# THE SCHOOL FOR EXCELLENCE
# UNIT 4 SPECIALIST MATHEMATICS 2006
# COMPLIMENTARY WRITTEN EXAMINATION 2 - SOLUTIONS

## SECTION 1 – MULTIPLE CHOICE QUESTIONS

| | |
|---|---|
| **QUESTION 1** | Answer is D |
| **QUESTION 2** | Answer is B |
| **QUESTION 3** | Answer is E |
| **QUESTION 4** | Answer is A |
| **QUESTION 5** | Answer is D |
| **QUESTION 6** | Answer is C |
| **QUESTION 7** | Answer is B |
| **QUESTION 8** | Answer is D |
| **QUESTION 9** | Answer is B |
| **QUESTION 10** | Answer is E |
| **QUESTION 11** | Answer is E |
| **QUESTION 12** | Answer is B |
| **QUESTION 13** | Answer is C |
| **QUESTION 14** | Answer is A |
| **QUESTION 15** | Answer is B |
| **QUESTION 16** | Answer is A |
| **QUESTION 17** | Answer is B |
| **QUESTION 18** | Answer is A |
| **QUESTION 19** | Answer is B |
| **QUESTION 20** | Answer is C |
| **QUESTION 21** | Answer is D |
| **QUESTION 22** | Answer is E |

# SECTION 2 – EXTENDED ANSWER QUESTIONS

**QUESTION 1**

**a.** (i) $T_0 = A + T_r$

$A = T_0 - T_r$
$A = T_0 - 20$

(ii) $A = 90 - 20 = 70$

$80 = 70e^{-5k} + 20$
$\frac{6}{7} = e^{-5k}$
$k = \frac{-1}{5}\log_e \frac{6}{7}$
$k = 0.03083$

(iii) $T = 70e^{-0.03083t} + 20$

$t = -\frac{1}{0.03083}\log_e\left(\frac{T-20}{70}\right)$ $t_1 = 22.48\text{sec}$
$t_2 = 33.40\text{sec}$

**b.** (i) $T_0 = \frac{5m + 90v}{250}$

$m + v = 250$
$v = 250 - m$

$T_0 = \frac{5m + 90(250-m)}{250} = \frac{5m + 22500 - 90m}{250}$

$T_0 = -\frac{17m}{50} + 90$

$A = T_0 - 20 = -\frac{17m}{50} + 70$

(ii) $T(t) = \left(-\frac{17m}{50} + 70\right)e^{-kt} + 20$

$T_0 = -\frac{17 \times 5}{50} + 90 = 88.3^\circ C$

$A = 68.3^\circ C$

(iii) $T(t) = 68.3e^{-kt} + 20$

$$k = -\frac{1}{10}\log_e \frac{78.3 - 20}{68.3}$$
$$k = 0.02480$$

**c.** Apply Euler's method with step size $\Delta m = 1$

$k(0) = 0.03083$ Use $k(0+h) \approx k(0) + h \times \frac{dk}{dm}(m=0)$

$$k(1) \approx k(0) + 1 \times a\log_e(1) = 0.03083$$

$$k(2) \approx k(1) + 1 \times a\log_e(2) = 0.03083 + 0.69315a$$

$$k(3) \approx k(2) + 1 \times a\log_e(3) = 0.03083 + 0.69315a + 1.09861a = 0.03083 + 1.79176a$$

$$k(4) \approx k(3) + 1 \times a\log_e(4) = 0.03083 + 1.79176a + 1.38629a = 0.03083 + 3.17805a$$

$$k(5) \approx k(4) + 1 \times a\log_e(5) = 0.03083 + 3.17805a + 1.60944a$$
$$= 0.03083 + 4.78749a = 0.02480$$

$$a = \frac{0.02480 - 0.03083}{4.78749} = -0.00126$$

**Alternatively:**

Use a solution expressed in integral form:

$$k = a\int_0^m \log_e(u+1)\,du + 0\cdot 03083.$$

Substitute $k = 0\cdot 02480$ when $m = 5$:

$$0\cdot 02480 = a\int_0^5 \log_e(u+1)\,du + 0\cdot 03083.$$

Solve for $a$:

$$a = \frac{0\cdot 02480 - 0\cdot 03083}{\int_0^5 \log_e(u+1)\,du} = \frac{0\cdot 02480 - 0\cdot 03083}{5\cdot 75056} = -0\cdot 00105$$

where the graphics or CAS calculator is used to find $\int_0^5 \log_e(u+1)\,du$ .

**d.** (i) $y=(x+1)\log_e(x+1)$ Use product rule.

$$\frac{dy}{dx}=\frac{x+1}{x+1}+1\times\log_e(x+1)=1+\log_e(x+1)$$

(ii) Hence:

$$\int 1+\log_e(x+1).dx=(x+1)\log_e(x+1)+c$$
$$\int \log_e(x+1).dx=(x+1)\log_e(x+1)-x+c$$

$$\frac{dk}{dm}=a\log_e(m+1)\Rightarrow k=a\int\log_e(m+1).dm=a[(m+1)\log_e(m+1)-m+c]$$

(iii) $k=a[(m+1)\log_e(m+1)-m+c]$

$m=0$, $k=0.03083$

$0.03083=ac$ (1)

$m=5$, $k=0.02480$

$0.02480=a[6\log_e(6)-5+c]$ (2)

$0.02480=5.75056a+ac$

(1) into (2) $a=\dfrac{0.02480-0.03083}{5.75056}=-0.00105$

**e.** (i) $\dfrac{T-20}{A}=e^{-kt}$

$$t=\frac{-1}{k}\log_e\left(\frac{T-20}{A}\right)$$

(ii) $t=\dfrac{-1}{k}\log_e\left(\dfrac{T-20}{A}\right)$

$$t=\frac{-1}{-0.00105[(m+1)\log_e(m+1)-m-29.3619]}\log_e\left(\frac{35}{-\frac{17m}{50}+70}\right)$$

**QUESTION 2**

**a.** (i) $p(z) = z^2 - 2z + 2$

$a^2 = -2i$

$p(1-i) = (1-i)^2 - 2(1-i) + 2 = -2i - 2 + 2i + 2 = 0$

Hence $(z-(1-i))$ is a factor of $p(z) = z^2 - 2z + 2$

(ii) As all coefficients are real, Fundamental Theorem of Algebra gives:

$(z-(1+i))$ is a factor of $p(z) = z^2 - 2z + 2$

(iii) $(z+(1-i))(z+(1+i)) = ((z+1)-i)((z+1)+i)$

$$= (z+1)^2 - i^2$$
$$= z^2 + 2z + 2$$

**b.** $(z^2 + 2z + 2)(z^2 - 2z + 2)$ since

$(z^2 + 2z + 2)(z^2 - 2z + 2) = ([z^2 + 2] + 2z)([z^2 + 2] - 2z) = (z^2 + 2)^2 - (2z)^2 = z^4 + 4$.

**c.** Write down the **factors of** $(z^2 + 2z + 2)$ and $(z^2 - 2z + 2)$:

$z = 1 - i$

$z = -1 + i$

$z = 1 + i$

$z = -1 - i$

**QUESTION 3**

**a.** 1.760 kg Mass:

Take up as positive;

$1.76g = T$

$T = 17.248N$

4.000 kg Mass:

Take down plane and away from plane as positive

$R = mg\cos\alpha = \dfrac{16g}{5} = 31.36N$

$4g\sin\alpha - T - \mu R = 0$

$\dfrac{12g}{5} - 1.76g - \dfrac{\mu \times 16g}{5} = 0$

$\dfrac{16g}{25} \times \dfrac{5}{16g} = \mu$

$\mu = 0.2$

**b.** (i) 0.500 kg Mass:

$$T - 0.5g = 0.5a$$
$$T = 0.5a + 0.5g$$

4.000 kg Mass (Friction still acts up the plane):

$$R = \frac{16g}{5}$$

$$4g\sin\alpha - T - \mu R = 4a$$
$$\frac{12g}{5} - 0.5a - 0.5g - \frac{16g}{25} = 4a$$
$$\frac{9a}{2} = \frac{63g}{50}$$
$$a = \frac{7g}{25} = 2.744ms^{-2}$$

(ii) Constant acceleration:

$$u = 0\ ms^{-1}$$
$$t = 2\ \text{sec}$$
$$a = 2.744\ ms^{-2}$$
$$s = ?$$
$$v = ?$$

$$v = u + at$$
$$v = 0 + 2.744 \times 2$$
$$v = 5.488\ ms^{-1}$$

$$s = ut + \frac{1}{2}at^2$$
$$s = \frac{1}{2} \times 2.744 \times 4$$
$$s = 5.488\ m$$

**c.** (i) $m = 0.5 + 0.05t$

(ii) Bucket:

$$T - mg = ma$$
$$T = mg + ma$$

4 kg Mass (Friction still up the plane):

$$R = \frac{16g}{5}$$

$$4g\sin\alpha - T - \mu R = 4a$$
$$\frac{12g}{5} - mg - ma - \frac{16g}{25} = 4a$$
$$g\left(\frac{44}{25} - m\right) = a(m+4)$$
$$a = g\left(\frac{1.76 - m}{4 + m}\right)$$
$$a = g\left(\frac{1.76 - 0.5 - 0.05t}{4 + 0.5 + 0.05t}\right) = g\left(\frac{1.26 - 0.05t}{4.5 + 0.05t}\right)$$

**d.** (i) $bt+c\overline{)-bt+a}$ with quotient $-1$

$rem=(-bt+a)-(-1(bt+c))=a+c$

Hence: $\dfrac{a-bt}{c+bt}=\dfrac{a+c}{c+bt}-1$

(ii) $a=g\left(\dfrac{1.26-0.05t}{4.5+0.05t}\right)=g\left(\dfrac{1.26+4.5}{4.5+0.05t}-1\right)=\dfrac{5.76g}{4.5+.05t}-g$

Therefore $a=\dfrac{dv}{dt}=\dfrac{5.76g}{4.5+.05t}-g$

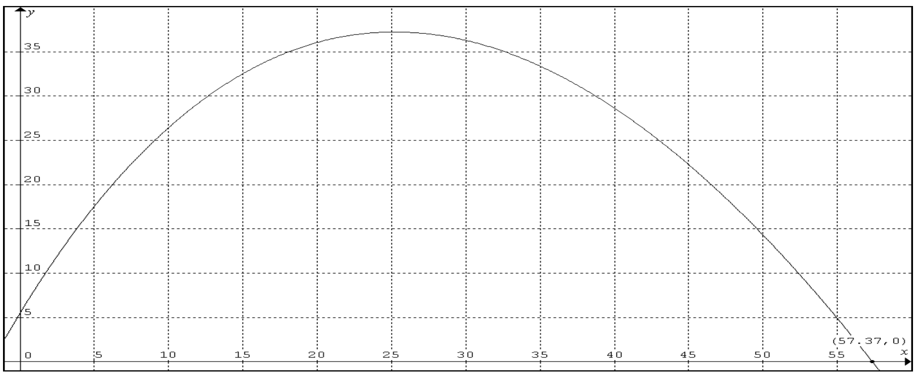
$v=\int\dfrac{5.76g}{4.5+.05t}-g.dt$

$v=\dfrac{5.76g}{0.05}\log_e(4.5+0.005t)-gt+c$
$v=115.2g\log_e(4.5+0.005t)-gt+c$

Set $v=5.488ms^{-1}$ at $t=0$

$5.488=115.2g\log_e(4.5)+c$
$c=-1692.56$
$v(t)=115.2g\log_e(4.5-0.05t)-gt-1692.56$

(iii) $v(t)=115.2g\log_e(4.5-0.05t)-gt-1692.56=0$




Examining graph gives the solution $t=57.37\sec$.

(iv) $v(t) = \frac{dx}{dt} = 115.2g\log_e(4.5 - 0.05t) - gt - 1692.56$

$$\Delta x = \int_0^{57.37} 115.2g\log_e(4.5 - 0.05t) - gt - 1692.56.dt$$

Therefore total displacement $= 5.488 + \int_0^{57.37} 115.2g\log_e(4.5 - 0.05t) - gt - 1692.56.dt$

Use the graphic calc or CAS to evaluate integral.

$$= 5.488 + 1464.81$$
$$= 1470.29m$$

**QUESTION 4**

**a.** $\overrightarrow{AB} = -\overrightarrow{OA} + \overrightarrow{OB}$

$\overrightarrow{AB} = (x-4)i - \underset{\sim}{j} + 2\underset{\sim}{k}$

**b.** $\left|\overrightarrow{AB}\right|^2 = 9 = (x-4)^2 + 1 + 4$

$x^2 - 8x + 21 = 9$

$x^2 - 8x + 12 = 0$

$\left|\overrightarrow{OB}\right| = \sqrt{x^2 + 1 + 4} < 6$

$\therefore x = 2$

$x = 2$ or $x = 6$

Hence $\overrightarrow{AB} = -2i - \underset{\sim}{j} + 2\underset{\sim}{k}$

**c.** (i) $\overrightarrow{AC} = -3i + (y-2)\underset{\sim}{j} + (z+4)\underset{\sim}{k}$

$\left|\overrightarrow{AC}\right| = \sqrt{9 + (y-2)^2 + (z+4^2)}$

(ii) $\overrightarrow{AC} \bullet \overrightarrow{AB} = 6 - (y-2) + 2(z+4) = 16 - y + 2z = 0$

$y = 2z + 16$

(iii) $\left|\overrightarrow{AC}\right| = \sqrt{9+(y-2)^2+\left(z+4^2\right)}$ Use $y = 2z+16$:

Minimum of $\left|\overrightarrow{AC}\right|$ corresponds to the minimum of $\left|\overrightarrow{AC}\right|^2$

$\left|\overrightarrow{AC}\right|^2 = 9+(2z+16-2)^2+(z+4)^2$

$5z^2+64z+221$

Minimum of $5z^2+64z+221$ occurs at $10z+64=0$,

$z=-6.4 \qquad y=5.2$

Hence $\overrightarrow{AC} = -3i+3.2\underset{\sim}{j}-2.4\underset{\sim}{k}$

(iv) $Area = \frac{1}{2}\left|\overrightarrow{AC}\right|\times\left|\overrightarrow{AB}\right| = \frac{1}{2}\times 5\times 3 = 7.5$ square units

**d.** (i) $\underset{\sim}{a} = -2\underset{\sim}{i}-\underset{\sim}{j}+2\underset{\sim}{k}$

$\left|\underset{\sim}{c}\right| = \sqrt{\frac{41}{4}} = \frac{\sqrt{41}}{2}$

$\hat{\underset{\sim}{c}} = \frac{\underset{\sim}{c}}{\left|\underset{\sim}{c}\right|} = \frac{1}{\sqrt{41}}\left(-\underset{\sim}{i}+2\underset{\sim}{j}-6\underset{\sim}{k}\right)$

Parallel Projection, $\underset{\sim}{u} = \underset{\sim}{a}\bullet\hat{\underset{\sim}{c}}\left(\hat{\underset{\sim}{c}}\right) = \frac{-12}{\sqrt{41}}\times\frac{1}{\sqrt{41}}\left(-\underset{\sim}{i}+2\underset{\sim}{j}-6\underset{\sim}{k}\right) = \frac{-12}{41}\left(-\underset{\sim}{i}+2\underset{\sim}{j}-6\underset{\sim}{k}\right)$

Perpendicular projection, $\underset{\sim}{w} = \underset{\sim}{a}-\underset{\sim}{u} = (-2\underset{\sim}{i}-\underset{\sim}{j}+2\underset{\sim}{k})-\left(\frac{-12}{41}\left(-\underset{\sim}{i}+2\underset{\sim}{j}-6\underset{\sim}{k}\right)\right)$

$= \frac{24}{41}\underset{\sim}{i}-\frac{17}{41}\underset{\sim}{j}-\frac{44}{41}\underset{\sim}{k}$

(ii) Area $= \frac{1}{2}\left|\underset{\sim}{c}\right|\left|\underset{\sim}{w}\right|$

$$\left|\underset{\sim}{c}\right| = \frac{\sqrt{41}}{2}$$

$$\left|\underset{\sim}{w}\right| = \sqrt{\left(\frac{24}{41}\right)^2 + \left(\frac{17}{41}\right)^2 + \left(\frac{44}{41}\right)^2} = \frac{1}{41}\sqrt{2801}$$

Area $= \frac{1}{2} \times \frac{\sqrt{41}}{2} \times \frac{\sqrt{2801}}{41} = \frac{\sqrt{2801}}{4\sqrt{41}} = 2.066$ square units