2007 Specialist Maths Trial Exam 2 Solutions

Section 1

1	2	3	4	5	6	7	8	9	10	11
D	D	C	C	C	E	E	A	E	B	A

12	13	14	15	16	17	18	19	20	21	22
B	C	B	A	B	A	C	C	B	E	D

Q1 Hyperbola in the form $\frac{x^2}{b^2}-\frac{y^2}{p^2}=1$, where b is the x-intercept and p a constant. $\frac{y^2}{p^2}=\frac{x^2}{b^2}-1$, $y^2=\frac{p^2}{b^2}\left(x^2-b^2\right)$,

$y=\pm\frac{p}{b}\sqrt{x^2-b^2}$.

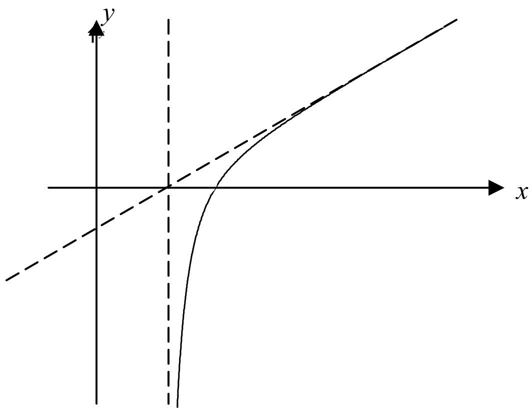
Q2 $\frac{1}{1+\sin x}=\frac{1(1-\sin x)}{(1+\sin x)(1-\sin x)}=\frac{1-\sin x}{1-\sin^2 x}=\frac{1-\sin x}{\cos^2 x}$

$=\frac{1}{\cos^2 x}-\frac{\sin x}{\cos^2 x}=\frac{1}{\cos^2 x}-\frac{\sin^2 x}{\sin x\cos^2 x}=\frac{1}{\cos^2 x}-\frac{1}{\sin x}\frac{\sin^2 x}{\cos^2 x}$

$=\sec^2 x-\operatorname{cosec} x\tan^2 x$

Q3 $\cos\left(\sin^{-1} ax\right)$ is a many-to-one function with domain $\left[-\frac{1}{a},\frac{1}{a}\right]$. $f(x)$ is one-to-one and has an inverse function if the domain is restricted to $\left[-\frac{1}{a},0\right]$ or $\left[0,\frac{1}{a}\right]$.

Q4



Q5 By definition $i^2=-1$, $\therefore i=\pm\sqrt{-1}$, $\therefore \sqrt{-1}=\pm i$.

Q6 $|z|^2=z\bar{z}=a^2+b^2$, $\therefore |z|=\sqrt{a^2+b^2}$.

Q7 $z^3-iz^2-a^2z+ia^2=z^3-a^2z-iz^2+ia^2$

$=\left(z^3-a^2z\right)-\left(iz^2-ia^2\right)=z\left(z^2-a^2\right)-i\left(z^2-a^2\right)$

$=z\left(z^2-a^2\right)-i\left(z^2-a^2\right)=(z-i)\left(z^2-a^2\right)=(z-i)(z-a)(z+a)$

Q8 $\left\{z: Arg(z-i)\le\frac{\pi}{4}\right\}$ does not include the horizontal ray $\{z: Arg(z-i)=\pi\}$.

Q9 Consider the following two examples.
$f(x)=(x-a)^3$ has an inflection point at $x=a$.
$f'(x)=3(x-a)^2$ and $f''(x)=6(x-a)$, $\therefore f''(a)=0$.
$f(x)=(x-a)^4$ has a turning point (local minimum) at $x=a$.
$f'(x)=4(x-a)^3$ and $f''(x)=12(x-a)^2$, $\therefore f''(a)=0$.
$\therefore f''(a)=0$, where $f(x)$ is a continuous function and $f'(x)$ is defined, indicates that $f(x)$ has a turning point or a point of inflection at $x=a$. To determine which one it is necessary to do a gradient check on both sides of $x=a$.

Q10 $A=4\pi r^2$, $r=\left(\frac{A}{4\pi}\right)^{\frac{1}{2}}$.

$V=\frac{4\pi}{3}r^3=\frac{4\pi}{3}\left(\frac{A}{4\pi}\right)^{\frac{3}{2}}=\frac{1}{6\sqrt{\pi}}A^{\frac{3}{2}}$. $\frac{dV}{dA}=\frac{1}{4\sqrt{\pi}}A^{\frac{1}{2}}=\frac{1}{4}\sqrt{\frac{A}{\pi}}$.

$\frac{dV}{dt}=\frac{dV}{dA}\times\frac{dA}{dt}=\frac{1}{4}\sqrt{\frac{A}{\pi}}\times\frac{dA}{dt}$.

$\therefore 0.25=\frac{1}{4}\sqrt{\frac{\pi}{\pi}}\times\frac{dA}{dt}$, $\therefore \frac{dA}{dt}=1$ m^{2}s^{-1}.

Q11

$\int\frac{-\sqrt{a}}{\sqrt{a-x}}dx=\int-\sqrt{a}(a-x)^{-\frac{1}{2}}dx=\frac{-\sqrt{a}(a-x)^{\frac{1}{2}}}{-\frac{1}{2}}=2\sqrt{a^2-ax}$.

Q12 Let $u=1+4x^2$, $\frac{du}{dx}=8x$, $\frac{1}{2}\frac{du}{dx}=4x$.

$\int\frac{4x}{1+4x^2}dx=\int\frac{\frac{1}{2}}{u}\frac{du}{dx}dx=\frac{1}{2}\int\frac{1}{u}du=\frac{1}{2}\log_e u+c$

$=\frac{1}{2}\log_e\left(1+4x^2\right)+c=\log_e\sqrt{1+4x^2}+c$.

Q13 The volume of the solid formed by revolving $y=\log_e(-x)$, $-2\le x\le -1$ equals the volume of the solid formed by revolving $y=\log_e(x)$, $1\le x\le 2$.

$\therefore V=\int_1^2\pi y^2dx=\pi\int_1^2\left[\log_e(x)\right]^2dx$.

Q14 Gradient $\propto y$, i.e. $\frac{dy}{dx}\propto y$.

Q15 $y=\int_0^{\frac{\pi}{2}}\sin\left(x^2\right)dx+2\approx 0.828+2=2.828$

↑ use graphics calculator

Q16 $v = 2\sin^{-1}(x)$, $\frac{1}{2}v^2 = 2\left(\sin^{-1}(x)\right)^2$,

$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = 4\sin^{-1}(x) \times \frac{1}{\sqrt{1-x^2}} = \frac{4\sin^{-1}(x)}{\sqrt{1-x^2}}$.

Q17 $\boldsymbol{a} = \sqrt{3}\,\boldsymbol{i} - 3\boldsymbol{j}$, $|\boldsymbol{a}| = 2\sqrt{3}$, $\hat{\boldsymbol{a}} = \frac{1}{2\sqrt{3}}\boldsymbol{a} = \frac{1}{2}\boldsymbol{i} - \frac{\sqrt{3}}{2}\boldsymbol{j}$.

$\therefore \cos\alpha° = \frac{1}{2}$, $\cos\beta° = -\frac{\sqrt{3}}{2}$ and $\cos\gamma° = 0$.
$\therefore \alpha = 60$, $\beta = 150$, and $\gamma = 90$.

Q18 From the diagram, $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{d} + \boldsymbol{c}$.
$\therefore$ $\boldsymbol{a} + \boldsymbol{b} - \boldsymbol{c} = \boldsymbol{d}$ and $\boldsymbol{c} - \boldsymbol{a} = \boldsymbol{b} - \boldsymbol{d}$ are true.
It is possible to make $p\boldsymbol{a} + q\boldsymbol{b} + r\boldsymbol{c} = \boldsymbol{0}$, $s\boldsymbol{a} + t\boldsymbol{b} + u\boldsymbol{c} + v\boldsymbol{d} = \boldsymbol{0}$ and $f\boldsymbol{b} + g\boldsymbol{c} + h\boldsymbol{d} = \boldsymbol{0}$ where $f, g, h, p, q, r, s, t, u$ and v are non-zero real numbers. $\therefore$C is false.

Q19 $\boldsymbol{v} = (0.2t)\boldsymbol{i} - \boldsymbol{j}$, $\therefore \boldsymbol{r} = (0.1t^2)\,\boldsymbol{i} - t\boldsymbol{j}$.

$\therefore x = 0.1t^2$, $y = -t$. $\therefore y = -\sqrt{\frac{x}{0.1}} = -\sqrt{10x}$.

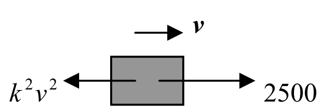
Q20 $\Delta \boldsymbol{p} = m\,\Delta\,\boldsymbol{v} = 0.1[(2\boldsymbol{j} + \sqrt{2}\,\boldsymbol{k}) - (5\boldsymbol{i} - \boldsymbol{j})]$
$= 0.1(-5\boldsymbol{i} + 3\boldsymbol{j} + \sqrt{2}\,\boldsymbol{k})$.
$|\Delta \boldsymbol{p}| = 0.1(6) = 0.6$

Q21 The resultant force on the particle is up the slope. Since the direction of motion is not given, it is not possible to say with certainty the state of motion of the particle.

Q22 The two particles are accelerated by the weight of the 2-kg particle, $\therefore 2 \times 9.8 = (M+2) \times 2.0$, $M = 7.8$

Section 2

Q1a.



Q1bi. $a = \frac{R}{m}$, $a = \frac{2500 - k^2v^2}{2000}$.

Q1bii. $\frac{dv}{dt} = \frac{2500 - k^2v^2}{2000}$, $\frac{dt}{dv} = \frac{2000}{2500 - k^2v^2}$,

$t = \int \frac{2000}{2500 - k^2v^2}dv = 2000\int \frac{1}{(50-kv)(50+kv)}dv$

$\therefore \frac{t}{20} = \int\left(\frac{1}{50-kv} + \frac{1}{50+kv}\right)dv$,

$\frac{t}{20} = -\frac{\log_e(50-kv)}{k} + \frac{\log_e(50+kv)}{k} + c$

$\frac{kt}{20} = \log_e\left(\frac{50+kv}{50-kv}\right) + c$.

When $t = 0$, $v = 0$. $\therefore c = 0$ and $\frac{kt}{20} = \log_e\left(\frac{50+kv}{50-kv}\right)$.

Hence $\frac{50+kv}{50-kv} = e^{\frac{kt}{20}}$, $50 + kv = e^{\frac{kt}{20}}(50 - kv)$, expand and

transpose to make v the subject, $v = \frac{50\left(e^{\frac{kt}{20}} - 1\right)}{k\left(e^{\frac{kt}{20}} + 1\right)}$.

Q1biii. When $t = \frac{20}{k}$, $v = 50$. $\therefore 50 = \frac{50(e-1)}{k(e+1)}$. $\therefore k = \frac{e-1}{e+1}$.

Q1c. $v = 50$ when $t = \frac{20}{k} = \frac{20}{\left(\frac{e-1}{e+1}\right)} = 43$.

Q1d. As $t \to \infty$, max $v \to \frac{50}{k} \approx 108.2$. $\therefore v = 108$.

Q2a. $y = A\cos^{-1}(bx - c) = A\cos^{-1} b\left(x - \frac{c}{b}\right)$ is the transformation of $y = \cos^{-1}(x)$.

The range changes from $[0, \pi]$ to $[0, 20]$, $\therefore A\pi = 20$, $A = \frac{20}{\pi}$.
Before translation the domain changes from $[-1, 1]$ to $[-10, 10]$,
$\therefore \frac{1}{b} = 10$, $b = \frac{1}{10}$. There is a translation of 10 to the right,
$\therefore \frac{c}{b} = 10$, $c = 10b = 1$.

Q2b $y = 5\cos^{-1}(0.125x - 1)$.
x-intercept: Let $y = 0$, $5\cos^{-1}(0.125x - 1) = 0$,
$\cos^{-1}(0.125x - 1) = 0$, $0.125x - 1 = 1$, $x = 16$.

Shaded area $= 2 \times \int_0^{16} 5\cos^{-1}(0.125x - 1)dx = 2 \times 125.664 \approx 251$

use graphics calculator

Q2c. $y = 5\cos^{-1}(0.125x - 1)$.
y-intercept: Let $x = 0$, $y = 5\cos^{-1}(-1) = 5\pi$.

$y = 5\cos^{-1}(0.125x - 1)$, $\cos\left(\frac{y}{5}\right) = 0.125x - 1$,

$x = 8\left(\cos\left(\frac{y}{5}\right) + 1\right)$, $x^2 = 64\left(\cos^2\left(\frac{y}{5}\right) + 2\cos\left(\frac{y}{5}\right) + 1\right)$.

Volume $= \int_0^{5\pi} \pi x^2 dy = 64\pi\int_0^{5\pi}\left(\cos^2\left(\frac{y}{5}\right) + 2\cos\left(\frac{y}{5}\right) + 1\right)dy$

$= 64\pi\int_0^{5\pi}\left(\frac{1}{2}\cos\left(\frac{2y}{5}\right) + 2\cos\left(\frac{y}{5}\right) + \frac{3}{2}\right)dy$

$= 64\pi\left[\frac{5\sin\left(\frac{2y}{5}\right)}{4} + 10\sin\left(\frac{y}{5}\right) + \frac{3y}{2}\right]_0^{5\pi}$

$= 480\pi^2$ m^3.

Q3ai. $\frac{d}{dx}\left(\frac{v^2}{2}\right) = -2x$, $\frac{v^2}{2} = \int(-2x)dx$, $\therefore \frac{v^2}{2} = -x^2 + c$.

$v = -1$ at $x = 0$. $\therefore c = \frac{1}{2}$.

$\therefore v^2 = -2x^2 + 1$, $v = \pm\sqrt{1-2x^2}$.

Q3aii. $\frac{dx}{dt} = \pm\sqrt{1-2x^2}$, $\frac{dt}{dx} = \pm\frac{1}{\sqrt{1-2x^2}}$, $t = \pm\int\frac{1}{\sqrt{1-2x^2}}dx$,

$t = \pm\int\frac{1}{\sqrt{1-\left(x\sqrt{2}\right)^2}}dx$, $t = \pm\frac{1}{\sqrt{2}}\sin^{-1}\left(x\sqrt{2}\right) + c$.

When $t = 0$, $x = 0$, $\therefore c = 0$. $\therefore t = \pm\frac{1}{\sqrt{2}}\sin^{-1}\left(x\sqrt{2}\right)$.

$\therefore x = \frac{1}{\sqrt{2}}\sin\left(\pm t\sqrt{2}\right) = \pm\frac{1}{\sqrt{2}}\sin\left(t\sqrt{2}\right)$.

Only $x = -\frac{1}{\sqrt{2}}\sin\left(t\sqrt{2}\right)$ satisfies the initial condition that

$v = -1$ when $t = 0$ $\because v = \frac{dx}{dt} = -\cos t\sqrt{2} = -\cos 0 = -1$.

Q3b. $\frac{d}{dx}\left(\frac{v^2}{2}\right) = -2x$, i.e. $a = -2x$,

$\therefore a = -2\left(-\frac{1}{\sqrt{2}}\sin\left(t\sqrt{2}\right)\right) = \sqrt{2}\sin\left(t\sqrt{2}\right)$.

Q3c. $x = -\frac{1}{\sqrt{2}}\sin\left(t\sqrt{2}\right)$, the particle oscillates about O with an amplitude of $\frac{1}{\sqrt{2}}$ and a period of $\frac{2\pi}{\sqrt{2}}$. The interval between $t = \frac{\pi}{\sqrt{2}}$ and $t = \frac{3\pi}{\sqrt{2}}$ is one period, so the total distance travelled by the particle is $4\times\frac{1}{\sqrt{2}} = 2\sqrt{2}$.

Q3d. $v = \frac{dx}{dt} = -\cos t\sqrt{2}$. When $t = 0$, $v = -1$. When $t = \frac{\pi}{\sqrt{2}}$, $v = 1$. $\therefore \Delta v = 1-(-1) = 2$ ms^{-1}. $\therefore \Delta p = m\Delta v = 5\times 2 = 10$ kgms^{-1}.

Q4a. At $t = 3$, $z = \frac{1.5}{2+\cos\left(\frac{\pi}{2}\right)}cis\left(\frac{\pi}{2}\right) = 0.75i$.

At $t = 9$, $z = \frac{1.5}{2+\cos\left(\frac{3\pi}{2}\right)}cis\left(\frac{3\pi}{2}\right) = -0.75i$.

Q4b. $|z| = \frac{1.5}{2+\cos\left(\frac{\pi}{6}t\right)}$. When $\cos\left(\frac{\pi}{6}t\right) = -1$, $|z| = 1.5$ is the maximum value. When $\cos\left(\frac{\pi}{6}t\right) = 1$, $|z| = 0.5$ is the minimum value.

Q4ci. $z = \frac{1.5}{2+\cos\left(\frac{\pi}{6}t\right)}\left[\cos\left(\frac{\pi}{6}t\right) + i\sin\left(\frac{\pi}{6}t\right)\right]$.

$\therefore \mathrm{Re}(z) = \frac{1.5}{2+\cos\left(\frac{\pi}{6}t\right)}\cos\left(\frac{\pi}{6}t\right) = |z|\cos\left(\frac{\pi}{6}t\right)$.

$\therefore \cos\left(\frac{\pi}{6}t\right) = \frac{\mathrm{Re}(z)}{|z|}$.

$\therefore |z| = \frac{1.5}{2+\cos\left(\frac{\pi}{6}t\right)} = \frac{1.5}{2+\frac{\mathrm{Re}(z)}{|z|}}$.

Q4cii. $|z| = \frac{1.5}{2+\frac{\mathrm{Re}(z)}{|z|}}$, $|z|\left(2+\frac{\mathrm{Re}(z)}{|z|}\right) = 1.5$, $2|z| + \mathrm{Re}(z) = 1.5$.

Let $z = x + yi$, $2\sqrt{x^2+y^2} + x = 1.5$, $4\left(x^2+y^2\right) = (1.5-x)^2$,

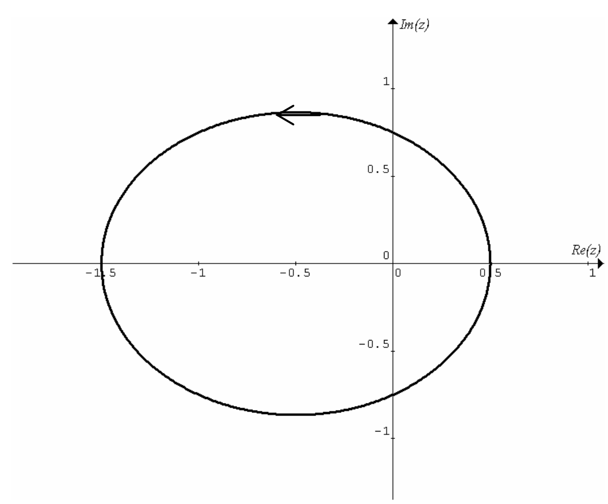
$\therefore 3x^2 + 3x + 4y^2 = 2.25$.

Q4ciii. $3\left(x^2+x\right) + 4y^2 = 2.25$,

$3\left(x^2+x+0.25\right) + 4y^2 = 2.25 + 0.75$, $3(x+0.5)^2 + 4y^2 = 3$,

$(x+0.5)^2 + \frac{4y^2}{3} = 1$, $\frac{(x+0.5)^2}{1^2} + \frac{y^2}{\left(\frac{\sqrt{3}}{2}\right)^2} = 1$.

Q4d. The spacecraft follows an elliptical path with a period $T = \frac{2\pi}{\frac{\pi}{6}} = 12$. At $t = 0$, $z = 0.5$, its position is (0.5, 0). At $t = 3$, $z = 0.75i$, its position is (0, 0.75). $\therefore$ its motion is anticlockwise.



Q5a. M is the midpoint of $\overline{BC}$.
$\overrightarrow{OM} = \overrightarrow{OB} + \overrightarrow{BM} = \overrightarrow{OB} + \frac{1}{2}\overrightarrow{BC} = \boldsymbol{b} + \frac{1}{2}(\boldsymbol{c} - \boldsymbol{b}) = \frac{1}{2}(\boldsymbol{b} + \boldsymbol{c})$.

Q5bi. $\overrightarrow{OM} = \frac{1}{2}(\boldsymbol{b} + \boldsymbol{c}) = -m\boldsymbol{a}$, $\overrightarrow{ON} = \frac{1}{2}(\boldsymbol{c} + \boldsymbol{a}) = -n\boldsymbol{b}$
$\therefore \boldsymbol{c} = -\boldsymbol{b} - 2m\boldsymbol{a}$ and $\boldsymbol{c} = -\boldsymbol{a} - 2n\boldsymbol{b}$
$\therefore -\boldsymbol{b} - 2m\boldsymbol{a} = -\boldsymbol{a} - 2n\boldsymbol{b}$
$\therefore (1-2m)\boldsymbol{a} - (1-2n)\boldsymbol{b} = \boldsymbol{0}$.

Q5bii. Since $\boldsymbol{a}$ and $\boldsymbol{b}$ are not parallel, they are linearly independent, $\therefore 1 - 2m = 0$ and $1 - 2n = 0$. $\therefore m = n = \frac{1}{2}$.
From Q5bi, $\boldsymbol{c} = -\boldsymbol{b} - 2m\boldsymbol{a}$, $\therefore \boldsymbol{c} = -\boldsymbol{b} - \boldsymbol{a}$, $\therefore \boldsymbol{b} = -(\boldsymbol{c} + \boldsymbol{a})$.

Q5ci. $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = -p\boldsymbol{c} - \boldsymbol{a}$.
$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \boldsymbol{b} - \boldsymbol{a} = -(\boldsymbol{c} + \boldsymbol{a}) - \boldsymbol{a} = -\boldsymbol{c} - 2\boldsymbol{a}$.

Q5cii. Since $\overrightarrow{AP} // \overrightarrow{AB}$, $\therefore \overrightarrow{AP} = q\overrightarrow{AB}$ where $q \in R^+$,
i.e. $-p\boldsymbol{c} - \boldsymbol{a} = q(-\boldsymbol{c} - 2\boldsymbol{a}) = -q\boldsymbol{c} - 2q\boldsymbol{a}$
$\therefore 2q = 1$ and $p = q$
i.e. $p = q = \frac{1}{2}$.
$\therefore P$ is the midpoint of $\overrightarrow{AB}$ and $\therefore \overrightarrow{CP}$ is a median.

Please inform mathline@itute.com re conceptual, mathematical and/or typing errors

