2007 VCAA Specialist Maths Exam 2 Solutions

SECTION 1

1	2	3	4	5	6	7	8	9	10	11
B	E	D	B	E	B	A	C	D	C	D

12	13	14	15	16	17	18	19	20	21	22
A	E	C	A	C	D	E	B	C	A	B

Q1 $\frac{(x-2)^2}{a^2}-\frac{4(y+3)^2}{a^2}=1$, $\frac{(x-2)^2}{a^2}-\frac{(y+3)^2}{\left(\frac{a}{2}\right)^2}=1$.

Asymptotes are $(y+3)=\pm\frac{\frac{a}{2}}{a}(x-2)=\pm\frac{1}{2}(x-2)$.

Product of gradients is $\frac{1}{2}\left(-\frac{1}{2}\right)=-\frac{1}{4}$.

Q2 Ellipse: $\frac{(x-1)^2}{3^2}+\frac{(y-3)^2}{2^2}=1$.

Let $\frac{x-1}{3}=\cos t$, $\frac{y-3}{2}=\sin t$.

$x=1+3\cos t$, $y=3+2\sin t$

Q3 For $\tan^{-1}x$, range is $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$,

$b\tan^{-1}x$, range is $\left(-\frac{b\pi}{2},\frac{b\pi}{2}\right)$,

$a+b\tan^{-1}x$, range is $\left(a-\frac{b\pi}{2},a+\frac{b\pi}{2}\right)$,

$a+b\tan^{-1}(x-c)$, range is still the same $\left(a-\frac{b\pi}{2},a+\frac{b\pi}{2}\right)$.

Q4 Given formula: $\cot^2 x+1=\operatorname{cosec}^2 x$.

$\therefore \cot^2 x-\operatorname{cosec}^2 x=-1$, $\therefore \cot^2(4\theta)-\operatorname{cosec}^2(4\theta)=-1$.

Q5 $T=\frac{2\pi}{n}=\frac{5a}{2}-\frac{a}{2}=2a$, $\therefore n=\frac{\pi}{a}$. The given graph is the translation to the left by $\frac{a}{2}$ of $y=\operatorname{cosec}\frac{\pi}{a}x$.

Q6 The equation has real coefficients, .: the complex roots are in conjugate pair(s).

Q7 $\frac{1}{1-z}=\frac{1}{-2+4i}=\frac{-1-2i}{10}$.

Q8 $z=2cis\left(-\frac{\pi}{3}\right)$,

$z^4=2^4 cis4\left(-\frac{\pi}{3}\right)=16cis\left(-\frac{4\pi}{3}\right)=16cis\left(\frac{2\pi}{3}\right)$.

Q9 $\frac{x}{3(x+c)^2}=\frac{\frac{x}{3}}{(x+c)^2}=\frac{A}{x+c}+\frac{B}{(x+c)^2}$.

Q10 $y=\sin^{-1}(2x)$, $x=\frac{1}{2}\sin y$, $x^2=\frac{1}{4}\sin^2 y=\frac{1}{8}(1-\cos 2y)$.

When $x=0$, $y=0$. When $x=\frac{1}{2}$, $y=\frac{\pi}{2}$.

$V=\int_0^{\frac{\pi}{2}}\pi x^2 dy=\frac{\pi}{8}\int_0^{\frac{\pi}{2}}(1-\cos 2y)dy$.

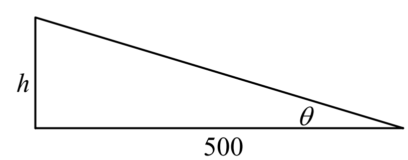
Q11 Refer to itute.com Specialist Maths summary sheets.

Numerical solution of $\frac{dy}{dx}=f(x)$ when $x=b$, given $y=k$ when $x=a$, is $y=\int_a^b f(x)dx+k$.

If $\frac{dy}{dx}=\sqrt{\sin x}$ and $y=1$ when $x=0$,

then $y=\int_0^{\frac{\pi}{3}}\sqrt{\sin x}dx+1$ when $x=\frac{\pi}{3}$.

Q12 Let *h* metres be the altitude of the shuttle at time *t* seconds.



$h=500\tan\theta$, $v=\frac{dh}{dt}=500\sec^2\theta\frac{d\theta}{dt}$.

When $\theta=\frac{\pi}{6}$, $\frac{d\theta}{dt}=0.5$, $\therefore v=500\left(\frac{2}{\sqrt{3}}\right)^2\times 0.5\approx 333\text{ ms}^{-1}$.

Q13 Let $u=\log_e x$. $\frac{du}{dx}=\frac{1}{x}$.

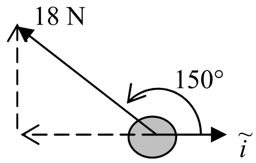
When $x=1$, $u=0$. When $x=e^3$, $u=3$.

$\therefore \int_1^{e^3}\left(\frac{(\log_e x)^3}{x}\right)dx=\int_1^{e^3}u^3\frac{du}{dx}dx=\int_0^3 u^3 du$.

Q14 $\frac{dN}{dt}\propto N(1000-N)$, $\therefore \frac{dN}{dt}=kN(1000-N)$.

Q15 Gradient of line = $-\frac{3}{2}$, $\therefore$ gradient of vector $\perp$ to line = $\frac{2}{3}$.
The vector is $3\tilde{i} + 2\tilde{j}$.

Q16



$\vec{F} = 18\cos 150°\tilde{i} + 18\sin 150°\tilde{j} = -9\sqrt{3}\tilde{i} + 9\tilde{j}$.

Q17 $\cos\theta = \frac{\tilde{a} \bullet \tilde{b}}{|\tilde{a}||\tilde{b}|} = \frac{-4}{9}$, $\therefore \theta = \cos^{-1}\left(\frac{-4}{9}\right) = \pi - \cos^{-1}\left(\frac{4}{9}\right)$.

Q18 $\hat{u} = \frac{1}{3}\left(2\tilde{i} - \tilde{j} - 2\tilde{k}\right)$, $v = a\tilde{i} + 2\tilde{j} - \tilde{k}$ and $\tilde{v} \bullet \hat{u} = 1$.
$\therefore \frac{1}{3}(2a - 2 + 2) = 1$, $\therefore a = \frac{3}{2}$.

Q19 $m = \frac{18}{1.5} = 12$ kg, $v = u + at = 5 + 1.5 \times 4 = 11$ ms^{-1}.
$p = mv = 12 \times 11 = 132$ kgms^{-1}.

Q20 The box slides up the plane, .: friction = 0.2N down the plane.

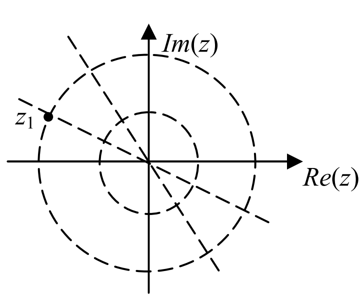
Q21 $v = \sqrt{3x^2 - x^3 + 16}$, $\therefore \frac{1}{2}v^2 = \frac{1}{2}\left(3x^2 - x^3 + 16\right)$.
$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{1}{2}\left(6x - 3x^2\right) = 3x - \frac{3x^2}{2}$,
$\therefore F = ma = 12\left(3x - \frac{3x^2}{2}\right)$.

Q22 $ma = F$, $\therefore m\frac{dv}{dt} = P - mkv^2$.

SECTION 2

Q1a $z_1 = -\sqrt{3} + i$ si in the second quadrant.
$r = \sqrt{\left(-\sqrt{3}\right)^2 + 1^2} = 2$ and $\theta = \tan^{-1}\left(\frac{1}{-\sqrt{3}}\right) = \frac{5\pi}{6}$.
$\therefore z_1 = 2cis\left(\frac{5\pi}{6}\right)$.

Q1b



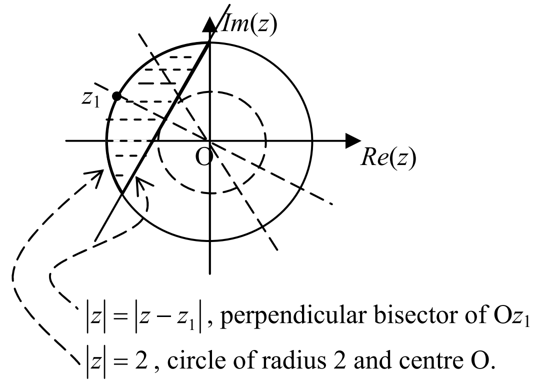
Q1c $z = \frac{2\sqrt{3} \pm \sqrt{12 - 16}}{2} = \sqrt{3} - i$ or $\sqrt{3} + i$.

Q1d $z_1 = -\sqrt{3} + i$
$\therefore \sqrt{3} - i = -z_1$ and $\sqrt{3} + i = -\bar{z}_1$ or $\overline{-z_1}$.

Q1e $|z| = |z - z_1|$. Let $z = x + yi$.
$\sqrt{x^2 + y^2} = \sqrt{\left(x + \sqrt{3}\right)^2 + (y - 1)^2}$, simplify to $y = \sqrt{3}x + 2$.

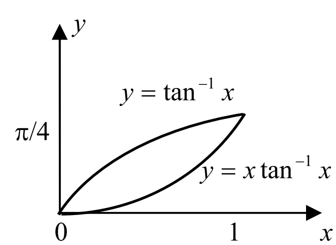
Q1f $\bar{z}_1 = -\sqrt{3} - i$, $x = -\sqrt{3}$ and $y = -1$ satisfy $y = \sqrt{3}x + 2$.
$\therefore \bar{z}_1 = -\sqrt{3} - i$ satisfies $|z| = |z - z_1|$.

Q1g



Q2a $f(x) = x\tan^{-1}x$, $f'(x) = \tan^{-1}x + \frac{x}{1 + x^2}$, $f'(0) = 0$.

Q2b



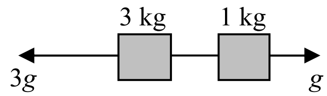
Q2ci and ii $\int_0^1 x\tan^{-1}x\,dx = 0.285$ by graphics calculator.

Q2d $f'(x) = \tan^{-1}x + \frac{x}{1 + x^2}$,
$\int f'(x)dx = \int \tan^{-1}x\,dx + \int \frac{x}{1 + x^2}dx$,
$f(x) = \int \tan^{-1}x\,dx + \int \frac{\frac{1}{2}}{u}du$, where $u = 1 + x^2$.
$\int \tan^{-1}x\,dx = x\tan^{-1}x - \frac{1}{2}\log_e\left(1 + x^2\right)$.

Q2e $\int_0^1 \tan^{-1}x\,dx = \left[x\tan^{-1}x - \frac{1}{2}\log_e\left(1 + x^2\right)\right]_0^1 = \frac{\pi}{4} - \frac{1}{2}\log_e 2$.

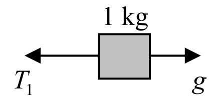
Q2f Enclosed area $= \frac{\pi}{4} - \frac{1}{2}\log_e 2 - 0.285 \approx 0.15$.

Q3a An equivalent diagram:



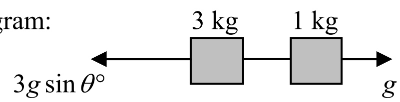
$a = \frac{R}{m} = \frac{3g - g}{4} = \frac{g}{2}$.

Q3b Consider the 1-kg mass:



$R = ma$, $T_1 - g = 1 \times \frac{g}{2}$, $T_1 = \frac{3g}{2}$.

Q3c An equivalent diagram:



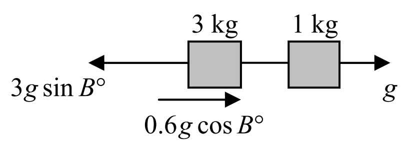
$a = \frac{R}{m}$, $b = \frac{3g \sin \theta° - g}{4} = \frac{(3 \sin \theta° - 1)g}{4}$.

When $\theta° = 30°$, $b = \frac{g}{8}$.

Q3d Let $b = \frac{(3 \sin \theta° - 1)g}{4} = 0$, $\therefore \sin \theta° = \frac{1}{3}$, $\theta° = 19.5°$.

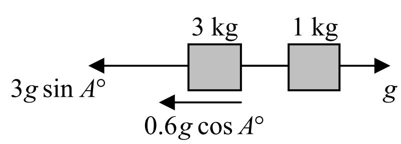
Q3e The normal reaction on the 3-kg mass $= 3g \cos \theta°$.
The force of friction $= \mu \times 3g \cos \theta° = 0.6g \cos \theta°$.

On the verge of sliding down, the force of friction points up along the plane.
An equivalent diagram:



$R = 3g \sin B° - 0.6g \cos B° - g = 0$, $B° = 30.4°$ by graphics calculator.

On the verge of sliding up, the force of friction points down along the plane.
An equivalent diagram:



$R = g - 3g \sin A° - 0.6g \cos A° = 0$, $A° = 7.8°$ by graphics calculator.

Q4a $\tilde{v} = 30\tilde{i} - 40\tilde{j} - 4\tilde{k}$.

$\tilde{r}(t) = \int \tilde{v} dt = 30t.\tilde{i} - 40t.\tilde{j} - 4t.\tilde{k} + \tilde{c}$.

Use $\tilde{r}(10) = -500\tilde{i} + 2500\tilde{j} + 200\tilde{k}$ to find

$\tilde{c} = -800\tilde{i} + 2900\tilde{j} + 240\tilde{k}$.

$\therefore \tilde{r}(t) = (30t - 800)\tilde{i} + (2900 - 40t)\tilde{j} + (240 - 4t)\tilde{k}$.

Q4b Landing when $240 - 4t = 0$, $t = 60$.

$\tilde{r}(60) = 1000\tilde{i} + 500\tilde{j}$, distance $= \sqrt{1000^2 + 500^2} = 1118$ m.

Q4c Consider the velocity vector $\tilde{v} = 30\tilde{i} - 40\tilde{j} - 4\tilde{k}$.

$\tan \theta° = \frac{4}{\sqrt{30^2 + 40^2}} = \frac{4}{50}$, $\theta° \approx 4.6°$.

Q4d Closest when $\tilde{v} \perp \tilde{r}(t)$, i.e. $\tilde{v} \bullet \tilde{r}(t) = 0$.

$\therefore 30(30t - 800) - 40(2900 - 40t) - 4(240 - 4t) = 0$, $\therefore t \approx 56$.

Q4e Speed $= |\tilde{v}| = \sqrt{30^2 + (-40)^2 + (-4)^2} = 50.1597$ ms^{-1}.
Distance = speed × time $= 50.1597 \times 60 \approx 3010$.

Q5a $v(t) = 20 - 2\tan^{-1} t$, $17 = 20 - 2\tan^{-1} t$, $t = \tan \frac{3}{2} \approx 14.1$

Q5b As $t \to \infty$, $\tan^{-1} t \to \frac{\pi}{2}$, $v \to (20 - \pi)^+$,

$\therefore v > 20 - \pi > 16$.

Q5c Distance (polluting car) $= \int_0^T (20 - 2\tan^{-1} t) dt$.

Q5d Distance (police car) $= \int_3^8 13\cos^{-1}\left(\frac{13 - 2t}{7}\right) dt$.

Polluting car ahead of police car by
$\int_0^8 (20 - 2\tan^{-1} t) dt - \int_3^8 13\cos^{-1}\left(\frac{13 - 2t}{7}\right) dt \approx 60.7$ by graphics calculator.

Q5e When $t \geq 8$, speed of police car $= 13\cos^{-1}\left(-\frac{3}{7}\right)$.

Between $t = 8$ and $t = T_c$,

total distance (police car) $= (T_c - 8) \times 13\cos^{-1}\left(-\frac{3}{7}\right)$,

total distance (polluting car) $= \int_8^{T_c} (20 - 2\tan^{-1} t) dt$.

$\therefore (T_c - 8) \times 13\cos^{-1}\left(-\frac{3}{7}\right) = 60.7 + \int_8^{T_c} (20 - 2\tan^{-1} t) dt$.

Q5f For $t \geq 8$, $20 - 2\tan^{-1} t \approx 17$,

$\therefore \int_8^{T_c} (20 - 2\tan^{-1} t) dt \approx 17(T_c - 8)$.

$\therefore (T_c - 8) \times 13\cos^{-1}\left(-\frac{3}{7}\right) \approx 60.7 + 17(T_c - 8)$.

$\therefore T_c \approx 15$.

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