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# SPECIALIST MATHS TRIAL EXAMINATION 2 SOLUTIONS 2008

## Section 1 – Multiple-choice answers

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1. | C | 7. | E | 13. | D | 19. | D |
| 2. | A | 8. | C | 14. | A | 20. | A |
| 3. | B | 9. | C | 15. | C | 21. | E |
| 4. | D | 10. | B | 16. | D | 22. | A |
| 5. | B | 11. | A | 17. | E | | |
| 6. | B | 12. | E | 18. | D | | |

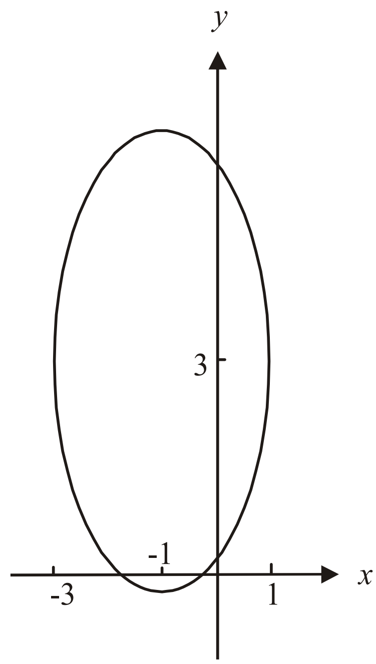
## Section 1- Multiple-choice solutions

**Question 1**

$$\frac{(x+1)^2}{2}+\frac{(y-3)^2}{5}=2$$

So $$\frac{(x+1)^2}{4}+\frac{(y-3)^2}{10}=1$$

The ellipse has its centre at $(-1,3)$ and a semi-major axis of length $\sqrt{10}$ units running parallel to the $y$-axis.
The minimum value of $y$ for the ellipse is therefore $3-\sqrt{10}$.
The answer is C.



**Question 2**

$$x = 3\sec(t) \qquad\qquad y = 1 + \tan(t)$$

$$\frac{x}{3} = \sec(t) \qquad\qquad y - 1 = \tan(t)$$

$$\frac{x^2}{9} = \sec^2(t) \qquad\qquad (y-1)^2 = \tan^2(t)$$

Since $\tan^2(t) + 1 = \sec^2(t)$ (from the formula sheet)

$$(y-1)^2 + 1 = \frac{x^2}{9}$$

$$\frac{x^2}{9} - (y-1)^2 = 1$$

Centre is located at $(0,1)$.
The answer is A.

**Question 3**

For the function $y = \arccos(x)$, the domain is $-1 \le x \le 1$.
For the function $y = \arccos(bx + c)$ the domain is

$$-1 \le bx + c \le 1$$

$$-1 - c \le bx \le 1 - c$$

$$\frac{-1-c}{b} \le x \le \frac{1-c}{b}$$

The implied domain of the function $y = a - \arccos(bx + c)$ is $\left[\frac{-1-c}{b}, \frac{1-c}{b}\right]$.

The answer is B.

**Question 4**

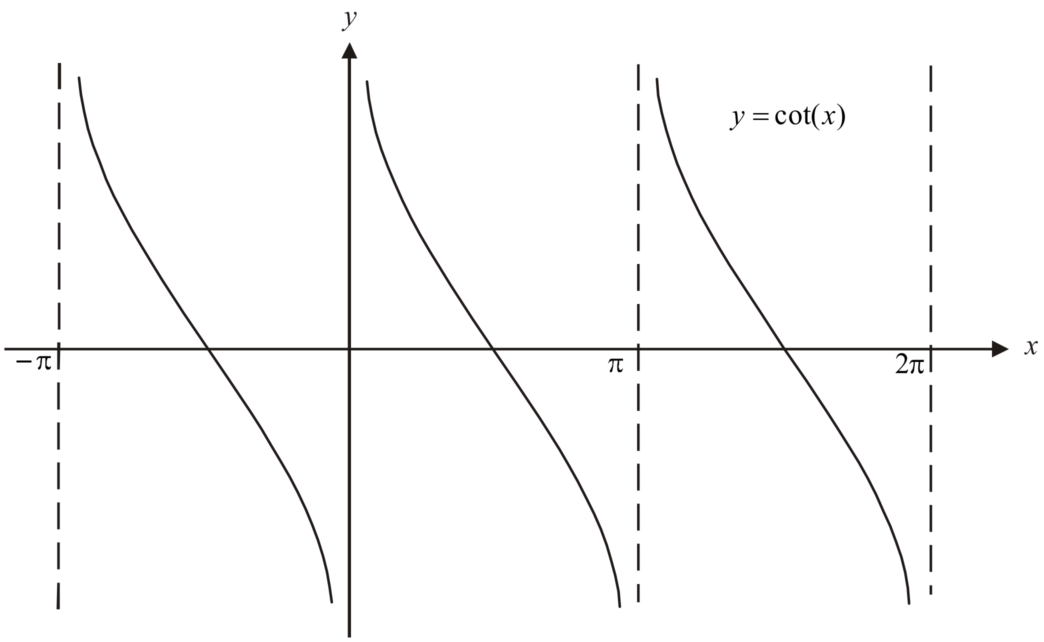
$$\text{period} = \frac{3a}{4} - \frac{a}{4}$$
$$= \frac{a}{2}$$

Now, for the graph of $y = -\tan(nx)$, period $= \dfrac{\pi}{n}$.

So $\quad \dfrac{a}{2} = \dfrac{\pi}{n}$

$$n = \frac{2\pi}{a}$$

So options **A.** and **B.** are incorrect.
The graph of $y = \cot(x)$ is shown.




The period of the graph of $y = \cot\left(\dfrac{2\pi x}{a}\right)$ will be $\dfrac{a}{2}$ but the graph of $y = \cot\left(\dfrac{2\pi x}{a}\right)$ must be translated $\dfrac{a}{4}$ units to the right or to the left to look like the graph shown in the question. The rule could therefore be $y = \cot\left(\dfrac{2\pi}{a}\left(x - \dfrac{a}{4}\right)\right)$.

The answer is D.

**Question 5**

$$\frac{u}{\bar{v}} = \frac{2i}{1-i}$$
$$= \frac{2i}{1-i} \times \frac{1+i}{1+i}$$
$$= \frac{2i-2}{1+1}$$
$$= -1 + i$$

The answer is B.

**Question 6**

Multiplying $z$ by $i$ rotates the complex number $z$ anticlockwise about the origin by $\frac{\pi^c}{2}$.

Multiplying $z$ by $i^3$ rotates the complex number $z$ anticlockwise about the origin by $\frac{3\pi^c}{2}$.

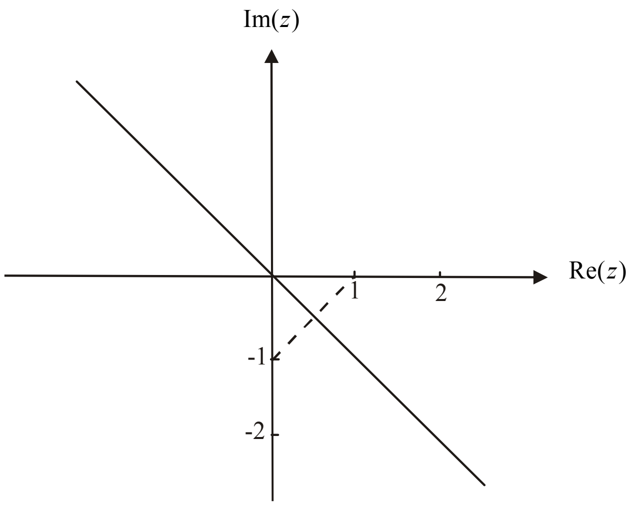
Since $-1 = i^2$, multiplying $z$ by $-i^3$ rotates the complex number $z$ anticlockwise about the origin by $\frac{5\pi^c}{2}$ or $\frac{\pi^c}{2}$.

The answer is B.

**Question 7**

Method 1

For $|z-1| = |z+i|$, we are looking for the set of points in the complex plane such that the distance from any one of them to the complex number $1+0i$ is the same as the distance to the complex number $-i$.




The answer is E.

Method 2

$|z-1| = |z+i|$

Let $z = x + yi, \quad x, y \in R$.

$$|x+yi-1| = |x+yi+i|$$
$$\sqrt{(x-1)^2+y^2} = \sqrt{x^2+(y+1)^2}$$
$$x^2-2x+1+y^2 = x^2+y^2+2y+1$$
$$-x = y$$

The answer is E.

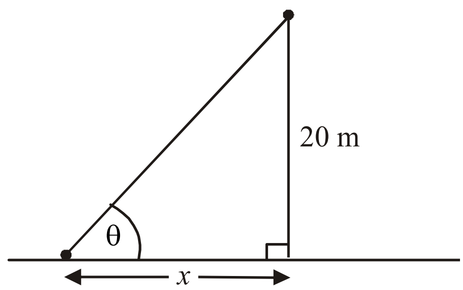
**Question 8**
We have a truncated ("chopped off") circle.
The full circle could be described by $\{z : |z-2| \le 2\}$.
The straight line could be described by $\{z : \text{Im}(z)=1\}$.
The region could be described by $\{z : |z-2| \le 2 \cap \text{Im}(z)<1\}$
The answer is C.

**Question 9**

If $F(x)$ is the antiderivative of $f(x)$ then the derivative of $F(x)$ is $f(x)$; that is, the graph of $y=f(x)$ shows the gradient function of $y=F(x)$. Therefore the graph of $y=F(x)$ has a stationary point at $x=-2,\ x=0,\ x=4$ and $x=8$. Options A and B are incorrect.

A stationary point of inflection can only occur where there is a stationary point and the gradient is positive on either side of the stationary point or negative on either side of the stationary point. This only occurs at $x=4$.
Note that at $x=-2$ and $x=8$ there would be a local maximum and at $x=0$ there would be a local minimum.
The answer is C.

**Question 10**
We need to find the speed at which the ranger is moving, that is, $\frac{dx}{dt}$.




$$\frac{dx}{dt}=\frac{dx}{d\theta}\cdot\frac{d\theta}{dt} \quad -(1)$$

Now, $\tan(\theta)=\frac{20}{x}$

$$x=20(\tan(\theta))^{-1}$$

$$\frac{dx}{d\theta}=-20(\tan(\theta))^{-2}\times\sec^2(\theta)$$

$$=-20\times\frac{\cos^2(\theta)}{\sin^2(\theta)}\times\frac{1}{\cos^2(\theta)}$$

$$=\frac{-20}{\sin^2(\theta)}$$

From (1), $\frac{dx}{dt}=-0.0625\times\frac{-20}{\sin^2(\theta)}$

$$=1.25\times\frac{1}{\sin^2(\theta)}$$

When $\theta=\frac{\pi}{4}$,

$$\frac{dx}{dt}=1.25\times\frac{1}{\left(\frac{1}{\sqrt{2}}\right)^2}$$

$$=1.25\times 2$$

$$=2.5\text{ms}^{-1}$$

The answer is B.

**Question 11**

$$\sin^2(x)\frac{dy}{dx}=\cos(x)$$

$$\frac{dy}{dx}=\frac{\cos(x)}{\sin^2(x)}$$

$$y=\int\frac{\cos(x)}{\sin^2(x)}dx$$

$$=\int u^{-2}\frac{du}{dx}dx \qquad \text{let } u=\sin(x)$$

$$=\int u^{-2}du \qquad \frac{du}{dx}=\cos(x)$$

$$y=\frac{u^{-1}}{-1}+c$$

$$=\frac{-1}{\sin(x)}+c$$

When $y=0, x=\frac{3\pi}{2}$

$$0=\frac{-1}{-1}+c$$

$$c=-1$$

$$y=\frac{-1}{\sin(x)}-1$$

The answer is A.

**Question 12**

$$\int_0^2\frac{\sqrt{\operatorname{arsin}\left(\frac{x}{2}\right)}}{\sqrt{4-x^2}}dx$$

$$=\int_0^{\frac{\pi}{2}}\frac{du}{dx}\cdot u^{\frac{1}{2}}dx$$

$$=\int_0^{\frac{\pi}{2}}u^{\frac{1}{2}}du$$

The answer is E.

Let $u=\operatorname{arsin}\left(\frac{x}{2}\right)$

$$\frac{du}{dx}=\frac{1}{\sqrt{4-x^2}}$$

$x=2$

$u=\operatorname{arsin}(1)$

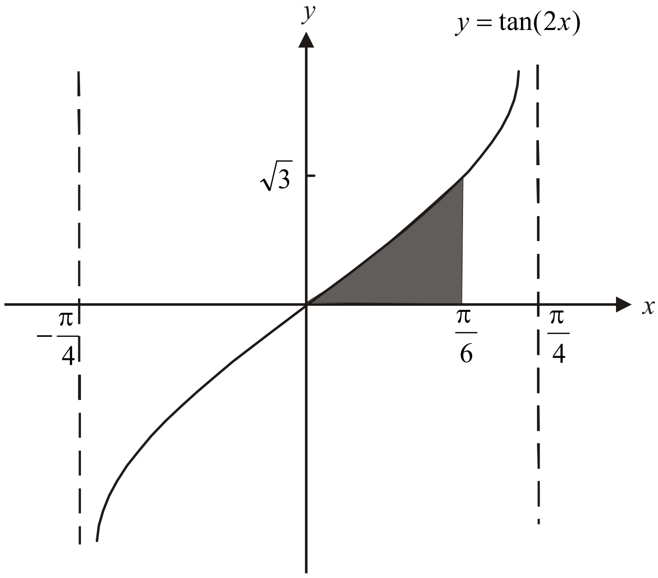
$=\frac{\pi}{2}$

$x=0$

$u=\operatorname{arsin}(0)$

$=0$

**Question 13**




$$\begin{aligned} \text{volume} &= \pi \int_0^{\frac{\pi}{6}} y^2\,dx \\ &= \pi \int_0^{\frac{\pi}{6}} \tan^2(2x)\,dx \\ &= \pi \int_0^{\frac{\pi}{6}} \left(\sec^2(2x) - 1\right)dx \qquad \text{since } \tan^2(x) + 1 = \sec^2(x) \text{ from the formula sheet} \end{aligned}$$

The answer is D.

**Question 14**

Using Eulers method,

$\frac{dy}{dx} = x^2$, $x_0 = 0$ and $y_0 = 1$ and $h = 0 \cdot 1$

So, $x_1 = 0 + 0 \cdot 1 = 0.1$ and $y_1 = 1 + 0 \cdot 1 \times 0$
$= 1$

Now, $y = \int x^2\,dx$

$y = \frac{x^3}{3} + c$

When $x = 0, y = 1$

$1 = c$

$y = \frac{x^3}{3} + 1$

When $x = 0 \cdot 1$, $y = \frac{0 \cdot 1^3}{3} + 1$

$y = 1.00033...$

The difference is $1.00033... - 1 = 0.00033...$

The answer is A.

**Question 15**

The rate of reproduction $= kN$ where $k$ is a constant.
The rate of death $= C$ where $C$ is also a constant.

The required differential equation is $\frac{dN}{dt} = kN - C$.

The answer is C.

**Question 16**

$$\underset{\sim}{u} = 2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k},\ \underset{\sim}{v} = \underset{\sim}{i} - 2\underset{\sim}{k}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = |\underset{\sim}{u}||\underset{\sim}{v}|\cos(\theta)$$

$$2 + 0 + 2 = \sqrt{4+1+1} \times \sqrt{1+4}\cos(\theta)$$

$$4 = \sqrt{30}\cos(\theta)$$

$$\cos(\theta) = \frac{4}{\sqrt{30}}$$

$$\theta = 43^\circ 5'$$

The answer is D.

**Question 17**

Let $\underset{\sim}{a} = 2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}$.

The vector resolute of $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{b}$ is $\underset{\sim}{a} - \left(\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}}\right)\hat{\underset{\sim}{b}}$.

Now, $\hat{\underset{\sim}{b}} = \frac{1}{\sqrt{1+4+4}}\left(\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}\right)$

$$= \frac{1}{3}\left(\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}\right)$$

So $\underset{\sim}{a} - \left(\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}}\right)\hat{\underset{\sim}{b}}$

$$= 2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k} - \frac{1}{3}(2-2-2) \times \frac{1}{3}\left(\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}\right)$$

$$= 2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k} + \frac{2}{9}\left(\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}\right)$$

$$= \frac{20}{9}\underset{\sim}{i} - \frac{5}{9}\underset{\sim}{j} + \frac{5}{9}\underset{\sim}{k}$$

$$= \frac{1}{9}\left(20\underset{\sim}{i} - 5\underset{\sim}{j} + 5\underset{\sim}{k}\right)$$

The answer is E.

**Question 18**

$$\underset{\sim}{r}(t)=e^{2t}\left(\underset{\sim}{i}+\underset{\sim}{j}-\underset{\sim}{k}\right)$$

$$\underset{\sim}{v}(t)=2e^{2t}\left(\underset{\sim}{i}+\underset{\sim}{j}-\underset{\sim}{k}\right)$$

$$\underset{\sim}{a}(t)=4e^{2t}\left(\underset{\sim}{i}+\underset{\sim}{j}-\underset{\sim}{k}\right)$$

$$\begin{aligned}\left|\underset{\sim}{a}(t)\right| &= \sqrt{3\times 16e^{4t}}\\ &= 4\sqrt{3}e^{2t}\end{aligned}$$

$$\left|\underset{\sim}{a}(1)\right| = 4\sqrt{3}e^{2}$$

The answer is D.

**Question 19**

Let $m$ be the mass of the particle in kg.
At $t=0$, momentum $=2m$.
At $t=5$, momentum $=8m$.
Change in momentum $=24$

$$\begin{aligned}\text{So}\quad 8m-2m&=24\\ 6m&=24\\ m&=4\text{kg}\end{aligned}$$

Change of momentum of the particle between $t=0$ and $t=10$ given by
$4\times 14-4\times 2=48\text{kg m/s}$.
The answer is D.

**Question 20**

In the first 15 secs, the particle has covered $10\times 10+\frac{1}{2}\times 10\times 5=125\text{m}$ (to the right).

At $t=15$ it changes direction and between $t=15$ and $t=30$ covers

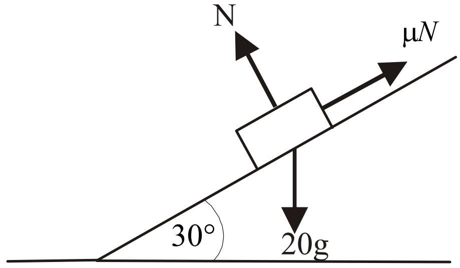
$\frac{1}{2}\times 10\times 20+5\times 20=200\text{m}$ (to the left).

So at $t=30$ the particle is $200-125=75\,\text{m}$ to the left of its starting point.

The answer is A.

**Question 21**




Acceleration is down the plane and hence the friction force $\mu N$ is directed up the plane.

$20g\sin(30^\circ) - \mu N = 20 \times 0\cdot 1$ and $N = 20g\cos(30^\circ) = 10\sqrt{3}g$

$$10g - 10\sqrt{3}g\mu = 2$$

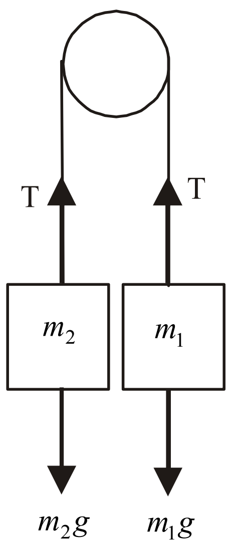
$$\mu = \frac{10g-2}{10\sqrt{3}g}$$

$$= \frac{5g-1}{5\sqrt{3}g}$$

The answer is E.

**Question 22**

Since $m_1 > m_2$ the $m_1$ particle will accelerate downwards and the $m_2$ particle will accelerate upwards.




<u>Around the $m_1$ particle</u>

$$m_1g - T = m_1a$$

$$T = m_1g - m_1a \qquad -(1)$$

<u>Around the $m_2$ particle</u>

$$T - m_2g = m_2a$$

$$T = m_2g + m_2a \qquad -(2)$$

Combining (1) and (2)

$$m_1g - m_1a = m_2g + m_2a$$

$$g(m_1 - m_2) = a(m_1 + m_2)$$

$$a = \frac{g(m_1 - m_2)}{m_1 + m_2}$$

The magnitude of the acceleration of the $m_1$ particle (and the $m_2$ particle) is $\dfrac{g(m_1 - m_2)}{m_1 + m_2}$ and the direction of the acceleration of the $m_1$ particle is downwards.

The answer is A.

## SECTION 2

### Question 1

**a.** Method 1

The graph of $y = \text{arsin}(x)$; which has a maximal domain of $[-1,1]$, has been dilated from the $y$- axis by a factor of 2 to produce the graph of $y = \text{arsin}\left(\frac{x}{2}\right)$.

So for $f(x) = \text{arsin}\left(\frac{x}{2}\right)$, $d_f = [-2,2]$. **(1 mark)**

Method 2

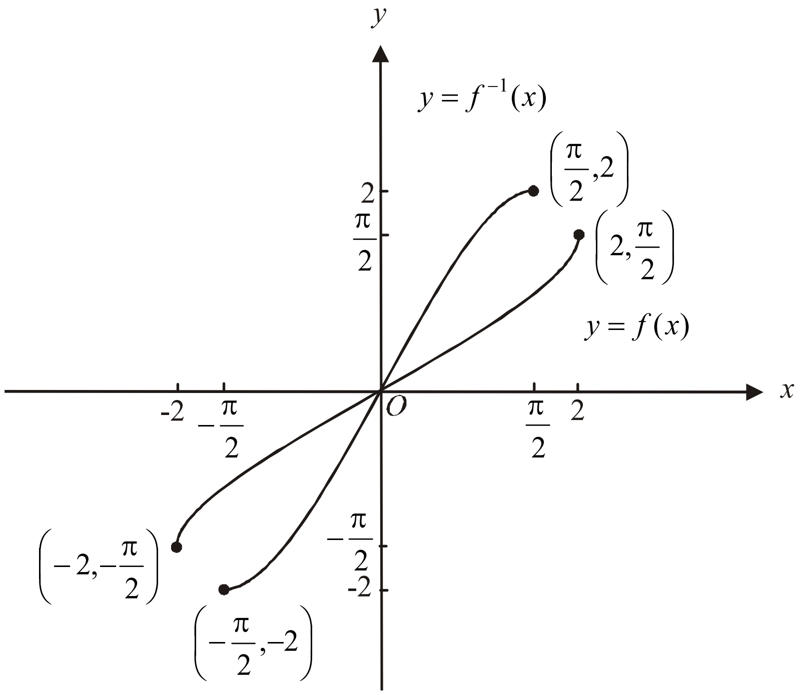
The graph of $y = \text{arsin}(x)$ has a maximal domain of $[-1,1]$ so $-1 \le x \le 1$.

For the graph of $y = \text{arsin}\left(\frac{x}{2}\right)$, $-1 \le \frac{x}{2} \le 1$

$$-2 \le x \le 2$$

So for $f(x) = \text{arsin}\left(\frac{x}{2}\right)$, $d_f = [-2,2]$. **(1 mark)**

**b.** **i.** and **ii.**




**(1 mark)** graph of $y = f(x)$

**(1 mark)** graph of $y = f^{-1}(x)$

**c.** $f(x) = \text{arsin}\left(\frac{x}{2}\right)$

Let $y = \text{arsin}\left(\frac{x}{2}\right)$

Swap $x$ and $y$ for inverse.

$$x = \text{arsin}\left(\frac{y}{2}\right)$$

Rearrange

$$\frac{y}{2} = \sin(x)$$

$$y = 2\sin(x)$$

so $f^{-1}(x) = 2\sin(x)$

**(1 mark)**

**d.** **i.**

$$\frac{d}{dx}\left(x \times \text{arsin}\left(\frac{x}{2}\right)\right)$$

$$= x \times \frac{1}{\sqrt{4-x^2}} + \text{arsin}\left(\frac{x}{2}\right) \qquad \text{(product rule)}$$

$$= \frac{x}{\sqrt{4-x^2}} + \text{arsin}\left(\frac{x}{2}\right)$$

**(1 mark)**

**ii.** From **i.**,

$$\text{arsin}\left(\frac{x}{2}\right) = \frac{-x}{\sqrt{4-x^2}} + \frac{d}{dx}\left(x \times \text{arsin}\left(\frac{x}{2}\right)\right)$$

$$\int \text{arsin}\left(\frac{x}{2}\right)dx = \int \frac{-x}{\sqrt{4-x^2}}dx + x \times \text{arsin}\left(\frac{x}{2}\right) \qquad c = 0 \text{ for "an antiderivative"}$$

let $u = 4 - x^2$

$\frac{du}{dx} = -2x$

$$= \int u^{-\frac{1}{2}} \times \frac{1}{2}\frac{du}{dx}dx + x \times \text{arsin}\left(\frac{x}{2}\right)$$

$$= \frac{1}{2}\int u^{-\frac{1}{2}}du + x \times \text{arsin}\left(\frac{x}{2}\right)$$

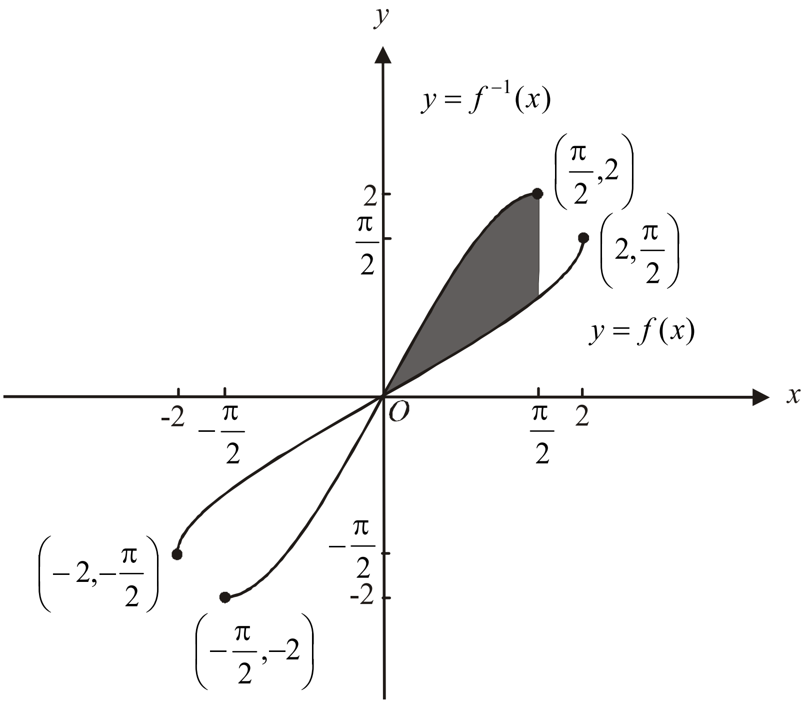
**(1 mark)**

$$= \frac{1}{2}u^{\frac{1}{2}} \times 2 + x \times \text{arsin}\left(\frac{x}{2}\right)$$

$$= \sqrt{4-x^2} + x \times \text{arsin}\left(\frac{x}{2}\right)$$

**(1 mark)**

**e.** From the graph,




$$\text{required area} = \int_0^{\frac{\pi}{2}} \left(2\sin(x) - \text{arsin}\left(\frac{x}{2}\right)\right)dx$$ **(1 mark)**

$$= \left[-2\cos(x) - \sqrt{4-x^2} - x \times \text{arsin}\left(\frac{x}{2}\right)\right]_0^{\frac{\pi}{2}}$$ **(1 mark) –** this step must be shown because of the term "hence" in the question

$$= \left\{\left(-2\cos\left(\frac{\pi}{2}\right) - \sqrt{4-\frac{\pi^2}{4}} - \frac{\pi}{2}\text{arsin}\left(\frac{\pi}{4}\right)\right) - \left(-2\cos(0) - \sqrt{4} - 0\right)\right\}$$

$$= 0 - \sqrt{4-\frac{\pi^2}{4}} - \frac{\pi}{2} \times \text{arsin}\left(\frac{\pi}{4}\right) + 2 + 2$$

$$= -\sqrt{4-\frac{\pi^2}{4}} - \frac{\pi}{2} \times \text{arsin}\left(\frac{\pi}{4}\right) + 4 \quad \text{square units}$$

**(1 mark)**
**Total 10 marks**

**Question 2**

**a.** Method 1 – use a calculator

$$2+2i=2\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$$

**(1 mark)**

Method 2 – by hand

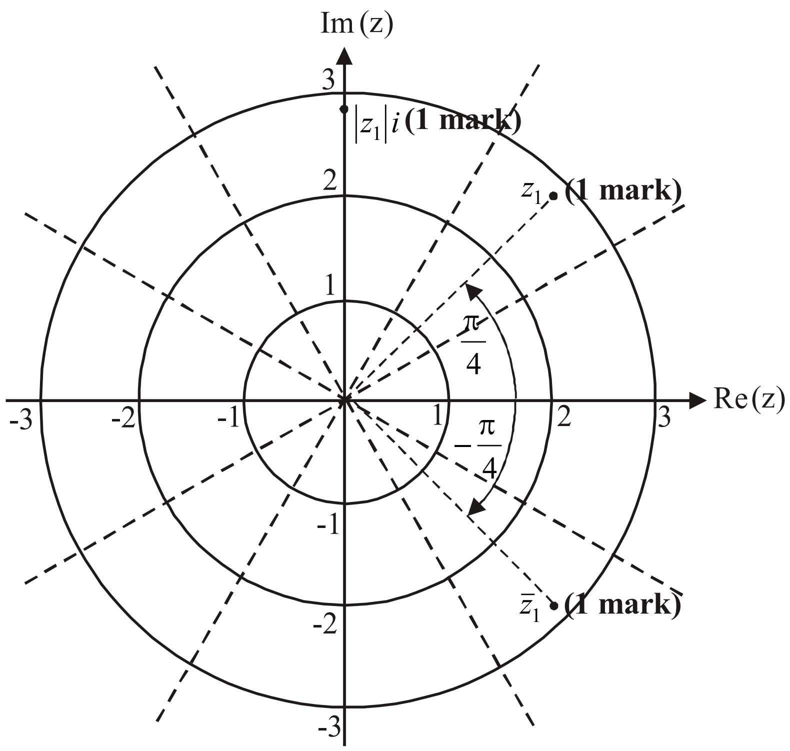
$$\begin{aligned}z_1&=2+2i\\ r&=\sqrt{2^2+2^2}\\ &=2\sqrt{2}\\ \text{cis}(\theta)&=\tan^{-1}\left(\frac{2}{2}\right)\\ &=\tan^{-1}(1)\\ &=\frac{\pi}{4}\quad (z_1 \text{ is in the } 1^{\text{st}} \text{ quadrant})\\ z_1&=2\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)\end{aligned}$$

**1 mark)**

**b.** **i.** $z_1=2\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$

**ii.** $\bar{z}_1=2\sqrt{2}\text{cis}\left(\frac{-\pi}{4}\right)$

**iii.** $|z_1|i=2\sqrt{2}i$



**c.**

$$\sin\left(\frac{\pi}{12}\right)=\sin\left(\frac{\pi}{3}-\frac{\pi}{4}\right)$$

$$=\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right)-\cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)$$ **(1 mark)**

$$=\frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}-\frac{1}{2}\times\frac{1}{\sqrt{2}}$$

$$=\frac{\sqrt{3}}{2\sqrt{2}}-\frac{1}{2\sqrt{2}}$$

$$=\frac{\sqrt{2}\left(\sqrt{3}-1\right)}{4}$$ as required

**(1 mark)**

**d.** Method 1

$z_1=2+2i$

Let $z^3=2+2i$ where $z=r\text{cis}(\theta)$

$$(r\text{cis}\,\theta)^3=2\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$$

$$r^3\text{cis}(3\theta)=2\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)\quad\text{(De Moivre)}$$

$$r^3=2\sqrt{2}\qquad 3\theta=\frac{\pi}{4}+2k\pi,\quad k\in z$$

$$r^3=2^{\frac{3}{2}}\qquad \theta=\frac{\pi}{12}+\frac{2k\pi}{3}$$ **(1 mark)**

$$r=\sqrt{2}$$ **(1 mark)**

When $k=0$, $z=\sqrt{2}\text{cis}\left(\frac{\pi}{12}\right)$

$$=\sqrt{2}\left(\cos\left(\frac{\pi}{12}\right)+i\sin\left(\frac{\pi}{12}\right)\right)$$

$$=\sqrt{2}\left(\frac{\sqrt{2}\left(1+\sqrt{3}\right)}{4}+i\frac{\sqrt{2}\left(\sqrt{3}-1\right)}{4}\right)$$

$$=\frac{\left(1+\sqrt{3}\right)}{2}+\frac{\left(\sqrt{3}-1\right)}{2}i$$

as required.

**(1 mark)**

Method 2

$$\left(\frac{1+\sqrt{3}}{2}+\frac{\sqrt{3}-1}{2}i\right)^3=\left(\sqrt{2}\left(\frac{\sqrt{2}\left(1+\sqrt{3}\right)}{4}+i\frac{\sqrt{2}\left(\sqrt{3}-1\right)}{4}\right)\right)^3$$ **(1 mark)**

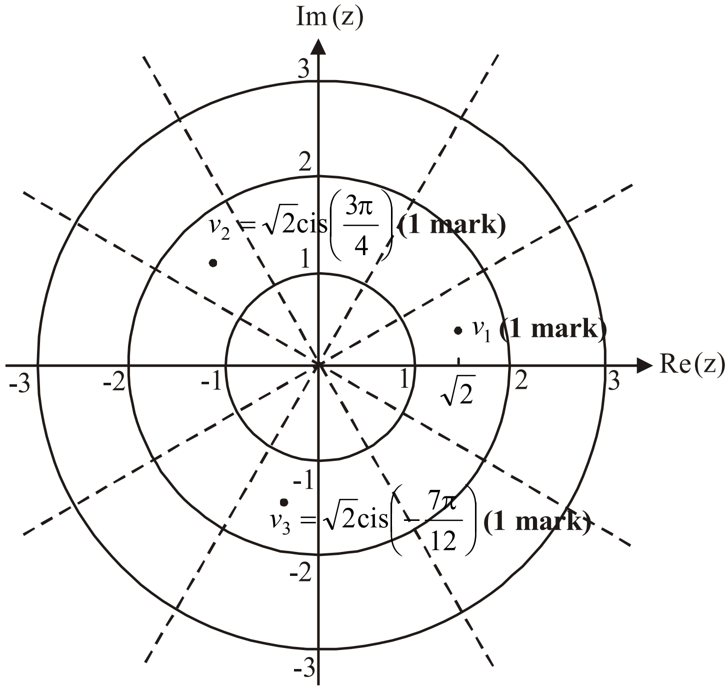
$$=\left(\sqrt{2}\text{cis}\left(\frac{\pi}{12}\right)\right)^3$$ **(1 mark)**

$$=\sqrt{8}\text{cis}\left(\frac{\pi}{4}\right)$$

$$=z_1$$

**(1 mark)**

**e.**




**i.** From part **d.**, when $k = 0$

$$\begin{aligned} z &= \sqrt{2}\text{cis}\left(\frac{\pi}{12}\right) \\ &= \frac{\left(1+\sqrt{3}\right)}{2} + \frac{\left(\sqrt{3}-1\right)}{2}i \\ &= v_1 \end{aligned}$$

So $v_1 = \sqrt{2}\text{cis}\left(\frac{\pi}{12}\right)$

**ii.** There are three cube roots of $z_1$ spaced evenly around a circle of radius $\sqrt{2}$ units. They are spaced $\frac{2\pi}{3}$ apart.

$$\begin{aligned} \text{So } v_2 &= \sqrt{2}\text{cis}\left(\frac{\pi}{12}+\frac{2\pi}{3}\right) \\ &= \sqrt{2}\text{cis}\left(\frac{3\pi}{4}\right) \\ \text{and } v_3 &= \sqrt{2}\text{cis}\left(\frac{\pi}{12}-\frac{2\pi}{3}\right) \\ &= \sqrt{2}\text{cis}\left(-\frac{7\pi}{12}\right) \end{aligned}$$

or vice-versa.

**f.** Show that $v_1{}^3 = z_1$, $v_2{}^3 = z_1$ and $v_3{}^3 = z_1$.

From **e.**, $v_1 = \sqrt{2}\text{cis}\left(\frac{\pi}{12}\right)$

$$\begin{aligned} v_1{}^3 &= \left(\sqrt{2}\text{cis}\left(\frac{\pi}{12}\right)\right)^3 \\ &= \sqrt{8}\text{cis}\left(\frac{3\pi}{12}\right) \\ &= 2\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right) \\ &= z_1 \end{aligned}$$

From **e.**, $v_2 = \sqrt{2}\text{cis}\left(\frac{3\pi}{4}\right)$

$$\begin{aligned} v_2{}^3 &= \left(\sqrt{2}\text{cis}\left(\frac{3\pi}{4}\right)\right)^3 \\ &= \sqrt{8}\text{cis}\left(\frac{9\pi}{4}\right) \\ &= 2\sqrt{2}\text{cis}\left(2\pi + \frac{\pi}{4}\right) \\ &= 2\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right) \\ &= z_1 \end{aligned}$$

From **e.**, $v_3 = \sqrt{2}\text{cis}\left(\frac{-7\pi}{12}\right)$

$$\begin{aligned} v_3{}^3 &= \left(\sqrt{2}\text{cis}\left(\frac{-7\pi}{12}\right)\right)^3 \\ &= \sqrt{8}\text{cis}\left(\frac{-21\pi}{12}\right) \\ &= 2\sqrt{2}\text{cis}\left(-2\pi + \frac{\pi}{4}\right) \\ &= 2\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right) \\ &= z_1 \end{aligned}$$

**(1 mark)** for correct attempt
**(1 mark)** for correct results
**Total 14 marks**

**Question 3**

**a.** **i.** We have constant (uniform) acceleration so we can use the formula

$$v^2 = u^2 + 2as \qquad \text{where } u = 0,\ v = 40 \text{ and } s = 600$$

$$1600 = 0 + 2 \times a \times 600$$

$$a = \frac{1600}{1200}$$

$$a = \frac{4}{3}\,\text{ms}^{-2}$$

**(1 mark)**

**ii.** Similarly, $v = u + at$

$$40 = 0 + \frac{4}{3} \times t$$

$$t = 30 \text{ secs}$$

**(1 mark)**

**b.** The equation of motion is given by

$$\underset{\sim}{R} = m\,\underset{\sim}{a}$$

where $\underset{\sim}{R}$ represents the resultant forces acting on the cyclist and his bicycle.

Because the cyclist's motion is along a straight line, we have

$$\underset{\sim}{R} = m\,\underset{\sim}{a}$$

$$(-0.2v - 500)\underset{\sim}{i} + (N - 120g)\underset{\sim}{j} = 120a\,\underset{\sim}{i}$$

$$\text{so,} \quad -0\cdot 2v - 500 = 120a$$

$$a = \frac{-0\cdot 2v - 500}{120}$$

**(1 mark)**

**c.**

$$a = \frac{-0\cdot 2v - 500}{120}$$

$$\frac{dv}{dt} = \frac{-0\cdot 2(v + 2500)}{120}$$

$$= \frac{-(v + 2500)}{600}$$

So $\dfrac{dt}{dv} = \dfrac{-600}{v + 2500}$ as required.

**(1 mark)**

**d.** Method 1

From part **c.**, $\frac{dt}{dv} = \frac{-600}{v+2500}$

$$t = -600\int \frac{1}{v+2500}\,dv$$

$$= -600\ln|v+2500| + c \qquad \textbf{(1 mark)}$$

$t = 0,\ v = 40$

$$0 = -600\ln(2540) + c$$

$$c = 600\ln(2540)$$

$$t = -600\ln|v+2500| + 600\ln(2540)$$

$$t = 600\ln\left|\frac{2540}{v+2500}\right| \qquad \textbf{(1 mark)}$$

When $v = 0$, $t = 600\ln\left(\frac{2540}{2500}\right)$

$$= 9\cdot 52401...$$

So $t = 9\cdot 5$ (correct to 1 decimal place)

**(1 mark)**

Method 2

The time it takes to come to rest is given by

$$t = \int_{40}^{0} \frac{-600}{v+2500}\,dv \qquad \textbf{(1 mark)}$$

$$= \int_{0}^{40} \frac{600}{v+2500}\,dv$$

$$= 600\left[\log_e|2500+v|\right]_0^{40} \qquad \textbf{(1 mark)}$$

$$= 600\left(\log_e(2540) - \log_e(2500)\right)$$

$$= 600\log_e\left(\frac{2540}{2500}\right)$$

$$= 9.52401...$$

So $t = 9.5$ (correct to 1 decimal place) **(1 mark)**

**e.** Method 1

From part **d.**,

$$t = 600\log_e\left|\frac{2540}{v+2500}\right|$$

$$\frac{2540}{v+2500} > 0$$
$$v+2500 > 0$$
$$v > -2500$$

$$\frac{t}{600} = \log_e\left|\frac{2540}{v+2500}\right|$$

$$e^{\frac{t}{600}} = \frac{2540}{v+2500}$$

$$v+2500 = \frac{2540}{e^{\frac{t}{600}}}$$

$$v = 2540e^{\frac{-t}{600}} - 2500$$

$$\frac{dx}{dt} = 2540e^{\frac{-t}{600}} - 2500$$

$$x = \int_0^{9\cdot 52401}\left(2540e^{\frac{-t}{600}} - 2500\right)dt$$ **(1 mark)**

$$= 189\cdot 976...$$
$$= 190 \text{ m (to the nearest metre)}$$

**(1 mark)**

Method 2

From part **b.**,

$$a = \frac{-(v+2500)}{600}$$

$$v\frac{dv}{dx} = \frac{-(v+2500)}{600}$$

$$\frac{dv}{dx} = \frac{-(v+2500)}{600v}$$

$$x = -600\int\frac{v}{v+2500}dv$$ **(1 mark)**

$$= -600\int\left(1 - \frac{2500}{v+2500}\right)dv$$

$$= -600\left(v - 2500\log_e|v+2500|\right) + c$$

$x = 0$, $v = 40$

$$0 = -600\left(40 - 2500\log_e(2540)\right) + c$$

$$c = 600\left(40 - 2500\log_e(2540)\right)$$

$$x = -600\left(v - 2500\log_e|v+2500|\right) + 600\left(40 - 2500\log_e(2540)\right)$$

$$= 600\left(40 - v + 2500\log_e\left|\frac{v+2500}{2540}\right|\right)$$

When $v = 0$

$$x = 600\left(40 + 2500\log_e\left(\frac{2500}{2540}\right)\right)$$

$$= 189\cdot 976$$
$$= 190\text{m (to the nearest metre)}$$

**(1 mark)**

Method 3

Required distance is given by

$$x = \int_{40}^{0} \frac{-600v}{v+2500}\,dv$$

$$= \int_{0}^{40} \frac{600v}{v+2500}\,dv$$

*let* $u = v + 2500$ *so* $v = u - 2500$

$$\frac{du}{dv} = 1$$

*Also,* $v = 40$ *so* $u = 2540$

*and* $v = 0$ *so* $u = 2500$

$$= 600 \int_{2500}^{2540} (u - 2500).\frac{1}{u}.\frac{du}{dv}\,dv$$

$$= 600 \int_{2500}^{2540} \left(1 - \frac{2500}{u}\right) du$$ **(1 mark)**

$$= 600\left[u - 2500\log_e |u|\right]_{2500}^{2540}$$

$$= 600\{(2540 - 2500\log_e(2540)) - (2500 - 2500\log_e(2500))\}$$

$$= 600\left(40 + 2500\log_e\left(\frac{2500}{2540}\right)\right)$$

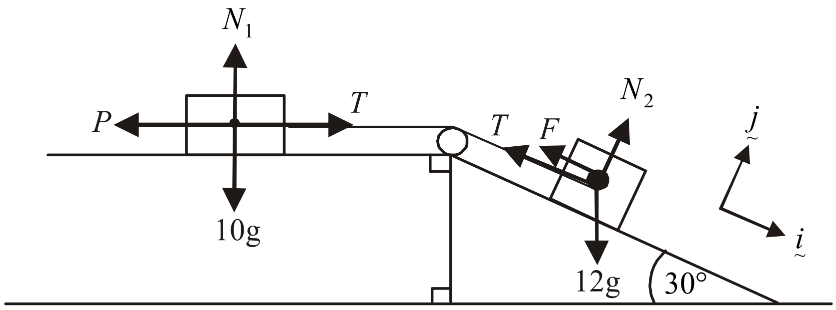
$$= 189.976...$$

$$= 190 \quad \text{(to the nearest metre)}$$

**(1 mark)**
**Total 9 marks**

**Question 4**

**a.** Mark in the forces operating.




$N_1$ and $N_2$ are the normal forces acting on the 10kg and 12kg masses respectively. $F$ is the friction force of the inclined plane on the 12kg mass.

Around the 10kg mass

The 10kg mass will accelerate to the right so there must be a net force to the right.

$$R = ma$$
$$T - P = 10a$$

Now, $P = 0$ so $T = 10a$

Around the 12kg mass

$$\underset{\sim}{R} = m\underset{\sim}{a} \qquad \text{(equation of motion)}$$

$$\left(12g\sin\left(30^\circ\right) - T - F\right)\underset{\sim}{i} + \left(N_2 - 12g\cos\left(30^\circ\right)\right)\underset{\sim}{j} = 12a\,\underset{\sim}{i}$$

**(1 mark)**

$$N_2 = 6\sqrt{3}g$$

Also, $\quad 6g - T - \mu N_2 = 12a$

so $\quad 6g - 10a - \dfrac{\sqrt{3}}{10} \times 6\sqrt{3}g = 12a$

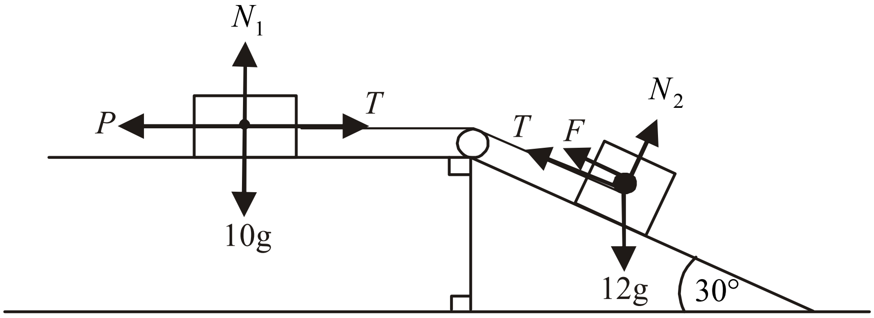
$$4.2g = 22a$$

$$a = \frac{42g}{220}$$

so $\quad a = \dfrac{21g}{110}\,\text{ms}^{-2}$ as required

**(1 mark)**

**b.** If $P=Q$, then the 12kg mass is at the point of sliding down the inclined plane (ie. limiting equilibrium).




The friction force $F$ is therefore directed up the inclined plane.
Now, resolving around the 10kg mass we have $P=T$ and $P=Q$ so $T=Q$.

Around the 12kg mass

$12g\sin(30^\circ)=T+F$ and $N_2=12g\cos(30^\circ)$

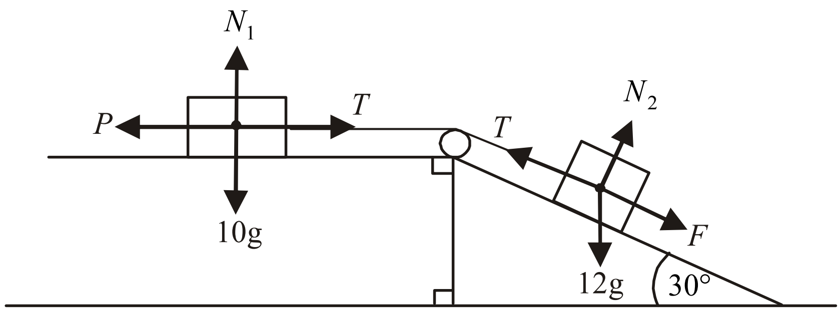
$6g=Q+\frac{\sqrt{3}}{10}\times 6\sqrt{3}g$ $\qquad$ $=6\sqrt{3}g$

$Q=6g-1\cdot 8g$

$Q=4\cdot 2g$

**(1 mark)**

If $P=R$, then the 12kg mass is at the point of sliding up the inclined plane and therefore the friction force is directed down the inclined plane.
Now, resolving around the 10kg mass we have $P=T$ and $P=R$ so $T=R$.




**(1 mark)** – giving some indication that the friction force is directed down the plane.

Around the 12kg mass

$12g\sin(30^\circ)+F=T$ and $N_2=12g\cos(30^\circ)$

$6g+\frac{\sqrt{3}}{10}\times 6\sqrt{3}g=R$ $\qquad$ $=6\sqrt{3}g$

$6g+1\cdot 8g=R$

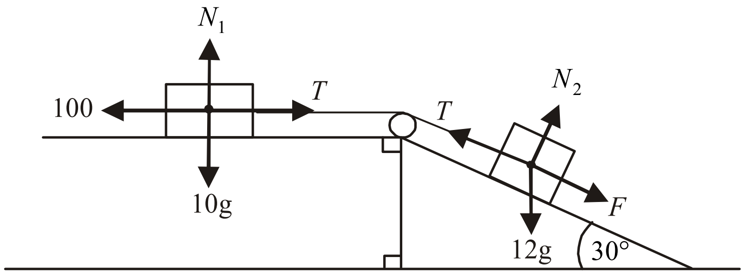
$R=7\cdot 8g$

**(1 mark)**

**c.** In part **b.**, we found $R = 7.8g = 76.44N$

When $P = R$, the 12kg mass is at the point of moving up the inclined plane and therefore when $P = 100N$, the 12kg mass will be accelerating up the inclined plane.

**(1 mark)**




<u>Around the 10kg mass</u>

$100 - T = 10a$

$T = 100 - 10a \qquad -(1)$

<u>Around the 12kg mass</u>

$T - 12g\sin(30^\circ) - F = 12a$ and $N_2 = 12g\cos(30^\circ)$

$T - 6g - \frac{\sqrt{3}}{10} \times 6\sqrt{3}g = 12a \qquad = 6\sqrt{3}g$

$T = 12a + 6g + 1 \cdot 8g$

$= 12a + 7 \cdot 8g$

Using (1)

$100 - 10a = 12a + 7 \cdot 8g$

$-22a = 7 \cdot 8g - 100$ **(1 mark)**

$a = \frac{7 \cdot 8g - 100}{-22}$

$= 1 \cdot 07091...$

The 12kg mass will accelerate at $1 \cdot 07\text{ms}^{-2}$ (to 3 significant figures) up the plane.

**(1 mark)**

**d.** Once the string is disconnected, $T = 0$.

<u>Around the 12kg mass</u>

$\underset{\sim}{R} = m\underset{\sim}{a}$

$(12g\sin(30°) - \mu N)\underset{\sim}{i} + (N - 12g\cos(30°))\underset{\sim}{j} = 12a\underset{\sim}{i}$

$6g - \frac{\sqrt{3}}{10}\left(6\sqrt{3}g\right) = 12a \qquad N = 6\sqrt{3}g$

$6g - 1.8g = 12a$

$4.2g = 12a$

$a = \frac{4.2g}{12}$

$= \frac{7g}{20}\text{ms}^{-2}$

**(1 mark)**

Since acceleration is constant, we can use the formula

$s = ut + \frac{1}{2}at^2$ where $s = 8$, $u = 1$ and $a = \frac{7g}{20}$

$8 = t + \frac{1}{2} \times \frac{7g}{20} t^2$

$\frac{7g}{40} t^2 + t - 8 = 0$ **(1 mark)**

$t = -2 \cdot 47093...$ or $1 \cdot 88784...$

but $t > 0$ so $t = 1 \cdot 9$ secs correct to 1 decimal place. **(1 mark)**

**e.** As with part **d.**, the acceleration is $\frac{7g}{20} \text{ms}^{-2}$ but $u = 0$ and $s = 8$.

So $8 = \frac{7g}{40} t^2$

$t = -2 \cdot 1598$ or $2 \cdot 1598...$

but $t > 0$ so $t = 2 \cdot 2$ secs correct to 1 decimal place. **(1 mark)**

**Total 12 marks**

**Question 5**

**a.**

When $t = 0$, $\quad 80 - 2t$

$$= 80 - 2 \times 0$$
$$= 80$$

The kite is 80m above the ground when the monitoring begins.

**(1 mark)**

**b.** **i.** The kite hits the ground when

$$80 - 2t = 0$$
$$t = 40 \text{ secs}$$

**(1 mark)**

**ii.** The position vector of the kite when it has hit the ground is

$$\underset{\sim}{r}(40) = (4 \times 40 - 20)\underset{\sim}{i} + 80\underset{\sim}{j} + (80 - 2 \times 40)\underset{\sim}{k}$$
$$= 140\underset{\sim}{i} + 80\underset{\sim}{j} + 0\underset{\sim}{k}$$

**(1 mark)**

The distance from the recording device is given by

$$\left|\underset{\sim}{r}(40)\right| = \sqrt{140^2 + 80^2 + 0^2}$$
$$= 161 \cdot 245...$$
$$= 161\text{m (to the nearest metre)}$$

**(1 mark)**

**c.**

$$\underset{\sim}{r} = (4t - 20)\underset{\sim}{i} + 2t\underset{\sim}{j} + (80 - 2t)\underset{\sim}{k}$$
$$\underset{\sim}{v} = 4\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}$$

**(1 mark)**

$$\left|\underset{\sim}{v}\right| = \sqrt{4^2 + 2^2 + (-2)^2}$$
$$= 4 \cdot 9\text{m/s (correct to 1 decimal place)}$$

**(1 mark)**

**d.** From part **c.** the kite is travelling at a constant speed of $\sqrt{4^2 + 2^2 + (-2)^2}$ m/s.
It travels for 40 secs before it hits the ground. **(1 mark)**

$$\text{distance travelled} = \sqrt{4^2 + 2^2 + (-2)^2} \times 40$$
$$= 196\text{m (to the nearest metre)}$$

(Check units $\frac{\text{m}}{\text{s}} \times \text{s} = \text{m}$ .)

**(1 mark)**

**e.** Method 1

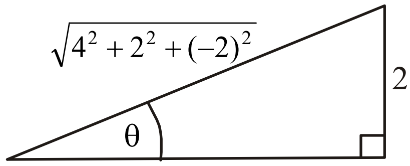
We need to find the angle that the velocity vector makes with the horizontal ground.

$\underset{\sim}{v} = 4\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}$

$$\sin(\theta) = \frac{2}{\sqrt{4^2 + 2^2 + (-2)^2}}$$

$\theta = 24\cdot1^\circ$ (correct to one decimal place)

The required angle is $24\cdot1^\circ$




**(1 mark)** using $\underset{\sim}{v}$

**(1 mark)** correct answer

Method 2

The required angle can be found by finding the angle between the vectors $4\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}$ and $4\underset{\sim}{i} + 2\underset{\sim}{j}$.

Now, $(4\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}) \bullet (4\underset{\sim}{i} + 2\underset{\sim}{j}) = |4\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}||4\underset{\sim}{i} + 2\underset{\sim}{j}|\cos(\theta)$ **(1 mark)**

$16 + 4 + 0 = \sqrt{24} \times \sqrt{20}\cos(\theta)$

$$\cos(\theta) = \frac{20}{\sqrt{480}}$$

$\theta = 24.1^\circ$ correct to 1 decimal place

**(1 mark)**

**f.** The vector $\underset{\sim}{i}$ is the vector in the east direction. When $(4t - 20)$ is a maximum, the kite will be furthest east. This happens at $t = 40$ when the kite has hit the ground.

**(1 mark)**

**g.** Method 1

The kite is closest to the recording device when

$\underset{\sim}{r} \cdot \underset{\sim}{v} = 0$ **(1 mark)**

$$4(4t - 20) + 2 \times 2t - 2(80 - 2t) = 0$$
$$16t - 80 + 4t - 160 + 4t = 0$$
$$24t = 240$$
$$t = 10 \text{ secs}$$

**(1 mark)**

Method 2

distance $= \sqrt{(4t - 20)^2 + (2t)^2 + (80 - 2t)^2}$ **(1 mark)**

$= \sqrt{16t^2 - 160t + 400 + 4t^2 + 6400 - 320t + 4t^2}$

$= \sqrt{24t^2 - 480t + 6800}$

Using a CAS calculator , the minimum distance is 66.33 when t=10 seconds.

**(1 mark)**

**Total 13 marks**