Insight Publications

INSIGHT

Trial Exam Paper

2006

SPECIALIST MATHEMATICS

Written examination 1

Worked solutions

This book presents:

- worked solutions, giving you a series of points to show you how to work through the questions
- mark allocation details.

Question 1

Consider the function $f(x) = \dfrac{1}{2x^2 - x - 3}$

1a. Determine the equations of the asymptotes of f.

Worked solution

Vertical asymptotes: $2x^2 - x - 3 = 0$

$(2x-3)(x+1) = 0$

$x = \dfrac{3}{2}$, and $x = -1$ 1A

Horizontal asymptote: $y = 0$ 1A

2 marks

1b. Find the coordinates of any intercepts and stationary points of f.

Worked solution

y-intercept: $x = 0$, $y = -\dfrac{1}{3}$ 1A

Stationary points: $f'(x) = -1(2x^2 - x - 3)^{-2}(4x-1) = 0$ 1M

$-\dfrac{4x-1}{(2x^2 - x - 3)^2} = 0$

$4x - 1 = 0$

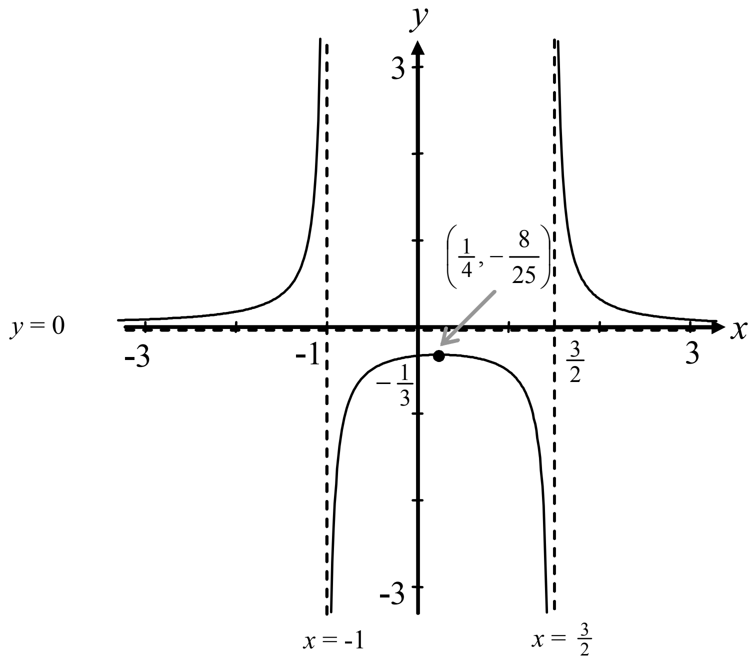
$x = \dfrac{1}{4}$, $y = \dfrac{1}{2\left(\frac{1}{4}\right)^2 - \frac{1}{4} - 3} = -\dfrac{8}{25}$

There is a maximum turning point at $\left(\dfrac{1}{4}, -\dfrac{8}{25}\right)$ 1A

3 marks

1c. Sketch f on the axes below labeling all key features.

Answer



Shape and features 1A

1 mark

Question 2

2a. Show that $\dfrac{1}{\cos^4 x - \sin^4 x} = \sec(2x)$

Worked solution

$$\frac{1}{\cos^4 x - \sin^4 x}$$

$$= \frac{1}{(\cos^2 x + \sin^2 x)(\cos^2 x - \sin^2 x)}$$ 1M

$$= \frac{1}{1 \times (\cos^2 x - \sin^2 x)}$$

$$= \frac{1}{\cos(2x)}$$ 1A

$$= \sec(2x)$$

2 marks

Question 2 – continued

2b. Hence find the exact values of *x* for which $\dfrac{1}{\cos^4 x - \sin^4 x} = 2$, $x \in [0, 2\pi]$

Worked solution

$$\frac{1}{\cos^4 x - \sin^4 x} = 2$$

$\sec(2x) = 2$ (from a.)

$\cos(2x) = \dfrac{1}{2}$ 1A

$2x = \dfrac{\pi}{3},\quad 2\pi - \dfrac{\pi}{3},\quad 2\pi + \dfrac{\pi}{3},\quad 4\pi - \dfrac{\pi}{3}$

$2x = \dfrac{\pi}{3},\quad \dfrac{5\pi}{3},\quad \dfrac{7\pi}{3},\quad \dfrac{11\pi}{3}$

$x = \dfrac{\pi}{6},\quad \dfrac{5\pi}{6},\quad \dfrac{7\pi}{6},\quad \dfrac{11\pi}{6}$ 1A

2 marks

Question 3

Find the fourth roots of $16i$ in exact polar form.

Worked solution

Let $z^4 = 0 + 16i$

$z^4 = 16\text{cis}\left(\dfrac{\pi}{2}\right)$

$z = \left(16\text{cis}\left(\dfrac{\pi}{2} + 2k\pi\right)\right)^{\frac{1}{4}}$, $k \in Z$ 1A

$z = 16^{\frac{1}{4}}\text{cis}\dfrac{1}{4}\left(\dfrac{\pi}{2} + 2k\pi\right)$ By De Movire's Theorem

$z = 2\text{cis}\left(\dfrac{\pi}{8} + \dfrac{2k\pi}{4}\right)$ 1M

$k = 0,\quad z = 2\text{cis}\left(\dfrac{\pi}{8}\right),\qquad k = 1,\quad z = 2\text{cis}\left(\dfrac{5\pi}{8}\right)$ 1A

$k = -1,\quad z = 2\text{cis}\left(-\dfrac{3\pi}{8}\right)\qquad k = -2,\quad z = 2\text{cis}\left(-\dfrac{7\pi}{8}\right)$

The four roots are: $2\text{cis}\left(-\dfrac{7\pi}{8}\right),\quad 2\text{cis}\left(-\dfrac{3\pi}{8}\right),\quad 2\text{cis}\left(\dfrac{\pi}{8}\right),\quad 2\text{cis}\left(\dfrac{5\pi}{8}\right)$ 1A

4 marks

Question 4

Determine the rate of change of y with respect to x on the curve $y = x - 5xy^2$ at the point where $y = 1$.

Worked solution

Find x when $y = 1$:

$$y = x - 5xy^2$$
$$1 = x - 5x(1)^2$$
$$1 = -4x$$
$$x = -0.25 \qquad \text{1A}$$

Find the rate of change using implicit differentiation:

$$\frac{dy}{dx} = 1 - 5\left(1 \times y^2 + x \times 2y\frac{dy}{dx}\right) \qquad \text{1M}$$
$$\frac{dy}{dx} = 1 - 5y^2 - 10xy\frac{dy}{dx}$$
$$(1 + 10xy)\frac{dy}{dx} = 1 - 5y^2$$
$$\frac{dy}{dx} = \frac{1 - 5y^2}{1 + 10xy} \qquad \text{1A}$$

When $y = 1, \quad x = -0.25$:

$$\frac{dy}{dx} = \frac{1 - 5(1)^2}{1 + 10 \times -0.25 \times 1}$$
$$\frac{dy}{dx} = \frac{-4}{-1.5}$$
$$\frac{dy}{dx} = \frac{8}{3} \qquad \text{1A}$$

4 marks

Question 5

5a. i. Give the domain over which $\frac{d}{dx}\left(\arcsin(2x) + 2x\sqrt{1 - 4x^2}\right)$ is defined.

Worked solution

Domain is $x \in \left(-\frac{1}{2}, \ \frac{1}{2}\right)$. 1A

The derivative is not defined at the end points of this range.

5a. ii. Show that $\frac{d}{dx}\left(\arcsin(2x) + 2x\sqrt{1-4x^2}\right) = 4\sqrt{1-4x^2}$

Worked solution

$$\frac{d}{dx}\left(\arcsin(2x) + 2x\sqrt{1-4x^2}\right)$$

$$= \frac{d}{dx}\left(\arcsin(2x)\right) + \frac{d}{dx}\left(2x\sqrt{1-4x^2}\right), \qquad x \in \left(-\frac{1}{2}, \ \frac{1}{2}\right)$$

$$= \frac{1}{\sqrt{1-(2x)^2}} \times 2 + 2\sqrt{1-4x^2} + 2x\left(\frac{1}{2}\left(1-4x^2\right)^{-\frac{1}{2}}\right)(-8x) \qquad \text{1M}$$

$$= \frac{2}{\sqrt{1-4x^2}} + 2\sqrt{1-4x^2} - \frac{8x^2}{\sqrt{1-4x^2}}$$

$$= \frac{2}{\sqrt{1-4x^2}} - \frac{8x^2}{\sqrt{1-4x^2}} + 2\sqrt{1-4x^2}$$

$$= \frac{2-8x^2}{\sqrt{1-4x^2}} + 2\sqrt{1-4x^2}$$

$$= \frac{2(1-4x^2)}{\sqrt{1-4x^2}} + 2\sqrt{1-4x^2}$$

$$= 2\sqrt{1-4x^2} + 2\sqrt{1-4x^2} \qquad \text{1A}$$

1 + 2 = 3 marks

5b. Hence, find the exact area enclosed by the curve $4x^2 + y^2 = 1$.

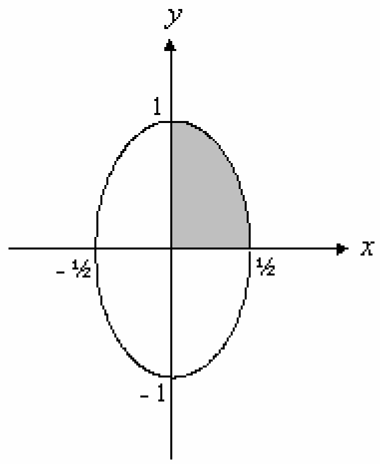
Worked solution

$4x^2 + y^2 = 1$ is an ellipse.

Express y in terms of x

$y = \pm\sqrt{1-4x^2}$

Sketch a graph.



Area of ellipse = 4 × shaded region

$$A = 4\int_0^{\frac{1}{2}} \sqrt{1-4x^2}\,dx \qquad \text{1A}$$

$$A = \int_0^{\frac{1}{2}} 4\sqrt{1-4x^2}\,dx$$

$$A = \left[\arcsin(2x) + 2x\sqrt{1-4x^2}\right]_0^{\frac{1}{2}} \qquad \text{1A}$$

$$A = \arcsin(1)$$

$$A = \frac{\pi}{2} \text{ square units} \qquad \text{1A}$$

3 marks

Question 6

Find constants m and n such that $y = \dfrac{\log_e |x|}{x}$, $x \neq 0$ is a solution of the differential equation

$$x^2 \frac{d^2 y}{dx^2} + mx\frac{dy}{dx} + ny = 0$$

Worked solution

$$y = \frac{\log_e |x|}{x}$$

Using the quotient rule

$$\frac{dy}{dx} = \frac{\frac{1}{x}.x - 1.\log_e |x|}{x^2}$$

$$\frac{dy}{dx} = \frac{1 - \log_e |x|}{x^2} \qquad \text{1A}$$

$$\frac{d^2 y}{dx^2} = \frac{-\frac{1}{x}.x^2 - \left(1 - \log_e |x|\right).2x}{x^4}$$

$$\frac{d^2 y}{dx^2} = \frac{-x - 2x + 2x\log_e |x|}{x^4}$$

$$\frac{d^2 y}{dx^2} = \frac{2\log_e |x| - 3}{x^3} \qquad \text{1A}$$

Substitute into the given differential equation

$$x^2\left(\frac{2\log_e |x| - 3}{x^3}\right) + mx\left(\frac{1 - \log_e |x|}{x^2}\right) + n\left(\frac{\log_e |x|}{x}\right) = 0 \qquad x \neq 0 \qquad \text{1M}$$

$$\frac{2\log_e |x| - 3}{x} + \frac{m\left(1 - \log_e |x|\right)}{x} + \frac{n\log_e |x|}{x} = 0$$

$$2\log_e |x| - 3 + m - m\log_e |x| + n\log_e |x| = 0$$

$$\left(2 - m + n\right)\log_e |x| + m - 3 = 0$$

Therefore:

$m - 3 = 0$ and $2 - m + n = 0$

$\therefore \quad m = 3$

$2 - 3 + n = 0$

$\therefore \quad n = 1$ 1A

4 marks

Question 7

Given $\frac{dy}{dx} = x\sqrt{1+x^2}$ and $y = 1$ when $x = 0$.

Find the value of y when $x = \sqrt{3}$.

Worked solution

$$\frac{dy}{dx} = x\sqrt{1+x^2}$$

$y = \int x\sqrt{1+x^2}\,dx$ Let $u = 1+x^2 \Rightarrow \frac{du}{dx} = 2x$

$y = \frac{1}{2}\int \sqrt{1+x^2}\ (2x\,dx)$ $du = 2x\,dx$ 1M

$$y = \frac{1}{2}\int u^{\frac{1}{2}}\,du$$

$y = \frac{1}{2}\times\frac{2}{3}u^{\frac{3}{2}} + c$ c is a constant 1A

$$y = \frac{1}{3}(1+x^2)^{\frac{3}{2}} + c$$

Find c: When $x = 0$, $y = 1$ $\Rightarrow$ $1 = \frac{1}{3}\left(1+(0)^2\right)^{\frac{3}{2}} + c$

$$1 = \frac{1}{3} + c$$

$c = \frac{2}{3}$ 1A

$$\therefore\ y = \frac{1}{3}(1+x^2)^{\frac{3}{2}} + \frac{2}{3}$$

When $x = \sqrt{3}$, $y = \frac{1}{3}\left(1+\left(\sqrt{3}\right)^2\right)^{\frac{3}{2}} + \frac{2}{3}$

$$y = \frac{1}{3}\times 4^{\frac{3}{2}} + \frac{2}{3}$$

$$y = \frac{1}{3}\times 8 + \frac{2}{3}$$

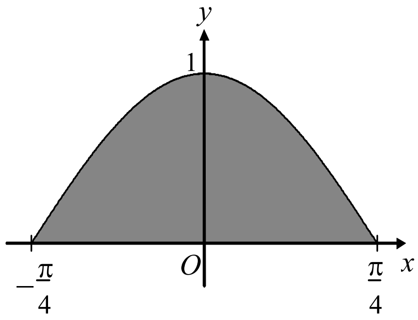
$y = \frac{10}{3}$ 1H

$$\therefore\ y\left(\sqrt{3}\right) = \frac{10}{3}$$

4 marks

Question 8

The graph below shows the region bounded by the curve $y = \cos(2x)$ and the x-axis.



Find the exact value of the volume of the solid of revolution formed when this region is rotated around the x-axis.

Worked solution

$$V = \pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} y^2 dx \quad = 2\pi \int_0^{\frac{\pi}{4}} y^2 dx$$

$$= 2\pi \int_0^{\frac{\pi}{4}} \cos^2(2x) dx$$ 1A

$$= 2\pi \int_0^{\frac{\pi}{4}} \frac{1}{2}\left(\cos(4x) + 1\right) dx$$ 1A

$$= \pi \int_0^{\frac{\pi}{4}} \left(\cos(4x) + 1\right) dx$$

$$= \pi \left[\frac{1}{4}\sin(4x) + x\right]_0^{\frac{\pi}{4}}$$ 1M

$$= \pi \left[\frac{1}{4}\sin(\pi) + \frac{\pi}{4}\right]$$

$$= \pi \left[0 + \frac{\pi}{4}\right]$$

$$= \frac{\pi^2}{4} \text{ cubic units.}$$ 1A

4 marks

Question 9

A particle moves in such a way that its position vector at time t seconds is given by

$\underset{\sim}{r} = 2t\,\underset{\sim}{i} + \cos(2\pi t)\,\underset{\sim}{j} + \sin(2\pi t)\,\underset{\sim}{k}$

9a. Find the constant speed at which the particle is moving.

Worked solution

$\underset{\sim}{r} = 2t\,\underset{\sim}{i} + \cos(2\pi t)\,\underset{\sim}{j} + \sin(2\pi t)\,\underset{\sim}{k}$

$\dot{\underset{\sim}{r}} = 2\,\underset{\sim}{i} - 2\pi\sin(2\pi t)\,\underset{\sim}{j} + 2\pi\cos(2\pi t)\,\underset{\sim}{k}$

1A

$$\begin{aligned}
\text{speed} &= \sqrt{2^2 + \left(-2\pi\sin(2\pi t)\right)^2 + \left(2\pi\cos(2\pi t)\right)^2} \\
&= \sqrt{4 + 4\pi^2\sin^2(2\pi t) + 4\pi^2\cos^2(2\pi t)} \\
&= \sqrt{4 + 4\pi^2\left(\sin^2(2\pi t) + \cos^2(2\pi t)\right)} \\
&= \sqrt{4 + 4\pi^2} \\
&= 2\sqrt{1+\pi^2} \text{ units/sec}
\end{aligned}$$

1A

2 marks

9b. Show that the velocity and acceleration are always perpendicular.

Worked solution

$\ddot{\underset{\sim}{r}}(t) = -4\pi^2\cos(2\pi t)\,\underset{\sim}{j} - 4\pi^2\sin(2\pi t)\,\underset{\sim}{k}$ 1A

$\dot{\underset{\sim}{r}}\,.\,\ddot{\underset{\sim}{r}} = \left(2\,\underset{\sim}{i} - 2\pi\sin(2\pi t)\,\underset{\sim}{j} + 2\pi\cos(2\pi t)\,\underset{\sim}{k}\right).\left(0\,\underset{\sim}{i} - 4\pi^2\cos(2\pi t)\,\underset{\sim}{j} - 4\pi^2\sin(2\pi t)\,\underset{\sim}{k}\right)$

$\dot{\underset{\sim}{r}}\,.\,\ddot{\underset{\sim}{r}} = 0 + 8\pi^3\sin(2\pi t)\cos(2\pi t) - 8\pi^3\cos(2\pi t)\sin(2\pi t)$

$\dot{\underset{\sim}{r}}\,.\,\ddot{\underset{\sim}{r}} = 0$

1A

$\therefore$ The velocity and acceleration are always perpendicular.

2 marks

END OF WORKED SOLUTIONS