VCAA Specialist Mathematics Exam 2 Solutions 2005
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Q1a 20 litres in and 10 litres out per minute. The volume increases by 10 litres per minute. At time t minutes, the volume of solution in the tank $= 10 + 10t$ litres.

$\therefore$ concentration $= \dfrac{x}{10+10t} = \dfrac{x}{10(1+t)}$ grams per litre.

Q1b Rate of inflow of chemical $= \dfrac{2}{1+t^2} \times 20 = \dfrac{40}{1+t^2}$ grams per minute. Rate of outflow of chemical $= \dfrac{x}{10(1+t)} \times 10 = \dfrac{x}{1+t}$ grams per minute.

Rate of change of chemicals = rate of inflow − rate of outflow

i.e. $\dfrac{dx}{dt} = \dfrac{40}{1+t^2} - \dfrac{x}{1+t}$. Hence $\dfrac{dx}{dt} + \dfrac{x}{1+t} = \dfrac{40}{1+t^2}$.

Q1ci $x = \dfrac{40}{1+t} Tan^{-1}(t) + \dfrac{20}{1+t} \log_e\left(1+t^2\right)$,

$$\frac{dx}{dt} = -\frac{40}{(1+t)^2} Tan^{-1}(t) + \frac{40}{1+t} \times \frac{1}{1+t^2} - \frac{20}{(1+t)^2}\log_e\left(1+t^2\right) + \frac{20}{1+t} \times \frac{2t}{1+t^2},$$

$$= \frac{40(1+t)}{(1+t)\left(1+t^2\right)} - \frac{40}{(1+t)^2} Tan^{-1}(t) - \frac{20}{(1+t)^2}\log_e\left(1+t^2\right)$$

$$= \frac{40}{\left(1+t^2\right)} - \frac{40}{(1+t)^2} Tan^{-1}(t) - \frac{20}{(1+t)^2}\log_e\left(1+t^2\right).$$

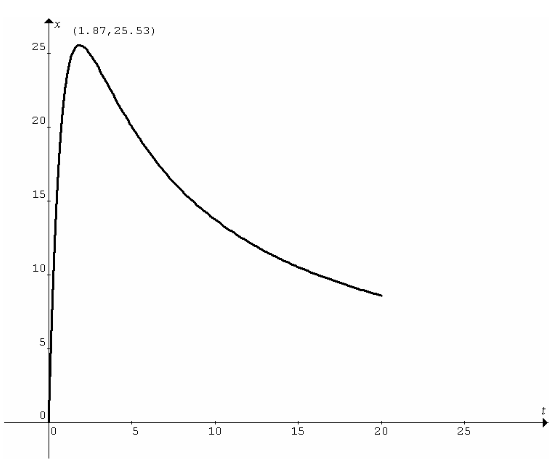
Q1cii $\dfrac{dx}{dt} = \dfrac{40}{1+t^2} - \dfrac{x}{1+t}$.

$$RHS = \frac{40}{1+t^2} - \frac{\dfrac{40}{1+t} Tan^{-1}(t) + \dfrac{20}{1+t}\log_e\left(1+t^2\right)}{1+t}$$

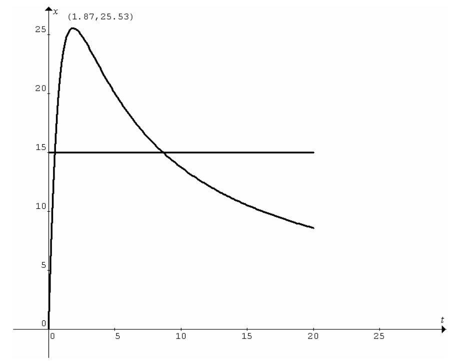
$$= \frac{40}{\left(1+t^2\right)} - \frac{40}{(1+t)^2} Tan^{-1}(t) - \frac{20}{(1+t)^2}\log_e\left(1+t^2\right) = LHS$$

$\therefore$ x satisfies the differential equation.

Q1d Sketch and find stationary points by graphics calculator.

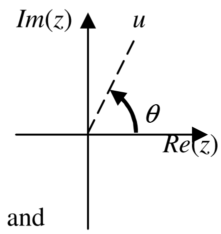


Q1ei Sketch $x = 15$ (horizontal line) on the last graph, find the first intersection $t = 0.485$ min.



Q1eii Find the second intersection $t = 8.655$ min.
Duration $= 8.655 - 0.485 = 8.17$ minutes.

Q2ai

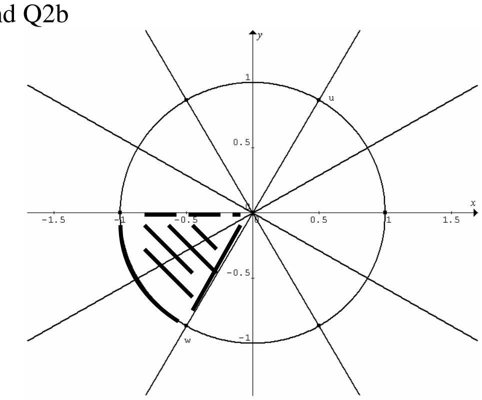


$u = rcis\theta$, where $r = \sqrt{\left(\dfrac{1}{2}\right)^2 + \left(\dfrac{\sqrt{3}}{2}\right)^2} = 1$ and

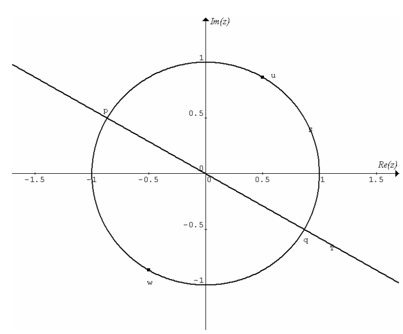
$\theta = Tan^{-1}\left(\dfrac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right) = \dfrac{\pi}{3}$. $\therefore u = cis\left(\dfrac{\pi}{3}\right)$.

Q2aii $u^6 = cis\left(6 \times \dfrac{\pi}{3}\right) = cis(2\pi) = 1$.

Q2aiii and Q2b

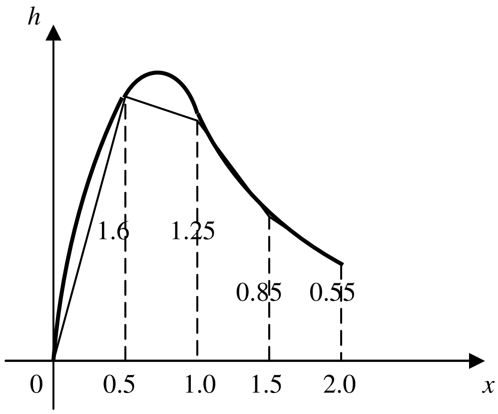


Q2ci and ii



Q2ciii Complex numbers $p = cis\left(\dfrac{5\pi}{6}\right)$ and $q = -p$. The coordinates are $\left(-\dfrac{\sqrt{3}}{2}, \dfrac{1}{2}\right)$ and $\left(\dfrac{\sqrt{3}}{2}, -\dfrac{1}{2}\right)$ respectively.

Q3a



Area $\approx \frac{1}{2}(0+1.6)0.5+\frac{1}{2}(1.6+1.25)0.5+\frac{1}{2}(1.25+0.85)0.5+\frac{1}{2}(0.85+0.55)0.5$

$=1.9875 \approx 2$ m^2.

Q3b

$$\frac{10x}{(x^2+1)(3x+1)}=\frac{x+A}{x^2+1}+\frac{B}{3x+1}=\frac{(x+A)(3x+1)+B(x^2+1)}{(x^2+1)(3x+1)}.$$

Equate the numerators: $10x=(x+A)(3x+1)+B(x^2+1)$

Let $x=0$, $0=A+B$ ………..(1)

Let $x=1$, $10=(1+A)4+2B$, $\therefore 3=2A+B$ ………..(2)

(2) − (1), $3=A$, $\therefore B=-3$.

Q3c Area $=\int_0^2 \frac{x+3}{x^2+1}-\frac{3}{3x+1}dx$

$=\int_0^2 \frac{x}{x^2+1}+\frac{3}{x^2+1}-\frac{3}{3x+1}dx$

$=\left[\frac{1}{2}\log_e(x^2+1)+3Tan^{-1}(x)-\log_e(3x+1)\right]_0^2$

$=\frac{1}{2}\log_e 5+3Tan^{-1}(2)-\log_e 7=2.18$ m^2.

Q3d $h(x)=\frac{10x}{(x^2+1)(3x+1)}$. At $x=2$, $h=0.57143$.

The other position where $h=0.57143$ is $x=0.06937$, found by sketching $h(x)=\frac{10x}{(x^2+1)(3x+1)}$ and $h=0.57143$, and finding the first intersection. At the base of a panel the length that is overlapped by the next panel is 0.06937. $\therefore$ distance between any two panels is $2.0-0.06937=1.93063$.

Minimum number of panels required $=\frac{100}{1.93063}=51.8$, i.e. 52.

Q4a $\underset{\sim}{u}=12\cos 60^o\,\underset{\sim}{i}+12\sin 60^o\,\underset{\sim}{j}=6\underset{\sim}{i}+6\sqrt{3}\,\underset{\sim}{j}$.

Q4b $\underset{\sim}{v}=\int \ddot{r}dt=-0.05t^2\underset{\sim}{i}-(gt-0.05t^2)\underset{\sim}{j}+\underset{\sim}{c}$.

At $t=0$, $\underset{\sim}{v}=6\underset{\sim}{i}+6\sqrt{3}\,\underset{\sim}{j}$, $\therefore \underset{\sim}{c}=6\underset{\sim}{i}+6\sqrt{3}\,\underset{\sim}{j}$.

Hence $\underset{\sim}{v}=-0.05t^2\underset{\sim}{i}-(gt-0.05t^2)\underset{\sim}{j}+6\underset{\sim}{i}+6\sqrt{3}\,\underset{\sim}{j}$

$\therefore \underset{\sim}{v}=(6-0.05t^2)\underset{\sim}{i}+(6\sqrt{3}-gt+0.05t^2)\underset{\sim}{j}$.

$\underset{\sim}{r}=\int \dot{r}dt=\left(6t-\frac{0.05t^3}{3}\right)\underset{\sim}{i}+\left(6\sqrt{3}t-\frac{gt^2}{2}+\frac{0.05t^3}{3}\right)\underset{\sim}{j}+\underset{\sim}{d}$.

At $t=0$, $\underset{\sim}{r}=\underset{\sim}{0}$, $\therefore \underset{\sim}{d}=\underset{\sim}{0}$.

$\therefore \underset{\sim}{r}=\left(6t-\frac{0.05t^3}{3}\right)\underset{\sim}{i}+\left(6\sqrt{3}t-\frac{gt^2}{2}+\frac{0.05t^3}{3}\right)\underset{\sim}{j}$ or

$\therefore \underset{\sim}{r}(t)=\left(6t-\frac{t^3}{60}\right)\underset{\sim}{i}+\left(6\sqrt{3}t-\frac{gt^2}{2}+\frac{t^3}{60}\right)\underset{\sim}{j}$, where $0\le t\le T$.

Q4c At $t=T$, skier lands on down-slope and skier's position vector makes -45^o with the horizontal (positive x-axis).

$\therefore \underset{\sim}{r}(T)=\left(6T-\frac{T^3}{60}\right)\underset{\sim}{i}+\left(6\sqrt{3}T-\frac{gT^2}{2}+\frac{T^3}{60}\right)\underset{\sim}{j}$,

$\frac{6\sqrt{3}T-\frac{gT^2}{2}+\frac{T^3}{60}}{6T-\frac{T^3}{60}}=\tan(-45^o)$, $\frac{6\sqrt{3}T-\frac{gT^2}{2}+\frac{T^3}{60}}{6T-\frac{T^3}{60}}=-1$,

$6\sqrt{3}T-\frac{gT^2}{2}+\frac{T^3}{60}=-6T+\frac{T^3}{60}$, $(6\sqrt{3}+6)T-\frac{gT^2}{2}=0$,

$T\left((6\sqrt{3}+6)-\frac{gT}{2}\right)=0$. Since $T\ne 0$, $\therefore T=\frac{12}{g}(\sqrt{3}+1)$.

Q4d $\underset{\sim}{v}(t)=(6-0.05t^2)\underset{\sim}{i}+(6\sqrt{3}-gt+0.05t^2)\underset{\sim}{j}$,

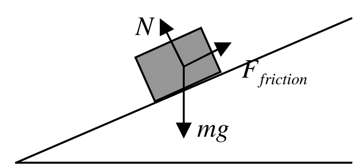
At $t=T=\frac{12}{g}(\sqrt{3}+1)=3.3454$, $\underset{\sim}{v}=5.44\underset{\sim}{i}-21.83\underset{\sim}{j}$,

Speed $=|\underset{\sim}{v}|=\sqrt{5.44^2+(-21.83)^2}=22.5$ ms^{-1}.

Q5a $R=0$,

$F_{friction}-5g\sin 30^o=0$,

$F_{friction}=\frac{5g}{2}=24.5$ newtons



Q5b In this situation it is the force of friction that accelerates the package up the belt. If the acceleration is greater than 0.8 ms^{-2}, the package will slip. This indicates that the friction force is at its maximum value μN. Same diagram as above.

Component $\perp$ to belt: $R=0$,

$N-5g\cos 30^o=0$ …….(1)

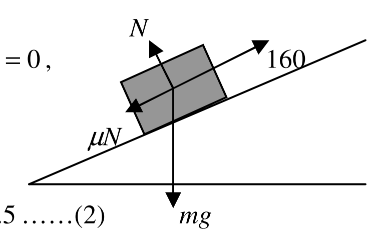
Component // to belt: $R=ma$,

$\mu N-5g\sin 30^o=5\times 0.8$ …….(2)

Solve (1) and (2), $\mu \approx 0.67$

Q5c Component $\perp$ to belt: $R=0$,

$N-mg\cos 30^o=0$ …….(1)



Component // to belt: $R=ma$,

$160-\mu N-mg\sin 30^o=m\times 0.5$ ……(2)

Solve (1) and (2), where $\mu \approx 0.67$, $m=14.4$ kg

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