an integral in terms of v.
- 1 mark for finding t in terms of v.
- 1 mark for finding the velocity of the mass after it has travelled 40 m.
- 1 mark for the answer.

Total 2 + 3 + 2 + 2 + 4 = 13 marks

Tips

- *Use equation* $x = 5\log_e\left(\frac{98}{98-v^2}\right)$ *to find the velocity when the displacement is 40 m.*
- *Store v = 9.8978343 m/s into the calculator memory.*

END OF SOLUTIONS BOOK