Trial Examination 2006

VCE Specialist Mathematics Units 3 & 4

Written Examination 2

Suggested Solutions

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SECTION 1

Question 1

$x = 0$ is a vertical asymptote.

$y = \frac{1}{4x^2} - \frac{x}{4}$

As $x \to \pm\infty$, $y \to -\frac{x}{4}$.

So $y = -\frac{x}{4}$ is a straight line asymptote.

Answer C

Question 2

For $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, the equations of the asymptotes are $y = \pm\frac{b}{a}(x-h) + k$.

Here, $h = -1$ and $k = 3$.

$\frac{b}{a} = \sqrt{2}$

$b = 2\sqrt{2}$ and $a = 2$ are possible values of *b* and *a*.

Hence $b^2 = 8$ and $a^2 = 4$.

The hyperbola has the equation $\frac{(x+1)^2}{4} - \frac{(y-3)^2}{8} = 1$.

Answer B

Question 3

A is correct, as $\operatorname{cosec}(x) = \frac{1}{\sin(x)}$.

B is correct, as $\operatorname{cosec}\left(x + \frac{\pi}{2}\right) = \sec(x)$.

C is correct, as $\operatorname{cosec}^2(x) = \cot^2(x) + 1$.

D is correct, as $\operatorname{cosec}(x) = \cot\left(\frac{x}{2}\right) - \cot(x)$.

E is incorrect.

Answer E

Question 4

$c - 3\arctan(x - 2) < 0$

For what values of c is the graph of $y = c - 3\arctan(x - 2)$ entirely below the x-axis?

The range of $y = c - 3\arctan(x - 2)$ is $\left(c - \frac{3\pi}{2}, c + \frac{3\pi}{2}\right)$.

We require $c + \frac{3\pi}{2} < 0$, i.e. $c < -\frac{3\pi}{2}$.

Answer D

Question 5

For **A**, $z - \bar{z} = 2i\text{Im}(z)$.
Hence **A** is purely imaginary.

For **B**, $z + \bar{z} = 2\text{Re}(z)$.

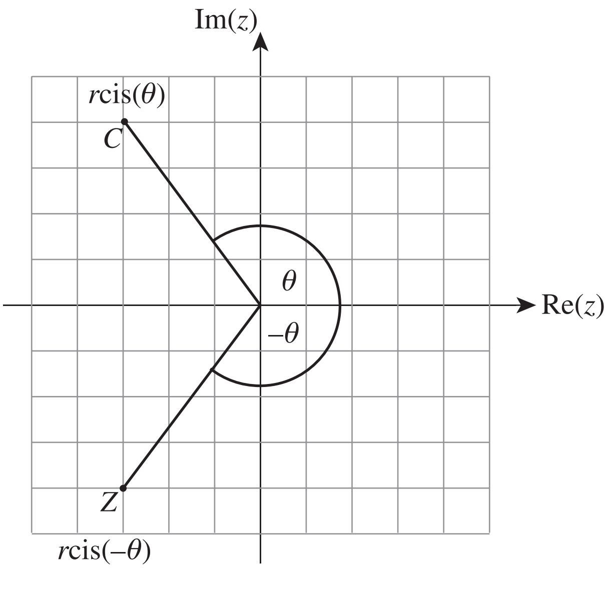
For **C**, $z\bar{z} = [\text{Re}(z)]^2 + [\text{Im}(z)]^2$.

For **D**, $\frac{1}{z} + \frac{1}{\bar{z}} = \frac{2\text{Re}(z)}{[\text{Re}(z)]^2 + [\text{Im}(z)]^2}$.

For **E**, $z^2 + \bar{z}^2 = 2[(\text{Re}(z))^2 - (\text{Im}(z))^2]$.

Answer A

Question 6



Answer C

Question 7

Let α and β be the roots.

$(z-\alpha)(z-\beta) = z^2 - (\alpha+\beta)z + \alpha\beta$

$\alpha = 2\text{cis}\left(-\frac{2\pi}{3}\right), \beta = 2\text{cis}\left(\frac{2\pi}{3}\right)$

$\alpha\beta = 4$

$\alpha + \beta = -2$

Hence $z^2 + 2z + 4 = 0$

Answer C

Question 8

$(x-1)^2 + y^2 = 1$ ... (1)

$x + y = 1$ ... (2)

Substitute $y = 1 - x$ into (1).

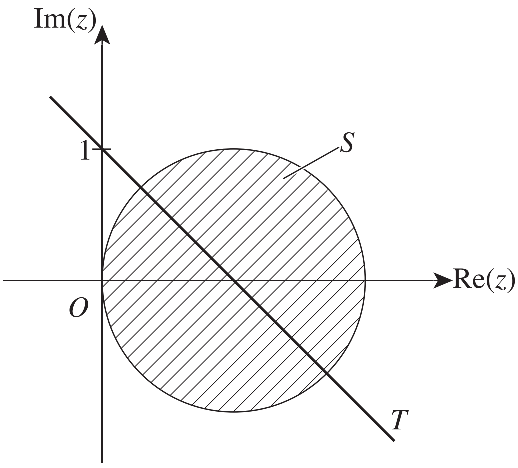
$(x-1)^2 + (1-x)^2 = 1$

$(x-1)^2 = \frac{1}{2}$

$x - 1 = \pm\frac{1}{\sqrt{2}}$

$x = 1 \pm \frac{1}{\sqrt{2}}$

A quick sketch illustrates that $1 - \frac{1}{\sqrt{2}} \le \text{Re}(z) \le 1 + \frac{1}{\sqrt{2}}$.



Answer B

Question 9

$x^2 + xy + y^2 = 12$

$2x + x\dfrac{dy}{dx} + y + 2y\dfrac{dy}{dx} = 0$

$(x + 2y)\dfrac{dy}{dx} = -(2x + y)$

$\dfrac{dy}{dx} = -\dfrac{(2x + y)}{x + 2y}$

For a horizontal tangent, $\dfrac{dy}{dx} = 0$.

So $y = -2x$.

When $x = 2$, $y = -4$.

Hence $k = -4$.

Answer D

Question 10

$\dfrac{dx}{dt} = 4$, $\dfrac{dy}{dt} = -\dfrac{4}{3}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dt} \times \frac{dt}{dx}\\ &= -\frac{4}{3} \times \frac{1}{4}\\ &= -\frac{1}{3}\end{aligned}$$

Answer A

Question 11

$$\int_0^{\frac{\pi}{4}} \left(\frac{\cos(x)}{\sin(x) + \cos^2(x)}\right) dx$$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{\cos(x)}{\sin(x) + 1 - \sin^2(x)}\right) dx$$

Let $u = \sin(x)$ and so $\dfrac{du}{dx} = \cos(x)$.

When $x = 0$, $u = 0$ and when $x = \dfrac{\pi}{4}$, $u = \dfrac{1}{\sqrt{2}}$.

$$\int_0^{\frac{\pi}{4}} \left(\frac{\cos(x)}{\sin(x) + 1 - \sin^2(x)}\right) dx = \int_0^{\frac{1}{\sqrt{2}}} \frac{1}{1 + u - u^2} du$$

Answer E

Question 12

$$\int_m^n \frac{1}{x-2}dx$$

$$= [\log_e|x-2|]_m^n$$

$$= \log_e|n-2| - \log_e|m-2|$$

Answer E

Question 13

$$V = \pi\int_{-4}^{4}(25-y^2)dx$$

$$= 2\pi\int_0^4(25-(x^2+9))dx$$

$$= 2\pi\int_0^4(16-x^2)dx$$

Answer B

Question 14

$x = 2e^t$

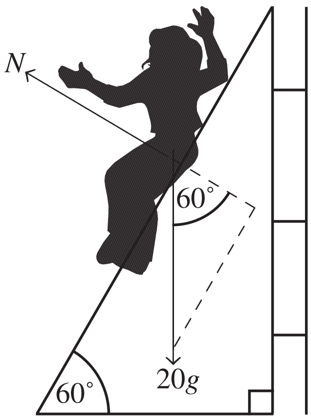
$y = 2e^{-t}$

$= \frac{4}{2e^t}$

$= \frac{4}{x}$

Answer C

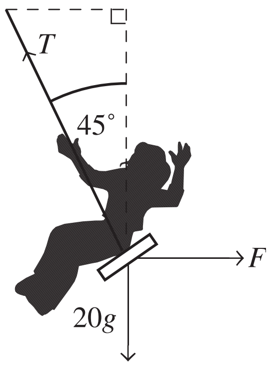
Question 15



$$N = 20g\cos(60°)$$
$$= 20g \times \frac{1}{2}$$
$$= 10g$$

Answer C

Question 16



Resolving vertically:

$$T\cos(45°) = 20g$$
$$T = \frac{20g}{\cos(45°)}$$
$$= 20\sqrt{2}g$$

Answer D

Question 17

$$\Sigma \underset{\sim}{F} = 5\underset{\sim}{a}$$
$$2\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{G} = 5\underset{\sim}{i} - 10\underset{\sim}{j}$$
$$\underset{\sim}{G} = 5\underset{\sim}{i} - 10\underset{\sim}{j} - 2\underset{\sim}{i} - \underset{\sim}{j}$$
$$= 3\underset{\sim}{i} - 11\underset{\sim}{j}$$

Answer C

Question 18

$u = 10$

$a = -9.8$

$s = 0$

$t = ?$

$s = ut + \frac{1}{2}at^2$

$0 = 10t - 4.9t^2$

$\quad = t(10 - 4.9t)$

So the rocket returns when $t = \frac{10}{4.9} = 2.04$ seconds.

Answer D

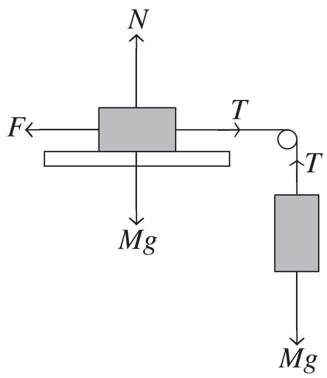
Question 19

If the system were to move or be on the point of moving, the body on the table would move or be on the point of moving to the right.

The forces acting on the body on the table are weight, *Mg*, vertically down, a normal reaction force, *N*, vertically up, friction, *F*, to the left and tension, *T*, to the right.

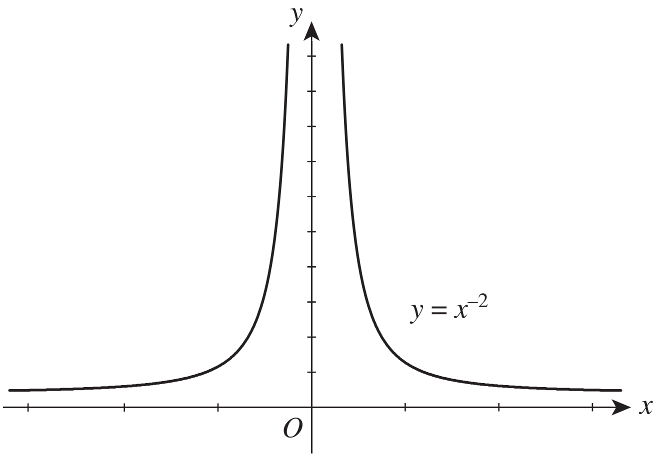
The forces acting on the body suspended from the string are weight, *Mg*, vertically down and Tension, *T*, vertically up.

The correct diagram is



Answer E

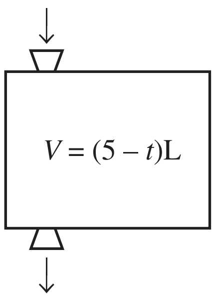
Question 20



Answer A

Question 21

inflow 10 g/L at a rate of 1 L/min



outflow $\frac{S}{V}$ g/L at a rate of 2 L/min

$$\frac{dS}{dt} = \text{inflow} - \text{outflow}$$

$$= 10 - \frac{2S}{5-t} \quad t = 0,\ S = 0$$

Answer C

Question 22

$$\frac{dy}{dx} = \frac{1}{\cos(x)}$$

$x_0 = 0,\ y_0 = 1,\ \Delta x = 0.1$

$$\Delta y_1 = \frac{1}{\cos(0)} \times 0.1 = 0.1$$

So $y_1 = 1.1$

$$\Delta y_2 = \frac{1}{\cos(0.1)} \times 0.1$$

So $y_2 = 1.1 + \frac{0.1}{\cos(0.1)}$

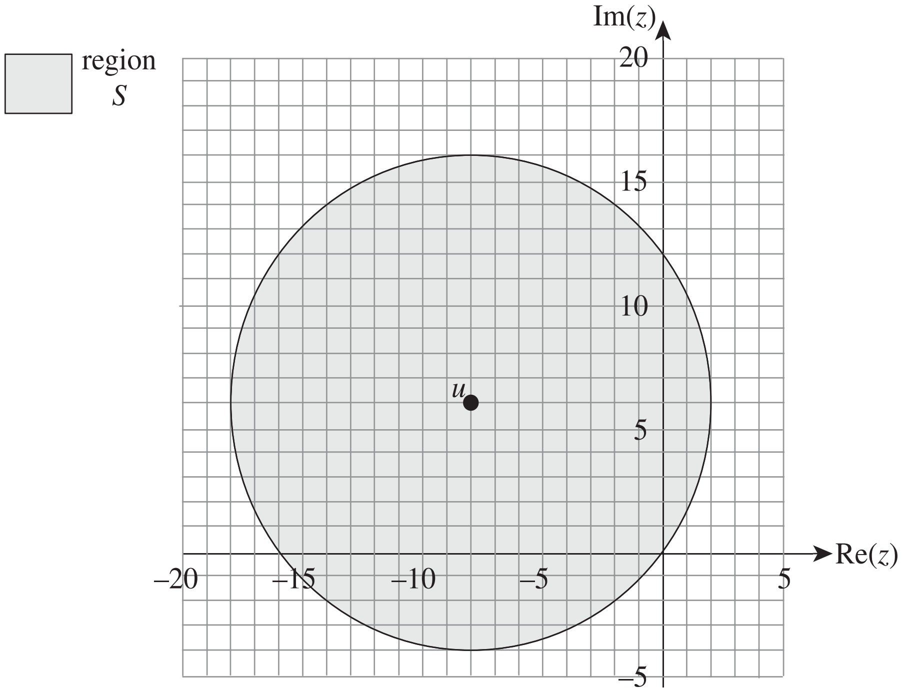
$$= 1.2005$$

Answer B

SECTION 2

Question 1

a.



Correct point u. M1
Correct region inside circle, with boundary included. M1
Correct radius of region. A1

b. If v lies on the boundary of S then $|v - u| = 10$.

$|v - u| = |-8 - 4i + 8 - 6i|$ M1

$= |-10i|$

$= 10$ A1

So v lies on the boundary of S.

c. $w = \dfrac{2 + 36i}{-8 + 6i}$

$= \dfrac{1 + 18i}{-4 + 3i} \times \dfrac{4 + 3i}{4 + 3i}$ M1

$= \dfrac{-50 + 75i}{-25}$

$= 2 - 3i$ A1

Note: this answer can be obtained using technology.

d. $|u - w| = |10 + 9i|$ M1

$= \sqrt{181}$

So w is $\sqrt{181} - 10$ units outside the region S. A1

Question 2

a. $[-1, \infty)$ A1

b. $(-1, 0)$ A1

c. **i.** $A = -\int_{-1}^{0} x\sqrt{1+x}\,dx$ A1

$x = u - 1$

$x = 0,\ u = 1$

$x = -1,\ u = 0$ M1

$\therefore A = -\int_{0}^{1} \sqrt{u}(u-1)\,du$ A1

ii. $A = \int_{0}^{1} \left(u^{\frac{1}{2}} - u^{\frac{3}{2}}\right) du$

$= \left[\frac{2}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}\right]_0^1$

$= \frac{2}{3} - \frac{2}{5}$

$= \frac{4}{15}$ units^2 A1

d. **i.** $V = \pi\int_{-1}^{1} (x^2 + x^3)\,dx$ A1

ii. $V = 2.09$ units^3 A1

Question 3

a. $x = 5\cos(t) + 5,\ y = 5\sin(t)$ M1

So $\cos(t) = \frac{x-5}{5}$

$\sin(t) = \frac{y}{5}$

$(x-5)^2 + y^2 = 25$ A1

b. $\overrightarrow{AB} = (x-5)\underset{\sim}{i} + y\underset{\sim}{j} = 5\cos(t)\underset{\sim}{i} + 5\sin(t)\underset{\sim}{j}$ M1

Now $\dot{\underset{\sim}{r}} = -5\sin(t)\underset{\sim}{i} + 5\cos(t)\underset{\sim}{j}$ A1

$\ddot{\underset{\sim}{r}} = -5\cos(t)\underset{\sim}{i} - 5\sin(t)\underset{\sim}{j}$ A1

$= -\overrightarrow{AB}$ M1

So $\ddot{\underset{\sim}{r}}$ is along the line $\overrightarrow{AB}$.

c. $x = t$

$y = 2t$

$\therefore y = 2x,\ x \geq 0$ A1

d. To find where the paths intersect, we solve $(x-5)^2 + 4x^2 = 25$ M1

$$5x^2 - 10x = 0$$

$$x = 0 \text{ or } x = 2$$

The paths intersect at $(0, 0)$ and (2, 4). A1

e. P is at (0, 0) when $\sin(t) = 0$, $\cos(t) = -1$, i.e. when $t = \pi$.

Q is at (0, 0) when $t = 0$. M1

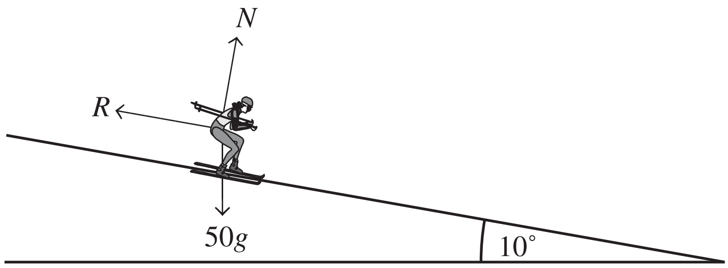
P is at (2, 4) when $\sin(t) = \frac{4}{5}$.

Q is at (2, 4) when $t = 2$. M1

Therefore, there is no collision.

Question 4

a.



A1

b. Parallel to the plane, $50g\sin(10°) - kv = 50a$. M1

Hence $a = g\sin(10°) - \frac{k}{50}v$. A1

c. The skier starts from rest, so when $v = 0$, $a = g\sin(10°)$ m/s^2. A1

d. As $v \to 25$, $a \to 0$.

We find the value of k by setting $a = 0$ when $v = 25$. M1

$$g\sin(10°) - \frac{25k}{50} = 0$$

Hence $k = 2g\sin(10°)$. A1

$$a = g\sin(10°) - \frac{g\sin(10°)}{25}v$$

$$a = \frac{g\sin(10°)}{25}(25 - v)$$ A1

e. $$\frac{dv}{dt} = \frac{g\sin(10°)}{25}(25 - v)$$ A1

$$t = \frac{25}{g\sin(10°)}\int_0^{10}\left(\frac{1}{25 - v}\right)dv$$ M1

$t = 7.50$ seconds (correct to two decimal places) A1

f. The terminal speed is 25 m/s. Hence 90% of this speed is 22.5 m/s. A1

Solving $25\left(1 - e^{-\frac{g\sin(10°)}{25}t}\right) = 22.5$ for t gives $t = 33.8$ seconds (correct to one decimal place). M1 A1

g. $x = \int_0^{33.827...} 25\left(1 - e^{-\frac{g\sin(10°)}{25}t}\right)dt$ M1

$x = 515$ metres (correct to the nearest metre) A1

h. $v\dfrac{dv}{dx} = -0.02v^2$

$\dfrac{dv}{dx} = -0.02v$ M1

$\dfrac{dx}{dv} = -\dfrac{50}{v}$

$x = \int_{22.5}^{5} -\dfrac{50}{v}dv$ M1

$= 75$ metres (correct to the nearest metre) A1

Question 5

a. $\underset{\sim}{a}.\underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos(\alpha)$

$= 2|\underset{\sim}{a}||\underset{\sim}{a}|\cos(\alpha)$

$= 2a^2\cos(\alpha)$ A1

b. $\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$

$= \underset{\sim}{a} + m\underset{\sim}{b}$ A1

c. $\overrightarrow{MC} = \overrightarrow{MA} + \overrightarrow{AO} + \overrightarrow{OC}$

$= -m\underset{\sim}{b} - \underset{\sim}{a} + \underset{\sim}{b}$

$= (1-m)\underset{\sim}{b} - \underset{\sim}{a}$ A1

d. $\overrightarrow{OM}.\overrightarrow{MC} = 0$

$(\underset{\sim}{a} + m\underset{\sim}{b}).[(1-m)\underset{\sim}{b} - \underset{\sim}{a}] = 0$

$(1-m)(\underset{\sim}{a}.\underset{\sim}{b}) - a^2 + m(1-m)b^2 - m(\underset{\sim}{a}.\underset{\sim}{b}) = 0$ M1

$(1-2m)(\underset{\sim}{a}.\underset{\sim}{b}) + 4a^2[m(1-m)] - a^2 = 0$ A1

$(1-2m)(\underset{\sim}{a}.\underset{\sim}{b}) + a^2(-4m^2 + 4m - 1) = 0$

$(1-2m)(\underset{\sim}{a}.\underset{\sim}{b}) - a^2(1 - 4m + 4m^2) = 0$

$(1-2m)(\underset{\sim}{a}.\underset{\sim}{b}) - a^2(1-2m)^2 = 0$ A1

e. $(1-2m)2a^2\cos(\alpha) - a^2(1-2m)^2 = 0$

$a^2(1-2m)[2\cos(\alpha) - (1-2m)] = 0$ M1

Hence $m = \dfrac{1}{2}$ or $m = \dfrac{1}{2} - \cos(\alpha)$ A1

f. When $m = \dfrac{1}{2}$, M is the midpoint of AB. A1

g. As $0 \le m \le 1$, $0 \le \frac{1}{2} - \cos(\alpha) \le 1$ M1

$-\frac{1}{2} \le \cos(\alpha) \le \frac{1}{2}$

$\frac{\pi}{3} \le \alpha \le \frac{2\pi}{3}$, $\alpha \ne \frac{\pi}{2}$ A1 A1

If $\alpha = \frac{\pi}{2}$, there is only one possible position for M.