Trial Examination 2006

VCE Specialist Mathematics Units 3 & 4

Written Examination 2

Question and Answer Booklet

Reading time: 15 minutes
Writing time: 2 hours

Student's Name: ______________________________

Teacher's Name: ______________________________

Structure of Booklet

Section	Number of questions	Number of questions to be answered	Number of marks
1	22	22	22
2	5	5	58
			Total 80

Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, one approved graphics calculator or CAS (memory DOES NOT need to be cleared) and, if desired, one scientific calculator.

Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Materials supplied

Question and answer booklet of 22 pages with a detachable sheet of miscellaneous formulas in the centrefold.

Answer sheet for multiple-choice questions.

Instructions

Detach the formula sheet from the centre of this book during reading time.
Write **your name** and your **teacher's name** in the space provided above on this page.
All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

Students are advised that this is a trial examination only and cannot in any way guarantee the content or the format of the 2006 VCE Specialist Mathematics Units 3 & 4 Written Examination 2.

SECTION 1

Instructions for Section 1

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Take the **acceleration due to gravity** to have magnitude g m/s^2, where $g = 9.8$.

Question 1

The graph of $y = \frac{1 - x^3}{4x^2}$ has

A. no straight line asymptotes.

B. $x = 0$ as its only straight line asymptote.

C. $x = 0$ and $y = -\frac{x}{4}$ as its only straight line asymptotes.

D. $x = 0$ and $y = -4x$ as its only straight line asymptotes.

E. $y = 0$ and $y = -\frac{x}{4}$ as its only straight line asymptotes.

Question 2

A hyperbola has asymptotes with equations $y = \sqrt{2}(x + 1) + 3$ and $y = -\sqrt{2}(x + 1) + 3$.
The equation of the hyperbola is

A. $\frac{(x-1)^2}{4} - \frac{(y+3)^2}{8} = 1$

B. $\frac{(x+1)^2}{4} - \frac{(y-3)^2}{8} = 1$

C. $\frac{(x+1)^2}{8} - \frac{(y-3)^2}{4} = 1$

D. $\frac{(x-1)^2}{8} - \frac{(y+3)^2}{4} = 1$

E. $\frac{(x+1)^2}{4} + \frac{(y-3)^2}{8} = 1$

 TEVSMU34EX2_QA_06.FM

Question 3

Which one of the following is **not** equal to $\operatorname{cosec}\left(\frac{\pi}{7}\right)$?

A. $\dfrac{1}{\sin\left(\frac{\pi}{7}\right)}$

B. $\sec\left(-\frac{5\pi}{14}\right)$

C. $\sqrt{\cot^2\left(\frac{\pi}{7}\right) + 1}$

D. $\cot\left(\frac{\pi}{14}\right) - \cot\left(\frac{\pi}{7}\right)$

E. $\cot^2\left(\frac{\pi}{7}\right) + 1$

Question 4

The set of values of c for which $c - 3\arctan(x - 2) < 0$ are

A. $c > 2$

B. $c < 2$

C. $c > -\frac{3\pi}{2}$

D. $c < -\frac{3\pi}{2}$

E. $-\frac{3\pi}{2} < c < \frac{3\pi}{2}$

Question 5

Given that $z = x + yi$, where x and y are real numbers, which one of the following complex numbers is purely imaginary?

A. $z - \bar{z}$

B. $z + \bar{z}$

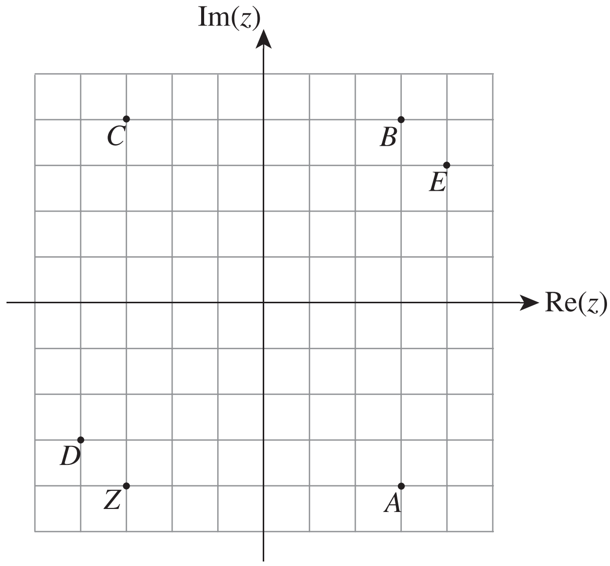
C. $z\bar{z}$

D. $\frac{1}{z} + \frac{1}{\bar{z}}$

E. $z^2 + \bar{z}^2$

Question 6

The point Z on the Argand diagram represents the complex number $z = r\text{cis}(-\theta)$.



The complex number $r\text{cis}(\theta)$ is represented by the point

A. A
B. B
C. C
D. D
E. E

Question 7

A quadratic equation whose roots are $2\text{cis}\left(-\frac{2\pi}{3}\right)$ and $2\text{cis}\left(\frac{2\pi}{3}\right)$ is

A. $z^2 + 4 = 0$
B. $z^2 + 2z - 4 = 0$
C. $z^2 + 2z + 4 = 0$
D. $z^2 - 2z + 4 = 0$
E. $z^2 - 2z - 4 = 0$

Question 8

The values of Re(z) such that z belongs to $S \cap T$ where $S = \{z: |z - 1| \le 1\}$ and $T = \{z: \text{Re}(z) + \text{Im}(z) = 1\}$ are

A. $-\frac{1}{\sqrt{2}} \le \text{Re}(z) \le \frac{1}{\sqrt{2}}$

B. $1 - \frac{1}{\sqrt{2}} \le \text{Re}(z) \le 1 + \frac{1}{\sqrt{2}}$

C. $\text{Re}(z) \le 1 - \frac{1}{\sqrt{2}}$

D. $\text{Re}(z) \ge 1 + \frac{1}{\sqrt{2}}$

E. $1 - \frac{1}{\sqrt{2}} < \text{Re}(z) < 1 + \frac{1}{\sqrt{2}}$

Question 9

Consider the curve with equation $x^2 + xy + y^2 = 12$. There is a point on the curve with coordinates $(2, k)$ at which the tangent to the curve is horizontal.
The value of k is

A. 0
B. –1
C. 1
D. –4
E. 4

Question 10

A point (x, y) is moving along a curve $y = g(x)$. At the instant when the x-coordinate of the point is increasing at the rate of 4 units per second, the y-coordinate is decreasing at the rate of $\frac{4}{3}$ units per second.
The rate of change, in units per second, of the gradient of the curve is

A. $-\frac{1}{3}$

B. $\frac{1}{3}$

C. $-\frac{16}{3}$

D. $\frac{16}{3}$

E. –3

Question 11

With a suitable substitution, $\int_0^{\frac{\pi}{4}} \left(\frac{\cos(x)}{\sin(x) + \cos^2(x)} \right) dx$ can be expressed as

A. $\int_0^{\frac{\pi}{4}} \left(\frac{1}{1 + u - u^2} \right) du$

B. $\int_0^{\frac{1}{\sqrt{2}}} \left(\frac{1}{u^2 - u - 1} \right) du$

C. $\int_0^{\frac{\pi}{4}} \left(\frac{1}{u^2 + u - 1} \right) du$

D. $\int_0^{\frac{1}{\sqrt{2}}} \left(\frac{1}{u^2 + u - 1} \right) du$

E. $\int_0^{\frac{1}{\sqrt{2}}} \left(\frac{1}{1 + u - u^2} \right) du$

Question 12

If $m < n < 0$, then $\int_m^n \frac{1}{x - 2} dx$ is equal to

A. $\log_e(n - 2) - \log_e(m - 2)$

B. $|\log_e(n - 2) - \log_e(m - 2)|$

C. $\log_e|n - 2| - \log_e(m - 2)$

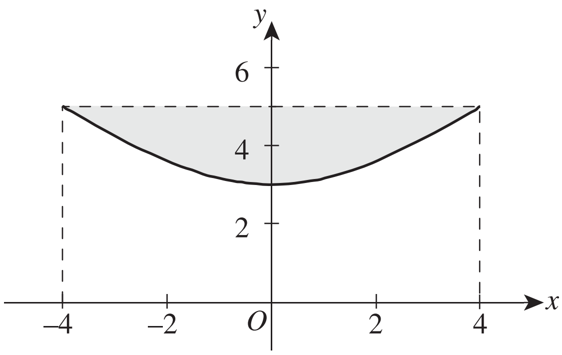
D. $\log_e(n - 2) - \log_e|m - 2|$

E. $\log_e|n - 2| - \log_e|m - 2|$

 TEVSMU34EX2_QA_06.FM

Question 13

The graph of $f : [-4, 4] \rightarrow R$, where $f(x) = \sqrt{x^2 + 9}$ is shown below.



The shaded region is rotated 360° about the x-axis to form a solid of revolution. The volume of this solid, in cubic units, is given by

A. $\pi \int_0^4 (16 - x^2)dx$

B. $2\pi \int_0^4 (16 - x^2)dx$

C. $\pi \int_{-4}^4 (25 - x^2)dx$

D. $2\pi \int_0^4 (x^2 + 9)dx$

E. $2\pi \int_0^4 (34 - x^2)dx$

Question 14

At time t, the position vector of a particle is given by $\underset{\sim}{r}(t) = 2e^t\underset{\sim}{i} + 2e^{-t}\underset{\sim}{j},\ t \geq 0$.

The Cartesian equation of its path is

A. $y = \frac{1}{x}$

B. $y = \frac{1}{e^x}$

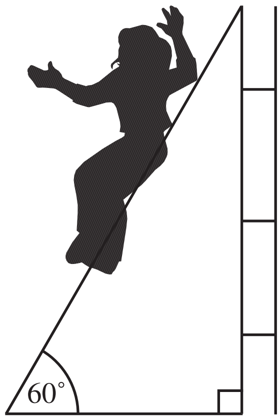
C. $y = \frac{4}{x}$

D. $y = \frac{x}{e}$

E. $y = \frac{e}{x}$

Question 15

Katie has a mass of 20 kg and slides down a smooth slide. The slide makes an angle of 60° with the ground as shown below.



The magnitude of the normal reaction force on Katie in newtons is

A. $10\sqrt{3}g$

B. $20\sqrt{3}g$

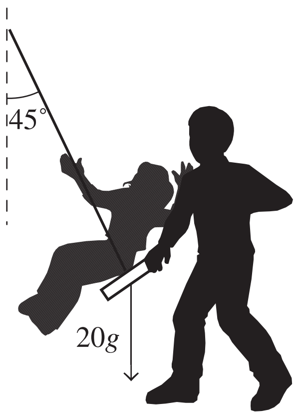
C. $10g$

D. $10\sqrt{3}$

E. $\dfrac{\sqrt{3}g}{2}$

Question 16

Katie sits on a swing while her brother exerts a horizontal force that holds her stationary so that the rope makes an angle of 45° with the vertical. The weight force of Katie and the swing is $20g$.



The magnitude, in newtons, of the tension of the rope is equal to

A. $20g$

B. $2g$

C. $\sqrt{2}g$

D. $20\sqrt{2}g$

E. $10\sqrt{2}g$

Question 17

A body of mass 5 kg is acted on by two coplanar forces, $\underset{\sim}{F}$ and $\underset{\sim}{G}$, where $\underset{\sim}{F} = 2\underset{\sim}{i} + \underset{\sim}{j}$ newtons.

If the resulting acceleration is $\underset{\sim}{i} - 2\underset{\sim}{j}$ m/s^2, then $\underset{\sim}{G}$ is

A. $-\underset{\sim}{i} - 3\underset{\sim}{j}$

B. $3\underset{\sim}{i} - \underset{\sim}{j}$

C. $3\underset{\sim}{i} - 11\underset{\sim}{j}$

D. $11\underset{\sim}{i} + 3\underset{\sim}{j}$

E. $5\underset{\sim}{i} - 10\underset{\sim}{j}$

Question 18

A rocket is fired vertically into the air with an initial velocity of 10 m/s.

Assuming that the propulsive force ceases as soon as it is released, and air resistance is negligible, the time in seconds that it takes to return to the ground is closest to

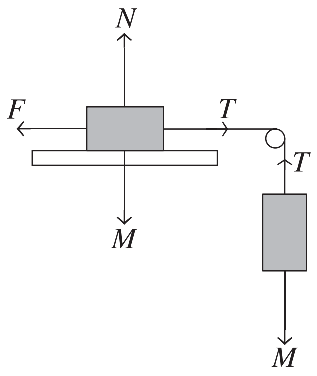
A. 0
B. 1
C. 1.5
D. 2
E. 2.5

Question 19

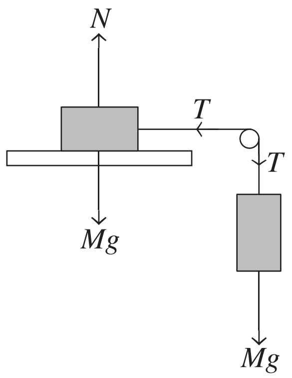
Two bodies of mass M are attached by a light, inextensible string. One body is initially at rest on a rough, horizontal table. The string passes over a smooth pulley and the other body is suspended from the string.

A diagram showing the forces acting on the bodies could be

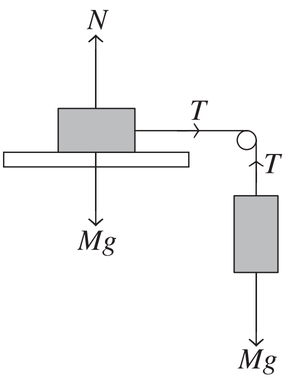
A.



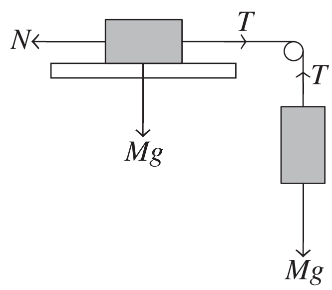
B.



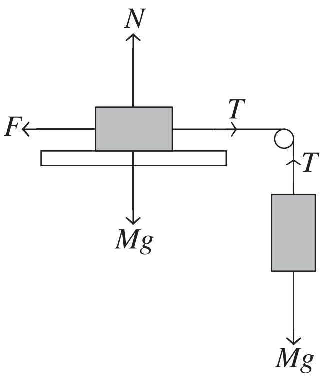
C.



D.

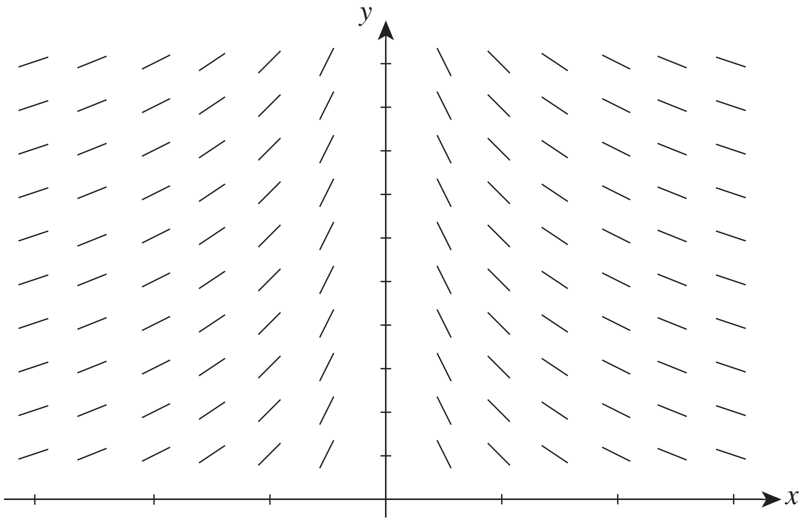


E.



Question 20

A slope field for a first-order differential equation is shown.



A solution could be

A. $y = \frac{1}{x^2}$

B. $y = -\log_e(x)$

C. $y = e^{-x}$

D. $y = \frac{1}{x}$

E. $y = \log_e(-x)$

Question 21

A 5 L container is initially full of fresh water. Saline solution with a concentration of 10 g/L flows into the container at 1 L/min. At the same time, the mixture in the container flows out at 2 L/min.

If S is the quantity of salt in the container at time t minutes, a differential equation describing the situation is

A. $\frac{dS}{dt} = 10 - \frac{2S}{5-t},\ t = 0,\ S = 5$

B. $\frac{dS}{dt} = -\frac{2S}{5-t},\ t = 0,\ S = 0$

C. $\frac{dS}{dt} = 10 - \frac{2S}{5-t},\ t = 0,\ S = 0$

D. $\frac{dS}{dt} = 10 - \frac{2S}{5},\ t = 0,\ S = 5$

E. $\frac{dS}{dt} = 10 - \frac{2S}{5},\ t = 0,\ S = 0$

Question 22

Euler's method, with a step size of 0.1, is used to solve the differential equation $\frac{dy}{dx} = \frac{1}{\cos(x)}$, with $y = 1$ at $x = 0$.

The value obtained for y at $x = 0.2$, correct to four decimal places, is

A. 0.1005
B. 1.2005
C. 1.1005
D. 1.3025
E. 1.1000

Trial Examination 2006

VCE Specialist Mathematics Units 3 & 4

Written Examination 2

Formula Sheet

Directions to students

Detach this formula sheet during reading time. This formula sheet is provided for your reference.

SPECIALIST MATHEMATICS FORMULAS

Mensuration

area of a trapezium:	$\frac{1}{2}(a+b)h$
curved surface area of a cylinder:	$2\pi rh$
volume of a cylinder:	$\pi r^2 h$
volume of a cone:	$\frac{1}{3}\pi r^2 h$
volume of a pyramid:	$\frac{1}{3}Ah$
volume of a sphere:	$\frac{4}{3}\pi r^3$
area of a triangle:	$\frac{1}{2}bc\sin(A)$
sine rule:	$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$
cosine rule:	$c^2 = a^2 + b^2 - 2ab\cos(C)$

Coordinate geometry

ellipse:	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$
hyperbola:	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

Circular (trigonometric) functions

$\cos^2(x) + \sin^2(x) = 1$

$1 + \tan^2(x) = \sec^2(x)$

$\cot^2(x) + 1 = \operatorname{cosec}^2(x)$

$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$

$\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$

$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$

$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$

$\tan(x+y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}$

$\tan(x-y) = \frac{\tan(x) - \tan(y)}{1 + \tan(x)\tan(y)}$

$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$

$\sin(2x) = 2\sin(x)\cos(x)$

$\tan(2x) = \frac{2\tan(x)}{1 - \tan^2(x)}$

function	$\sin^{-1}$	$\cos^{-1}$	$\tan^{-1}$
domain	$[-1, 1]$	$[-1, 1]$	R
range	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$[0, \pi]$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Algebra (Complex numbers)

$z = x + yi = r(\cos(\theta) + i\sin(\theta)) = r\operatorname{cis}(\theta)$

$|z| = \sqrt{x^2 + y^2} = r$

$-\pi < \operatorname{Arg}(z) \le \pi$

$z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$

$\frac{z_1}{z_2} = \frac{r_1}{r_2}\operatorname{cis}(\theta_1 - \theta_2)$

$z^n = r^n \operatorname{cis}(n\theta)$ (de Moivre's theorem)

Calculus

$\frac{d}{dx}(x^n) = nx^{n-1}$ | $\int x^n dx = \frac{1}{n+1}x^{n+1} + c,\ n \neq -1$

$\frac{d}{dx}(e^{ax}) = ae^{ax}$ | $\int e^{ax} dx = \frac{1}{a}e^{ax} + c$

$\frac{d}{dx}(\log_e(x)) = \frac{1}{x}$ | $\int \frac{1}{x} dx = \log_e(x) + c,\ \text{for } x > 0$

$\frac{d}{dx}(\sin(ax)) = a\cos(ax)$ | $\int \sin(ax) dx = -\frac{1}{a}\cos(ax) + c$

$\frac{d}{dx}(\cos(ax)) = -a\sin(ax)$ | $\int \cos(ax) dx = \frac{1}{a}\sin(ax) + c$

$\frac{d}{dx}(\tan(ax)) = a\sec^2(ax)$ | $\int \sec^2(ax) dx = \frac{1}{a}\tan(ax) + c$

$\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}$ | $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c,\ a > 0$

$\frac{d}{dx}(\cos^{-1}(x)) = \frac{-1}{\sqrt{1-x^2}}$ | $\int \frac{-1}{\sqrt{a^2-x^2}} dx = \cos^{-1}\left(\frac{x}{a}\right) + c,\ a > 0$

$\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}$ | $\int \frac{a}{a^2+x^2} dx = \tan^{-1}\left(\frac{x}{a}\right) + c$

product rule: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

chain rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

Euler's method: If $\frac{dy}{dx} = f(x)$, $x_0 = a$ and $y_0 = b$, then $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + hf(x_n)$

acceleration: $a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

constant (uniform) acceleration: $v = u + at$ $\quad s = ut + \frac{1}{2}at^2$ $\quad v^2 = u^2 + 2as$ $\quad s = \frac{1}{2}(u+v)t$

Vectors in two and three dimensions

$\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$

$|\underset{\sim}{r}| = \sqrt{x^2 + y^2 + z^2} = r$

$\underset{\sim}{r}_1 \cdot \underset{\sim}{r}_2 = r_1 r_2 \cos(\theta) = x_1 x_2 + y_1 y_2 + z_1 z_2$

$\dot{\underset{\sim}{r}} = \frac{d\underset{\sim}{r}}{dt} = \frac{dx}{dt}\underset{\sim}{i} + \frac{dy}{dt}\underset{\sim}{j} + \frac{dz}{dt}\underset{\sim}{k}$

Mechanics

momentum:	$\underset{\sim}{p} = m\underset{\sim}{v}$
equation of motion:	$\underset{\sim}{R} = m\underset{\sim}{a}$
sliding friction:	$F \le \mu N$

END OF FORMULA SHEET

TEVSMU34EX2_FS_06.FM

SECTION 2

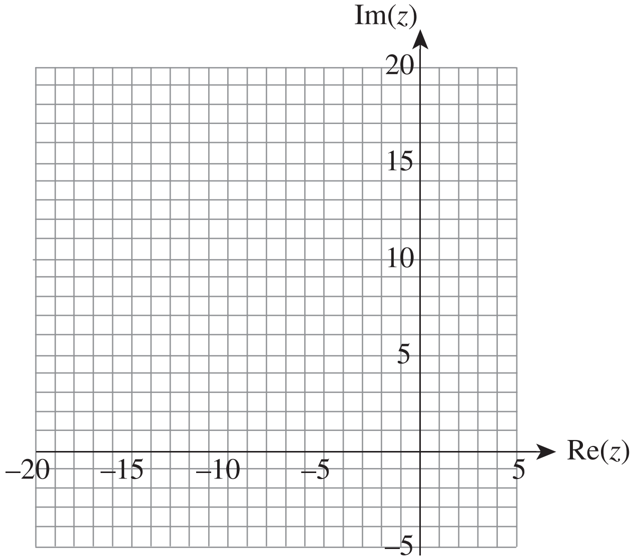
Instructions for Section 2

Answer **all** questions in the spaces provided.
A decimal approximation will not be accepted if an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this booklet are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g m/s^2, where $g = 9.8$.

Question 1

A complex number is given by $u = -8 + 6i$.

a. On the Argand diagram below, plot the point u and shade the region $S = \{z: |z - u| \leq 10\}$.



3 marks

b. Verify that the point $v = -8 - 4i$ lies on the boundary of S.

2 marks

c. Find the complex number w that satisfies the equation $uw = 2 + 36i$.

2 marks

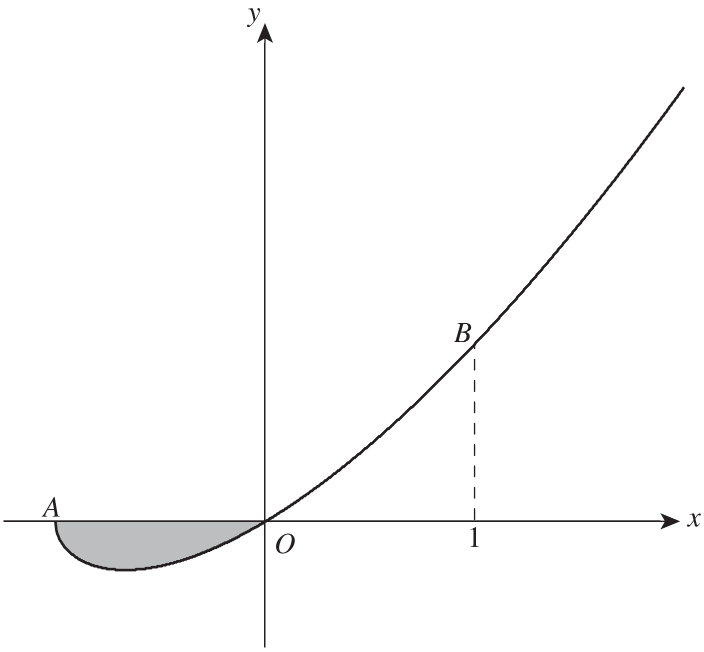
d. Find the exact minimum distance between the boundary of S and w.

2 marks

Total 9 marks

Question 2

A section of the graph with the rule $y = x\sqrt{1 + x}$ is shown below.



a. State the largest domain for which the function is defined.

1 mark

b. State the coordinates of A.

1 mark

c. **i.** Using the substitution $u = x + 1$, write an integral in terms of u to represent the shaded area.

ii. Find the exact area of the shaded region.

3 + 1 = 4 marks

d. The section of the curve from A to B is rotated 360° about the x-axis.

i. Write an integral that represents the volume of revolution formed.

ii. Find the volume correct to two decimal places.

1 + 1 = 2 marks
Total 8 marks

Question 3

Two particles P and Q have position vectors $\underset{\sim}{r}$ and $\underset{\sim}{s}$ respectively, given by $\underset{\sim}{r} = (5\cos(t) + 5)\underset{\sim}{i} + 5\sin(t)\underset{\sim}{j}$ and $\underset{\sim}{s} = t\underset{\sim}{i} + 2t\underset{\sim}{j}$ where t is the time elapsed since the start of the motion.

a. Find the Cartesian equation of the path of P.

2 marks

b. If A is the point (5, 0) and $B(x, y)$ is a point on the path of P, show that the acceleration of particle P is always directed along the line AB.

4 marks

c. Find the Cartesian equation of the path of Q and state its domain.

1 mark

d. Find the coordinates of the point(s) where the paths of P and Q intersect.

2 marks

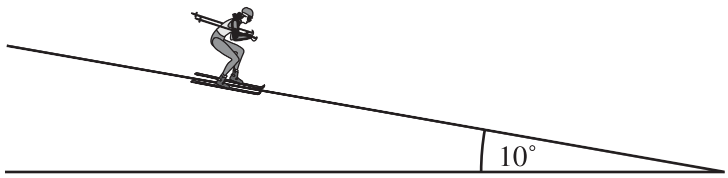
e. Show that the particles do not collide.

2 marks
Total 11 marks

Question 4

A skier of mass 50 kg starts skiing from rest on a straight downhill slope that is inclined at an angle of 10° to the horizontal. While in motion, the skier experiences a retarding force, *R* newtons, that is proportional to the skier's speed, *v* m/s. The snow surface is considered to be smooth enough to ignore friction between the snow and the skis. Time is measured in seconds.

a. On the diagram below, mark in the forces acting on the skier.



1 mark

b. The skier accelerates down the slope at a m/s^2. Show that $a = g\sin(10°) - \frac{k}{50}v$.

2 marks

c. State the skier's exact initial acceleration.

1 mark

d. After a period of time skiing, the skier reaches a terminal speed of 25 m/s.

Show that $a = \frac{g\sin(10°)}{25}(25 - v)$.

3 marks

e. By expressing an appropriate definite integral, find how long it takes for the skier to reach a speed of 10 m/s. Express your answer correct to two decimal places.

3 marks

f. Given that $v = 25\left(1 - e^{-\frac{g\sin(10°)}{25}t}\right)$, find the time taken for the skier's speed to first reach 90% of the terminal speed. Express your answer correct to one decimal place.

3 marks

g. By expressing an appropriate definite integral, find to the nearest metre, the distance travelled by the skier in reaching a speed that is 90% of the terminal speed.

2 marks

h. When travelling at 90% of terminal speed, the skier applies a retarding force to the skis in order to slow down to a safer speed of 5 m/s.

The skier now experiences an overall deceleration of $0.02v^2$.

Find the distance travelled by the skier before reaching the safe speed of 5 m/s.

Give your answer correct to the nearest metre.

3 marks
Total 18 marks

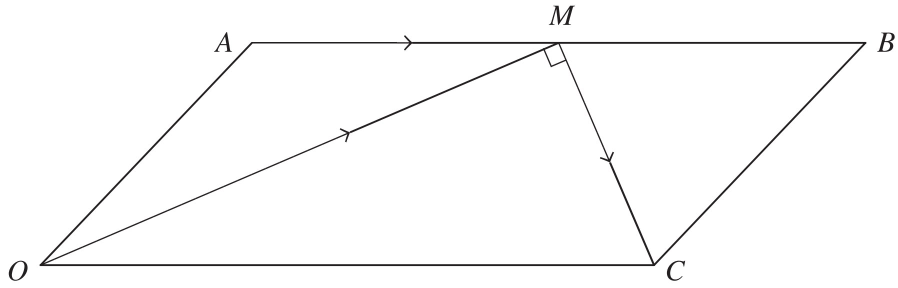
Question 5

$OABC$ is a parallelogram where $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OC} = \underset{\sim}{b}$ and $|\underset{\sim}{b}| = 2|\underset{\sim}{a}|$. The angle between $\overrightarrow{OA}$ and $\overrightarrow{OC}$ is α as shown below.

a. Show that $\underset{\sim}{a}.\underset{\sim}{b} = 2a^2\cos(\alpha)$.

1 mark

Let M be a point on AB such that $\overrightarrow{AM} = m\underset{\sim}{b}$, $0 \le m \le 1$ and $\overrightarrow{OM}.\overrightarrow{MC} = 0$.



b. Write down $\overrightarrow{OM}$ in terms of $\underset{\sim}{a}$, $\underset{\sim}{b}$ and m.

1 mark

c. Show that $\overrightarrow{MC} = (1 - m)\underset{\sim}{b} - \underset{\sim}{a}$.

1 mark

d. **Hence** show that $(1 - 2m)\underset{\sim}{a}.\underset{\sim}{b} - a^2(1 - 2m)^2 = 0$.

3 marks

e. By solving the equation in **d.**, show that $m = \frac{1}{2}$ or $m = \frac{1}{2} - \cos(\alpha)$.

2 marks

f. State the position of M when $m = \frac{1}{2}$.

1 mark

g. Find the set of values for α such that there are **two** possible positions for M.

3 marks

Total 12 marks

END OF QUESTION AND ANSWER BOOKLET