Specialist Mathematics

Written examination 2

2005 Trial Examination

SOLUTIONS

Question 1

a. **i.** $\sin\left(\frac{5\pi}{12}\right)=\sin\left(\frac{\pi}{4}+\frac{\pi}{6}\right)=\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right)+\cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)$

$$=\frac{\sqrt{2}}{2}\times\frac{\sqrt{3}}{2}+\frac{\sqrt{2}}{2}\times\frac{1}{2}=\frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}+\frac{1}{2}\right)=\frac{\sqrt{2}(\sqrt{3}+1)}{4}$$

ii. $\cos\left(\frac{5\pi}{12}\right)=\cos\left(\frac{\pi}{4}+\frac{\pi}{6}\right)=\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right)-\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)$

$$=\frac{\sqrt{2}}{2}\times\frac{\sqrt{3}}{2}-\frac{\sqrt{2}}{2}\times\frac{1}{2}=\frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}-\frac{1}{2}\right)=\frac{\sqrt{2}(\sqrt{3}-1)}{4}$$

b. $u-v=\left[\cos\left(\frac{5\pi}{12}\right)+i\sin\left(\frac{5\pi}{12}\right)\right]-\left[\cos\left(\frac{5\pi}{12}\right)-i\sin\left(\frac{5\pi}{12}\right)\right]$

$$=0+2i\sin\left(\frac{5\pi}{12}\right)$$

$\therefore Arg(u-v)=\frac{\pi}{2}$ (as $2\sin\left(\frac{5\pi}{12}\right)>0$)

$u+v=\left[\cos\left(\frac{5\pi}{12}\right)+i\sin\left(\frac{5\pi}{12}\right)\right]+\left[\cos\left(\frac{5\pi}{12}\right)-i\sin\left(\frac{5\pi}{12}\right)\right]$

$$=2\cos\left(\frac{5\pi}{12}\right)+0i$$

$\therefore Arg(u+v)=0$ (as $2\cos\left(\frac{5\pi}{12}\right)>0$)

c. Note that $Arg(u)=5\pi/12$ and $Arg(v)=Arg\left(\overline{u}\right)=-5\pi/12$.

$$\therefore Arg(uv)=Arg(u)+Arg(v)=\frac{5\pi}{12}+\left(\frac{-5\pi}{12}\right)=0$$

$$\therefore Arg\left(\frac{u}{v}\right)=Arg(u)-Arg(v)=\frac{5\pi}{12}-\left(\frac{-5\pi}{12}\right)=\frac{5\pi}{6}$$

d. Note that $u+v=2\cos(5\pi/12)=2\times\frac{\sqrt{2}(\sqrt{3}-1)}{4}=\frac{\sqrt{2}(\sqrt{3}-1)}{2}$ and $uv=1$.

Now, u and v are roots of a quadratic polynomial of the form

$$P(z)=(z-u)(z-v)=z^2-(u+v)z+uv$$

$$\therefore P(z)=z^2-\left(\frac{\sqrt{2}(\sqrt{3}-1)}{2}\right)z+1$$

Question 2

a. $x = 0 \Rightarrow f(0) = \sin(0) = 0 \;\therefore x = 0$ is a solution to the equation $f(x) = x$.

$x = 1 \Rightarrow f(1) = \sin\left(\frac{\pi \times 1}{2}\right) = 1 \;\therefore x = 1$ is a solution to the equation $f(x) = x$.

b. Domain of $f = [0,1]$ and range of $f = [0,1]$.

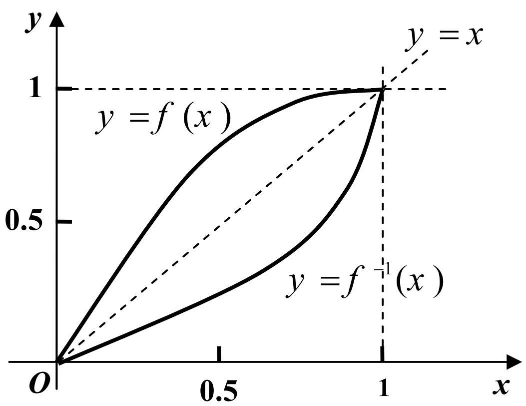
c. Yes, f^{-1}, the inverse function of f, exist because f is a one-to-one function.

d. For f, $y = \sin\left(\frac{\pi x}{2}\right)$, therefore, for f^{-1},

$$x = \sin\left(\frac{\pi y}{2}\right) \Rightarrow \frac{\pi y}{2} = \text{Sin}^{-1} x \Rightarrow y = \frac{2}{\pi}\text{Sin}^{-1} x \Rightarrow f^{-1}(x) = \frac{2}{\pi}\text{Sin}^{-1} x$$

Domain of f^{-1} = range of $f^{-1} = [0,1]$.

e.



f. The area between the graphs of f and f^{-1} is twice the area between the graph of f and the line $y = x$. Therefore,

$$Area = 2\int_0^1 [\sin(\pi x/2) - x]dx = 2\left[-\frac{2}{\pi}\cos(\pi x/2) - \frac{x^2}{2}\right]_0^1$$

$$= 2\left[-\frac{2}{\pi}\cos(\pi/2) - \frac{1^2}{2}\right] - 2\left[-\frac{2}{\pi}\cos(0) - \frac{0^2}{2}\right]$$

$$= 2 \times -\frac{1}{2} - 2 \times -\frac{2}{\pi} = \frac{4}{\pi} - 1 \text{ square units.}$$

g. The required volume is the same as the volume of the solid resulting from rotating the area between the graph of f , the *x*-axis and the line $x = 1$, about the *y*-axis. Therefore,

$$Volume = \pi\int_0^1 \sin^2(\pi x/2)dx = \frac{\pi}{2}\int_0^1 [1-\cos(2\times \pi x/2)]dx$$

$$= \frac{\pi}{2}\int_0^1 [1-\cos(\pi x)]dx = \frac{\pi}{2}\left[x - \frac{1}{\pi}\sin(\pi x)\right]_0^1$$

$$= \frac{\pi}{2}\left[(1-0)-(0)\right] = \frac{\pi}{2} \text{ cubic units}$$

Question 3

a. $V = V_0 + R_{in}\times t - R_{out}\times t \Rightarrow V = V_0 + (R_{in} - R_{out})t$

b. $C_{out} = \dfrac{Q}{V} \Rightarrow C_{out} = \dfrac{Q}{V_0 + (R_{in} - R_{out})t}$

c. $\dfrac{dQ}{dt} = \left(\dfrac{dQ}{dt}\right)_{in} - \left(\dfrac{dQ}{dt}\right)_{out} = R_{in}\times C_{in} - R_{out}\times C_{out}$

$$\frac{dQ}{dt} = R_{in}\times C_{in} - R_{out}\times\frac{Q}{V_0 + (R_{in} - R_{out})t} \Rightarrow \frac{dQ}{dt} = R_{in}C_{in} - \frac{R_{out}Q}{V_0 + (R_{in} - R_{out})t}$$

The initial condition is $Q = Q_0, t = 0$.

d. $R_{out} = R_{in} \Rightarrow \dfrac{dQ}{dt} = R_{in}C_{in} - \dfrac{R_{in}}{V_0}Q \Rightarrow \dfrac{dt}{dQ} = \dfrac{1}{R_{in}C_{in} - (R_{in}/V_0)Q}$

Integrating, we get

$$t = \int \frac{1}{R_{in}C_{in} - (R_{in}/V_0)Q}dQ = -\frac{R_{in}}{V_0}\log_e[R_{in}C_{in} - (R_{in}/V_0)Q] + c$$

To determine c , $Q = Q_0, t = 0 \Rightarrow 0 = -\dfrac{R_{in}}{V_0}\log_e[R_{in}C_{in} - (R_{in}/V_0)Q_0] + c$

$$\therefore c = \frac{R_{in}}{V_0}\log_e[R_{in}C_{in} - (R_{in}/V_0)Q_0]$$

$$t = -\frac{R_{in}}{V_0}\log_e\left(\frac{R_{in}C_{in} - (R_{in}/V_0)Q}{R_{in}C_{in} - (R_{in}/V_0)Q_0}\right)$$

$$\Rightarrow -V_0 t/R_{in} = \log_e\left(\frac{R_{in}C_{in} - (R_{in}/V_0)Q}{R_{in}C_{in} - (R_{in}/V_0)Q_0}\right)$$

$$\Rightarrow \frac{R_{in}C_{in} - (R_{in}/V_0)Q}{R_{in}C_{in} - (R_{in}/V_0)Q_0} = e^{-V_0 t/R_{in}}$$

$$\Rightarrow R_{in}C_{in} - (R_{in}/V_0)Q = [R_{in}C_{in} - (R_{in}/V_0)Q_0]\, e^{-V_0 t/R_{in}}$$

$$\Rightarrow (R_{in}/V_0)Q = R_{in}C_{in} - [R_{in}C_{in} - (R_{in}/V_0)Q_0]\, e^{-V_0 t/R_{in}}$$

$$\therefore\ Q = V_0 C_{in} - (V_0 C_{in} - Q_0)\, e^{-V_0 t/R_{in}}$$

e. $Q_0 = 200, V_0 = 10, C_{in} = 3$ and $R_{out} = R_{in} = 5$

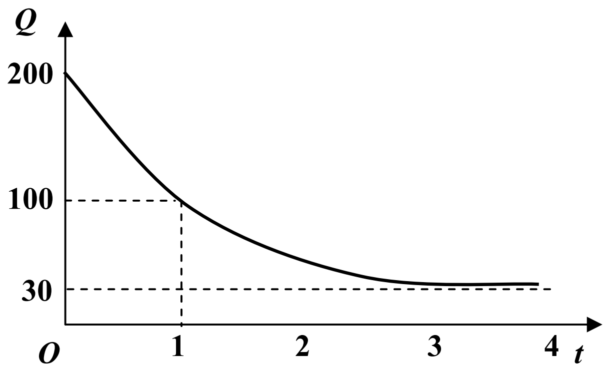
i. $\therefore Q = 10\times 3 - (10\times 3 - 200)\, e^{-10t/5} = 30 + 170e^{-2t}$

$Q = 100/2 = 50 \Rightarrow 50 = 30 + 170e^{-2t} \Rightarrow 20 = 170e^{-2t}$

$\Rightarrow 2 = 17e^{-2t} \Rightarrow e^{2t} = 17/2 \Rightarrow 2t = \log_e(17/2) \Rightarrow t = \frac{1}{2}\log_e\left(\frac{17}{2}\right)$

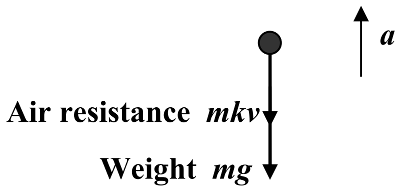
ii. Note that $Q = 30 + 170e^{-2t}$

$t = 0 \Rightarrow Q = 200,\ t = \frac{1}{2}\log_e\left(\frac{17}{2}\right) \approx 1.07 \Rightarrow Q = 100$ & the horizontal asymptote is $Q = 30$.



Question 4

a.



b. The equation of motion is $ma = -mg - mkv \Rightarrow a = -(g + kv)$

c. By choosing $a = \dfrac{dv}{dt}$, we have $\dfrac{dv}{dt} = -(g + kv) \Rightarrow \dfrac{dt}{dv} = -\dfrac{1}{g + kv}$

$$t = -\int_u^0 \frac{1}{g + kv} dv = -\frac{1}{k}\left[\log_e(g + kv)\right]_u^0 = -\frac{1}{k}\left[\log_e(g + 0) - \log_e(g + ku)\right]$$

should be u $\Rightarrow t = \dfrac{1}{k}\log_e\left(\dfrac{g + ku}{g}\right)$

d. By choosing $a = v\dfrac{dv}{dx}$, we have

$$v\frac{dv}{dx} = -(g + kv) \Rightarrow \frac{dv}{dx} = -\frac{g + kv}{v} \Rightarrow \frac{dx}{dv} = -\frac{v}{g + kv}$$

e. Integrating, we have

$$\int_0^h dx = -\int_u^0 \frac{v}{g + kv} dv = -\frac{1}{k}\int_u^0 \frac{kv}{g + kv} dv = -\frac{1}{k}\int_u^0 \frac{-g + (g + kv)}{g + kv} dv$$

$$\Rightarrow [x]_0^h = -\frac{1}{k}\int_u^0 \left(\frac{-g}{g + kv} + 1\right) dv = -\frac{1}{k}\left[\frac{-g}{k}\log_e(g + kv) + v\right]_u^0$$

$$\Rightarrow h = -\frac{1}{k}\left[\frac{-g}{k}\log_e(g + 0) + 0\right] + \frac{1}{k}\left[\frac{-g}{k}\log_e(g + ku) + u\right]$$

$$\Rightarrow h = \frac{g}{k^2}\log_e(g) - \frac{g}{k^2}\log_e(g + ku) + \frac{u}{k} = \frac{1}{k^2}\left[ku - g\log_e\left(\frac{g + ku}{g}\right)\right]$$

f. $h = 33, u = 30, g = 9.8 \Rightarrow \dfrac{1}{k^2}\left[30k - 9.8\log_e\left(\dfrac{9.8 + 30k}{9.8}\right)\right] = 33.$

Use the graphics calculator to sketch the two graphs

$y = \dfrac{1}{x^2}\left[30x - 9.8\log_e\left(\dfrac{9.8 + 30x}{9.8}\right)\right]$ and $y = 33$ for $0 \le x \le 1$. Then find the point of intersection. This gives $x = 0.20$ or $k = 0.20$.

Question 5

a. $\underset{\sim}{r}(0)=a\cos(0)\underset{\sim}{i}+b\sin(0)\underset{\sim}{j}=a\underset{\sim}{i}$. The particle returns to its initial position after one period, i.e. after $\dfrac{2\pi}{\pi/3}=6$ seconds.

b. $\because \underset{\sim}{r}(t)=a\cos\left(\dfrac{\pi t}{3}\right)\underset{\sim}{i}+b\sin\left(\dfrac{\pi t}{3}\right)\underset{\sim}{j}$

$\therefore \underset{\sim}{v}(t)=\dfrac{d}{dt}\left(\underset{\sim}{r}(t)\right)=-\dfrac{a\pi}{3}\sin\left(\dfrac{\pi t}{3}\right)\underset{\sim}{i}+\dfrac{b\pi}{3}\cos\left(\dfrac{\pi t}{3}\right)\underset{\sim}{j}$

Since $\underset{\sim}{v}$ is perpendicular to $\underset{\sim}{r}$, then

$\underset{\sim}{r}\cdot\underset{\sim}{v}=0\Rightarrow -\dfrac{\pi a^2}{3}\sin\left(\dfrac{\pi t}{3}\right)\cos\left(\dfrac{\pi t}{3}\right)+\dfrac{\pi b^2}{3}\sin\left(\dfrac{\pi t}{3}\right)\cos\left(\dfrac{\pi t}{3}\right)=0$

$\Rightarrow \dfrac{\pi(b^2-a^2)}{3}\sin\left(\dfrac{\pi t}{3}\right)\cos\left(\dfrac{\pi t}{3}\right)=0\Rightarrow \dfrac{\pi(b^2-a^2)}{6}\sin\left(\dfrac{2\pi t}{3}\right)=0$

$\Rightarrow \sin\left(\dfrac{2\pi t}{3}\right)=0\Rightarrow \dfrac{2\pi t}{3}=n\pi,\ n=0,1,2,\cdots\Rightarrow t=3n/2,\ n=0,1,2,\cdots$

c. $\underset{\sim}{a}(t)=\dfrac{d}{dt}\left(\underset{\sim}{v}(t)\right)=-\dfrac{a\pi^2}{9}\cos\left(\dfrac{\pi t}{3}\right)\underset{\sim}{i}-\dfrac{b\pi^2}{9}\sin\left(\dfrac{\pi t}{3}\right)\underset{\sim}{j}$

$|\underset{\sim}{a}(t)|=\dfrac{\pi^2}{9}\sqrt{a^2\cos^2\left(\dfrac{\pi t}{3}\right)+b^2\sin^2\left(\dfrac{\pi t}{3}\right)}=\dfrac{\pi^2}{9}\sqrt{a^2\cos^2\left(\dfrac{\pi t}{3}\right)+b^2\left[1-\cos^2\left(\dfrac{\pi t}{3}\right)\right]}$

$=\dfrac{\pi^2}{9}\sqrt{(a^2-b^2)\cos^2\left(\dfrac{\pi t}{3}\right)+b^2}$

Therefore, the magnitude of the acceleration is maximum when

$\cos\left(\dfrac{\pi t}{3}\right)=1\Rightarrow \dfrac{\pi t}{3}=2n\pi,\ n=0,1,2,\cdots\Rightarrow t=6n,\ n=0,1,2,\cdots$

$\max|\underset{\sim}{a}(t)|=\dfrac{\pi^2}{9}\sqrt{a^2-b^2+b^2}=\dfrac{\pi^2 a}{9}.$