SPECIALIST MATHEMATICS

Units 3 & 4 – Written examination 2

2007 Trial Examination

SOLUTIONS

SECTION 1: Multiple-choice questions (1 mark each)

Question 1

Answer: E

Explanation:

By completing the square

$x^2 - 6x + 9 + y^2 = 7 + 9$

$(x-3)^2 + y^2 = 16$ - standard equation of a circle, centre $(3, 0)$, $r = 4$

Question 2

Answer: C

Explanation:

The parabola in the denominator has a minimum at $(-a, b)$. Its reciprocal will have a maximum but the reflection due to the negative sign changes this to a minimum.

When the reciprocal is taken the y ordinate becomes $-\dfrac{1}{b}$, therefore there is a minimum at $\left(-a, \dfrac{1}{b}\right)$

Question 3

Answer: B

Explanation:

Non real (complex) roots come in conjugate pairs, so it is not possible for there to be three non real roots.

Question 4

Answer: A

Explanation:

Let $\cos^{-1}\left(\frac{1}{4}\right) = x$. Then $\cos x = \frac{1}{4}$

$$\cos\left(2\cos^{-1}\frac{1}{4}\right) = \cos 2x$$

$$= 2\cos^2 x - 1 = \frac{2}{16} - 1 = -\frac{7}{8}$$

Question 5

Answer: B

Explanation:

$Arg(z) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$. As z lies in the third quadrant,

$Arg(z) = -\frac{5\pi}{6}$ and $Arg(z^3) = 3\left(-\frac{5\pi}{6}\right) = \frac{-5\pi}{2}$

But $-\pi < Argz \le \pi$, so correct answer is $-\frac{\pi}{2}$

Question 6

Answer: D

Explanation:

An argument less than $\frac{\pi}{3}$ will extend to $-\pi$ but not include this value.

Question 7

Answer: D

Explanation:

Given $\dfrac{dV}{dt} = 6m^3 / s$, using the chain rule $\dfrac{dx}{dt} = \dfrac{dx}{dV} \times \dfrac{dV}{dt}$

$V = x^3$, $\dfrac{dV}{dx} = 3x^2$

$\dfrac{dx}{dt} = \dfrac{1}{3x^2} \times 6$ When $x = 0.4m$, $\dfrac{dx}{dt} = \dfrac{2}{0.4^2} = \dfrac{25}{2} m / s$

Question 8

Answer: E

Explanation:

$$\frac{1}{y}\frac{dy}{dx} + y + x\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-y}{x + \frac{1}{y}} = \frac{-y^2}{1 + xy}$$

When $x = 2$ and $y = 1$ $\dfrac{dy}{dx} = -\dfrac{1}{3}$.

Question 9

Answer: C

Explanation:

The substitution is $u = sin\ 2x$, so $\dfrac{du}{dx} = 2\cos 2x$ so $\cos(2x)dx = \dfrac{du}{2}$. At the upper terminal, $\sin\dfrac{2\pi}{6} = \dfrac{\sqrt{3}}{2}$.

Question 10

Answer: A

Explanation:

$$\int \frac{4}{2+x^2}dx = \frac{4}{\sqrt{2}}\int \frac{\sqrt{2}}{2+x^2}dx = \frac{4}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+c = 2\sqrt{2}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+c$$

Question 11

Answer: C

Explanation:

$$\begin{aligned}\overrightarrow{PQ} &= \overrightarrow{PA}+\overrightarrow{AQ}\\ &= \frac{1}{2}(\underset{\sim}{a}-\underset{\sim}{b})-\frac{1}{4}\underset{\sim}{a}\\ &= \frac{1}{4}\underset{\sim}{a}-\frac{1}{2}\underset{\sim}{b}\end{aligned}$$

Question 12

Answer: D

Explanation:

The rate of heating is proportional to $28 - T$, or equal to $-k(T - 28)$.

Question 13

Answer: C

Explanation:

Acceleration is given by $a = v\dfrac{dv}{dx} = e^{4x} \times 4e^{4x} = 4e^{8x}$.

Question 14

Answer: B

Explanation:

Particles meet if $t = 8 - t$ ($\underset{\sim}{i}$ components equated), or $t = 4$.

When $t = 4$, both $\underset{\sim}{j}$ components are 9.

Question 15

Answer: A

Explanation:

A unit vector parallel to $2\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$ is $\frac{1}{3}(2\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k})$. Taking the scalar product of this with $-3\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$ gives $-\frac{1}{3}$.

Question 16

Answer: E

Explanation:

$3\underset{\sim}{j} + \underset{\sim}{k}$ and $-\underset{\sim}{j} + m\underset{\sim}{k}$ are linearly dependent if their coefficients are proportional,

$3 : -1 = 1 : m$. Thus $m = -\frac{1}{3}$.

Question 17

Answer: D

Explanation:

Slopes are positive where x and y have the same sign. Slopes are vertical when $y = 0$, so y must be in the denominator of the differential equation. Slopes are horizontal when $x = 0$, so x must be in the numerator of the differential equation.

Question 18

Answer: D

Explanation:

Volume is obtained by subtracting the squares of the y values of the two curves. Upper terminal is where $e^x = 4 \Rightarrow \mathrm{x} = \log_e 4$.

Question 19

Answer: B

Explanation:

Resolving horizontally: $T_1 \cos 30 = T_2 \cos 60$ or $\sqrt{3}T_1 = T_2$.
Resolving vertically: $T_1 \sin 30 + T_2 \sin 60 = mg$ or $T_1 + \sqrt{3}T_2 = 2mg$.
Eliminating T_2, $T_1 + 3T_1 = 2mg$ or $T_1 = \frac{1}{2}mg$.

Question 20

Answer: A

Explanation:

Resolving vertically: $5g = 2g \sin 30 + R$, so $R = 4g$.
Resolving horizontally: $F = 2g \cos 30 = \sqrt{3}g$.
If $F = \mu R$, $\mu = \frac{\sqrt{3}}{4}$. If μ is greater than or equal to this value there is no motion.

Question 21

Answer: B

Explanation:

The equation of motion (downwards) is $70g - 546 = 70a$. This gives $a = 2$.

Question 22

Answer: E

Explanation:

The equations of motion are for the hanging mass $g - T = a$ and for the mass on the table $T = 3a$.
Elimination of T gives $a = \frac{g}{4}$.

SECTION 2: Short-answer questions

Question 1

a.

i.

$$u=4\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)$$

$$u=4\left(-\frac{\sqrt{3}}{2}+i\frac{1}{2}\right), \qquad u=-2\sqrt{3}+2i$$ A1

ii.

$$|w|=\sqrt{1^2+(-1)^2}=\sqrt{2}, \qquad Arg(w)=\tan^{-1}(-1),$$

As w lies in the fourth quadrant, $Arg(w)=-\frac{\pi}{4}$

$$w=\sqrt{2}cis\left(-\frac{\pi}{4}\right)$$ A1

iii. Cartesian form

$$uw=(-2\sqrt{3}+2i)(1-i)$$

$$uw=-2\sqrt{3}+2\sqrt{3}i+2i+2$$

$$uw=(-2\sqrt{3}+2)+(2\sqrt{3}+2)i$$

A1

Polar form

$$uw=4cis\left(\frac{5\pi}{6}\right)\times\sqrt{2}cis\left(-\frac{\pi}{4}\right)$$

$$uw=4\sqrt{2}cis\left(\frac{5\pi}{6}-\frac{\pi}{4}\right)$$

$$uw=4\sqrt{2}cis\left(\frac{7\pi}{12}\right)$$

A1

iv. From **ii**. and **iii.**

$$\left(-2\sqrt{3}+2\right)+\left(2\sqrt{3}+2\right)i=4\sqrt{2}\cos\left(\frac{7\pi}{12}\right)+4\sqrt{2}i\sin\left(\frac{7\pi}{12}\right)$$ M1

By equating imaginary parts, $\sin\left(\frac{7\pi}{12}\right)=\frac{\sqrt{3}+1}{2\sqrt{2}}=\frac{\sqrt{6}+\sqrt{2}}{4}$ A1

b. Let $u = -1 - i$

$|u| = \sqrt{2}$, $Arg(u) = \tan^{-1}(1) = -\frac{3\pi}{4}$ (u is in the third quadrant and $-\pi < Arg(u) \le \pi$) A1

By de Moivre's theorem $z = \sqrt[8]{2}cis\left(\frac{-\frac{3\pi}{4} + 2k\pi}{4}\right)$, $k = 0, 1, 2, 3$ M1

$k = 0 \quad z = \sqrt[8]{2}cis\left(-\frac{3\pi}{16}\right)$

$k = 1 \quad z = \sqrt[8]{2}cis\left(\frac{5\pi}{16}\right)$

$k = 2 \quad z = \sqrt[8]{2}cis\left(\frac{13\pi}{16}\right)$

$k = 3 \quad z = \sqrt[8]{2}cis\left(\frac{21\pi}{16}\right) = \sqrt[8]{2}cis\left(-\frac{11\pi}{16}\right)$ A1

c. The polynomial has real coefficients. As the given solution is a complex number, there must exist another complex solution, conjugate to $-3 + 2i$, which is $-3 - 2i$.

A1

As $(-3 + 2i)(-3 - 2i) = 13$ and the last term of the polynomial is -26, $(z - 2)$ has to be the third factor and $z = 2$ third solution. A1

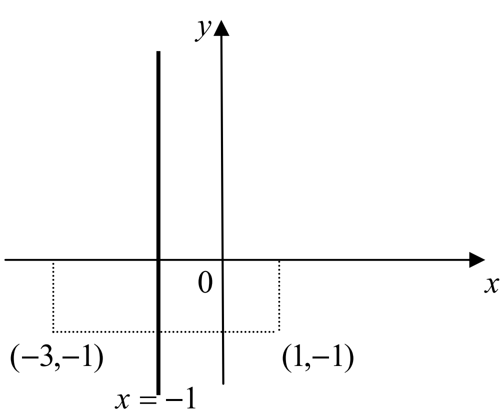
Alternatively

$(z - (-3 + 2i))(z - (-3 - 2i)) = z^2 + 6z + 13$

Dividing $z^3 + 4z^2 + z - 26$ by $z^2 + 6z + 13$ gives $z - 2$, $z = 2$ is a solution.

Or by using Factor Theorem $p(2) = 2^3 + 4 \times 2^2 + 2 - 26 = 0$, so $z - 2$ is a factor and $z = 2$ is a solution.

d. The locus defined by $|z - 1 + i| = |z + 3 + i|$ is the perpendicular bisector of the line segment connecting points $(1, -1)$ and $(-3, -1)$.



A1

Total 12 marks

Question 2

a.

$\ddot{\underset{\sim}{r}} = -g\,\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(t) = -gt\,\underset{\sim}{j} + c,$

$\dot{\underset{\sim}{r}}(0) = 10\cos 60^0\,\underset{\sim}{i} + 10\sin 60^0\,\underset{\sim}{j}$

$= 5\,\underset{\sim}{i} + 5\sqrt{3}\,\underset{\sim}{j}$ M1

$\dot{\underset{\sim}{r}}(t) = 5\,\underset{\sim}{i} + (5\sqrt{3} - gt)\,\underset{\sim}{j}$

A1

b.

$\underset{\sim}{r}(t) = 5t\,\underset{\sim}{i} + (5\sqrt{3}t - \frac{gt^2}{2})\,\underset{\sim}{j} + c$

$t = 0 \quad \underset{\sim}{r} = 0, \quad c = 0$ A1

$\underset{\sim}{r}(t) = 5t\,\underset{\sim}{i} + (5\sqrt{3}t - 4.9t^2)\,\underset{\sim}{j}$

c. Maximum height occurs when $\underset{\sim}{j}$ component of $\dot{\underset{\sim}{r}}$ is 0.

$5\sqrt{3} - gt = 0$ M1

$t = \frac{5\sqrt{3}}{g}$ A1

Maximum height = $\underset{\sim}{j}$ component of $\underset{\sim}{r}$ when $t = \frac{5\sqrt{3}}{g}$

$= 5\sqrt{3} \times \frac{5\sqrt{3}}{g} - 4.9 \times \frac{(5\sqrt{3})^2}{g^2}$

$= 3.83m$

A1

d.

i. $\underset{\sim}{j}$ component of $\underset{\sim}{r}$ must equal 2.44 m M1

$4.9t^2 - 5\sqrt{3}t + 2.44 = 0$

$t = \frac{5\sqrt{3} \pm \sqrt{75 - 4 \times 4.9 \times 2.44}}{2 \times 4.9}$ M1

$t = 0.3517544..., t = 1.4156443...$

Choose later time $t = 1.42$ seconds (2 dp)

A1

ii. Horizontal distance = $\underset{\sim}{i}$ component of $\underset{\sim}{r}$ when $t = 1.42$

$5 \times 1.4156443 = 7.08\ m$ (2 dp) A1

Total 10 marks

Question 3

a. Domain: $[-1,\ 3]$ Range: $[-3\pi,\ 0]$ A1 , A1

b.

$$-3\cos^{-1}\left(\frac{x-1}{2}\right) = -\frac{\pi}{2}$$

$$\frac{x-1}{2} = \cos\left(\frac{\pi}{6}\right), \quad \frac{x-1}{2} = \frac{\sqrt{3}}{2}$$ M1

$$x = \sqrt{3} + 1$$

A1

c.

$$\frac{dy}{dx} = \frac{1}{2}\frac{3}{\sqrt{1-\frac{(x-1)^2}{4}}},$$

$$= \frac{3}{\sqrt{4-(x-1)^2}}$$ M1, A1

$$= \frac{3}{\sqrt{3+2x-x^2}}$$

d.

$$\frac{d^2y}{dx^2} = -\frac{3}{2}(2-2x)(3+2x-x^2)^{-\frac{3}{2}}$$

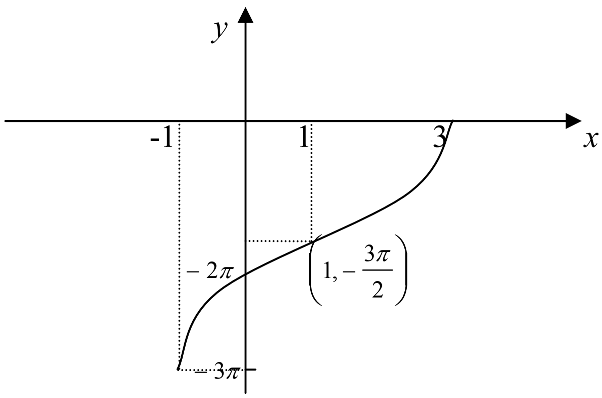
$$= -\frac{3}{2}\frac{2-2x}{(3+2x-x^2)^{\frac{3}{2}}}$$

For inflexion point $\frac{d^2y}{dx^2} = 0$ M1

$$2-2x = 0, x = 1, f(1) = -\frac{3\pi}{2}$$

Point of inflexion $\left(1, -\frac{3\pi}{2}\right)$ A1

e.



A1

f. Area = 18.85 units squared A1

Total 10 marks

Question 4

a.

$$ma = \frac{2m}{v} - 2mv, \quad t = 0, \quad v = 0.5ms^{-1}$$

$$\frac{dv}{dt} = \frac{2(1-v^2)}{v}$$

$$t = \int \frac{v}{2(1-v^2)} dv = -\frac{1}{4}\ln\left|1-v^2\right| + c$$

$$c = \frac{1}{4}\ln\frac{3}{4}$$

$$t = \frac{1}{4}\ln\frac{3}{4(1-v^2)}, \quad 0 < v < 1$$

M2 ,A1

$$\frac{3}{4(1-v^2)} = e^{4t}$$

$$1 - v^2 = \frac{3}{4e^{4t}}$$

$$v^2 = 1 - \frac{3}{4}e^{-4t} = \frac{1}{4}(4 - 3e^{-4t})$$

As $0 < v < 1, \quad v = \frac{1}{2}\sqrt{4 - 3e^{-4t}}$

M1, A1

b.

i.

$v = \frac{3}{4} \qquad t = \frac{1}{4}\ln\frac{3}{4\left(1-\frac{9}{16}\right)}$ M1

$= \frac{1}{4}\ln\frac{12}{7}$ A1

ii. For terminal velocity $t \to \infty$, $e^{-4t} \to 0$, $v = \frac{1}{2}\sqrt{4-3\times 0}$, $v \to 1 m/s$ A1

c.

$a = \frac{dv}{dt}$, and also $a = v\frac{dv}{dx}$ so $v\frac{dv}{dx} = \frac{2(1-v^2)}{v}$

M1

$\frac{dv}{dx} = 2\left(\frac{1-v^2}{v^2}\right) \qquad x = \frac{1}{2}\int \frac{v^2}{1-v^2}\,dv$

$\frac{v^2}{1-v^2} = -1 + \frac{1}{1-v^2}$ M1

Using partial fractions $\frac{v^2}{1-v^2} = -1 + \frac{1}{2(1-v)} + \frac{1}{2(1+v)}$

$x = \frac{1}{2}\int\left(-1 + \frac{1}{2(1-v)} + \frac{1}{2(1+v)}\right)dv$

$x = \frac{1}{2}\left(-v + \frac{1}{2}\ln\frac{1+v}{1-v}\right) + c$ M2

When $x = 0$, $v = 0.5$, $c = \frac{1}{4} - \frac{1}{4}\ln 3$.

$x = -\frac{1}{2}v + \frac{1}{4}\ln\frac{1+v}{3(1-v)} + \frac{1}{4}$ A1

d. For $v = \frac{3}{4}$, $x = -\frac{1}{2}\times\frac{3}{4} + \frac{1}{4} + \frac{1}{4}\ln\frac{7}{3} = 0.0868$

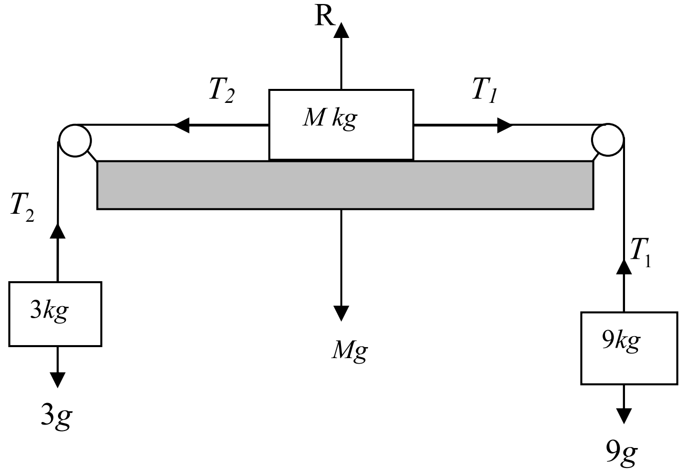
The body is 0.09 m right from the origin. A1

Total 14 marks

Question 5

a.

i.



A1

ii.

$9g - T_1 = 1.5 \times 9$ $\quad T_1 = 74.7N$

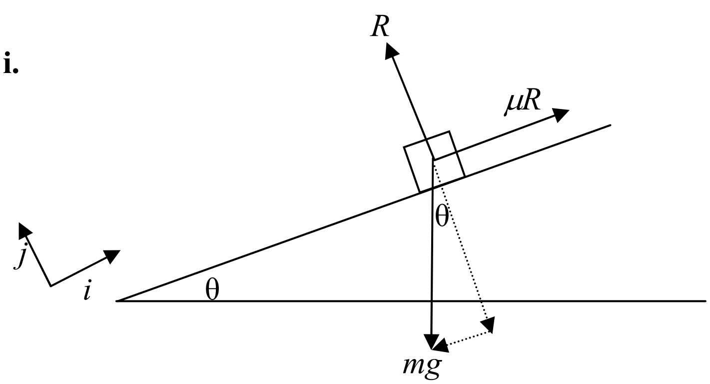
$T_2 - 3g = 1.5 \times 3$ $\quad T_2 = 33.9N$

$T_1 - T_2 = 1.5M$ $\quad M = \dfrac{40.8}{1.5} = 27.2kg$

M1, A2

b.

i.



A1

ii.

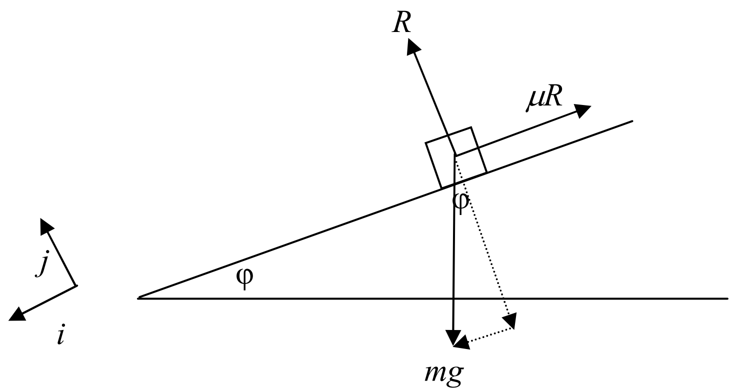
$R - mg\cos\theta = 0$

$\mu R - mg\sin\theta = 0$ M1

$\mu = \dfrac{mg\sin\theta}{mg\cos\theta} = \tan\theta$

A1

iii.



A1

$mg\sin\varphi - \mu R = ma$ A1

Substitute $\mu = \tan\theta$ and $R = mg\cos\varphi$

$$ma = mg\sin\varphi - mg\tan\theta\cos\varphi$$

$$a = g\left(\sin\varphi - \frac{\sin\theta}{\cos\theta}\cos\varphi\right)$$

$$a = g\left(\frac{\sin\varphi\cos\theta - \sin\theta\cos\varphi}{\cos\theta}\right) = g\,\frac{\sin(\varphi-\theta)}{\cos\theta}$$

M1, A1

iv. $a = g$ when

$$\frac{\sin(\varphi-\theta)}{\cos\theta} = 1$$

$$\sin(\varphi-\theta) = \cos\theta$$ M1

$$\varphi - \theta = \sin^{-1}(\cos\theta)$$

$$\varphi = \theta + \sin^{-1}(\cos\theta)$$

A1

Total 12 marks