Specialist Mathematics Trial Exam 1 Solutions 2007
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Q1a $\frac{x^2}{3} - \frac{y^2}{2} = 1$, $x, y \in R$.

Implicit differentiation, $\frac{2x}{3} - y\frac{dy}{dx} = 0$, $\therefore \frac{dy}{dx} = \frac{2x}{3y}$.

Q1b $\frac{dy}{dx} = -1$, $\therefore \frac{2x}{3y} = -1$, $\therefore y = -\frac{2x}{3}$.

Since $\frac{x^2}{3} - \frac{y^2}{2} = 1$, $\therefore \frac{x^2}{3} - \frac{\left(-\frac{2x}{3}\right)^2}{2} = 1$, $\frac{x^2}{3} - \frac{2x^2}{9} = 1$,

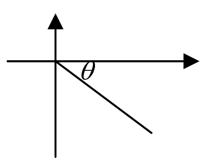
$\frac{x^2}{9} = 1$, $\therefore x = \pm 3$ and $y = \mp 2$.

$\therefore (3,-2)$ or $(-3,2)$.

Q2a $\boldsymbol{a} = 12\boldsymbol{i} - 5\boldsymbol{j}$, resultant force $\boldsymbol{R} = m\boldsymbol{a} = 24\boldsymbol{i} - 10\boldsymbol{j}$.

$|\boldsymbol{R}| = \sqrt{24^2 + (-10)^2} = 26$ N.

Direction: At $\theta = \tan^{-1}\left(\frac{-10}{24}\right) = \tan^{-1}\left(\frac{-5}{12}\right)$ with $\boldsymbol{i}$.



Q2b At $t = 0$, velocity $\boldsymbol{v} = 0$. $\therefore$ $\boldsymbol{v} = \int 12\,\boldsymbol{i} - 5\boldsymbol{j}\,dt = 12t\boldsymbol{i} - 5t\boldsymbol{j}$.

At $t = 2$, displacement $\boldsymbol{s} = \int_0^2 12t\,\boldsymbol{i} - 5t\,\boldsymbol{j}\,dt = [6t^2\,\boldsymbol{i} - 2.5t^2\boldsymbol{j}]_0^2$

$= 24\boldsymbol{i} - 10\boldsymbol{j}$.

Q3a $\boldsymbol{r}(t) = 2t\boldsymbol{i} - (5t + 1)\boldsymbol{j} + 2\boldsymbol{k}$, $\boldsymbol{v} = \frac{d}{dt}\boldsymbol{r}(t) = 2\boldsymbol{i} - 5\boldsymbol{j}$.

Velocity $\boldsymbol{v}$ is constant, $\therefore$ the particle moves in a straight line.

Q3b Speed = $|\boldsymbol{v}| = \sqrt{2^2 + (-5)^2} = \sqrt{29}$.

Q4 Let $u = \log_e(2x)$, $\frac{du}{dx} = \frac{1}{x}$.

$x\frac{dy}{dx} - \log_e(2x) = 0$, $\frac{dy}{dx} = \frac{\log_e(2x)}{x}$,

$y = \int \frac{\log_e(2x)}{x}dx = \int u\frac{du}{dx}dx = \int u\,du = \frac{1}{2}u^2 + c = \frac{1}{2}(\log_e(2x))^2 + c$

Since $f\left(\frac{1}{2}\right) = 0$, $\therefore c = 0$, $\therefore y = \frac{1}{2}(\log_e(2x))^2$.

Q5a $T_{CD} = 5g = 5 \times 9.8 = 49$ N

Q5b $\frac{T_{BC}}{T_{CD}} = \frac{0.6}{1.0}$, $T_{BC} = 0.6T_{CD} = 0.6 \times 49 = 29.4$ N

Q6a $P(-1,0,1)$, $Q(1,-2,2)$ and $R(2,1,0)$.

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}) - (-\boldsymbol{i} + \boldsymbol{k}) = 2\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$

Q6b $\left|\overrightarrow{PQ}\right| = \sqrt{2^2 + (-2)^2 + 1^2} = 3$.

Unit vector in the direction of $\overrightarrow{PQ} = \frac{1}{3}(2\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k})$.

$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = (2\boldsymbol{i} + \boldsymbol{j}) - (-\boldsymbol{i} + \boldsymbol{k}) = 3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$.

Scalar resolute of $\overrightarrow{PR}$ in the direction of $\overrightarrow{PQ}$

$= (3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}) \bullet \frac{1}{3}(2\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}) = 1$.

Vector resolute of $\overrightarrow{PR}$ in the direction of $\overrightarrow{PQ}$

$= 1 \times \frac{1}{3}(2\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}) = \frac{1}{3}(2\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k})$.

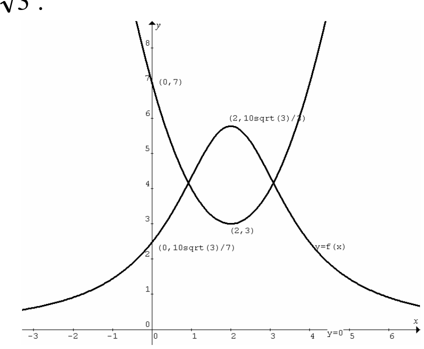
Vector resolute of $\overrightarrow{PR}$ perpendicular to $\overrightarrow{PQ}$

$= (3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}) - \frac{1}{3}(2\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}) = \frac{1}{3}(7\boldsymbol{i} + 5\boldsymbol{j} - 4\boldsymbol{k})$.

$\therefore$ shortest distance $= \frac{1}{3}\sqrt{7^2 + 5^2 + (-4)^2} = \sqrt{10}$.

Q7a $f(x) = \frac{10\sqrt{3}}{x^2 - 4x + 7} = \frac{10\sqrt{3}}{3 + x^2 - 4x + 4} = \frac{10\sqrt{3}}{3 + (x-2)^2}$.

Q7b Sketch the graph of the quadratic function $y = x^2 - 4x + 7$, then the graph of the reciprocal with dilation factor of $10\sqrt{3}$.



Q7c Area

$= \int_{-1}^{3} \frac{10\sqrt{3}}{x^2 - 4x + 7}dx = 10\int_{-1}^{3} \frac{\sqrt{3}}{3 + (x-2)^2}dx = 10\left[\tan^{-1}\left(\frac{x-2}{\sqrt{3}}\right)\right]_{-1}^{3}$

$= 10\left[\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) - \tan^{-1}\left(-\sqrt{3}\right)\right] = 10\left(\frac{\pi}{6} + \frac{\pi}{3}\right) = 5\pi$.

Q8a Use the quadratic formula to find the zeros of $x^2+i2\sqrt{3}x-4$.

$$x=\frac{-i2\sqrt{3}\pm\sqrt{(i2\sqrt{3})^2-4(1)(-4)}}{2(1)}=\frac{-i2\sqrt{3}\pm 2}{2}=\pm 1-i\sqrt{3}.$$

$\therefore x^2+i2\sqrt{3}x-4=(x-1+i\sqrt{3})(x+1+i\sqrt{3})$.

Q8b Express $1-i\sqrt{3}$ in polar form. $1-i\sqrt{3}=2cis\left(-\frac{\pi}{3}\right)$.

$$\sqrt{1-i\sqrt{3}}=\left[2cis\left(-\frac{\pi}{3}+2k\pi\right)\right]^{\frac{1}{2}}=\sqrt{2}cis\left(-\frac{\pi}{6}+k\pi\right)$$

$=\sqrt{2}cis\left(-\frac{\pi}{6}\right)=\sqrt{2}\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)=\frac{\sqrt{2}}{2}(\sqrt{3}-i)$ when $k=0$,

or $=\sqrt{2}cis\left(\frac{5\pi}{6}\right)=\sqrt{2}\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)=-\frac{\sqrt{2}}{2}(\sqrt{3}-i)$ when $k=1$.

Similarly, $-1-i\sqrt{3}=2cis\left(-\frac{2\pi}{3}\right)$.

$$\sqrt{-1-i\sqrt{3}}=\left[2cis\left(-\frac{2\pi}{3}+2k\pi\right)\right]^{\frac{1}{2}}=\sqrt{2}cis\left(-\frac{\pi}{3}+k\pi\right)$$

$=\sqrt{2}cis\left(-\frac{\pi}{3}\right)=\sqrt{2}\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)=\frac{\sqrt{2}}{2}(1-i\sqrt{3})$ when $k=0$,

or $=\sqrt{2}cis\left(\frac{2\pi}{3}\right)=\sqrt{2}\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=-\frac{\sqrt{2}}{2}(1-i\sqrt{3})$ when $k=1$.

Q8c $x^4+i2\sqrt{3}x^2-4=(x^2-1+i\sqrt{3})(x^2+1+i\sqrt{3})$

$$=\left(x-\frac{\sqrt{2}}{2}(\sqrt{3}-i)\right)\left(x+\frac{\sqrt{2}}{2}(\sqrt{3}-i)\right)\left(x-\frac{\sqrt{2}}{2}(1-i\sqrt{3})\right)\left(x+\frac{\sqrt{2}}{2}(1-i\sqrt{3})\right)$$

Q9a $v(t)=\frac{5(1-2t)}{1+2t}$, $t\ge 0$.

When $v=0$, $\frac{5(1-2t)}{1+2t}=0$, $1-2t=0$, $t=\frac{1}{2}$.

Q9b At $0\le t<\frac{1}{2}$, $v>0$, the particle is moving *forwards*.

$$\text{Displacement}=\int_0^{\frac{1}{2}}\frac{5(1-2t)}{1+2t}dt=\int_0^{\frac{1}{2}}\left(\frac{10}{2t+1}-5\right)dt$$

$=\left[5\log_e|2t+1|-5t\right]_0^{\frac{1}{2}}=5\left(\log_e 2-\frac{1}{2}\right)$.

Distance = |displacement| $=5\left(\log_e 2-\frac{1}{2}\right)$

At $\frac{1}{2}<t\le 1$, $v<0$, the particle is moving *backwards*.

$$\text{Displacement}=\int_{\frac{1}{2}}^{1}\left(\frac{10}{2t+1}-5\right)dt$$

$=\left[5\log_e|2t+1|-5t\right]_{\frac{1}{2}}^{1}=5(\log_e 3-1)-5\left(\log_e 2-\frac{1}{2}\right)=5\left(\log_e\frac{3}{2}-\frac{1}{2}\right)$

Distance = |displacement| $=-5\left(\log_e\frac{3}{2}-\frac{1}{2}\right)$.

Total distance $=5\left(\log_e 2-\frac{1}{2}\right)-5\left(\log_e\frac{3}{2}-\frac{1}{2}\right)=5\log_e\frac{4}{3}$.

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