2005 Specialist Mathematics
Written Examination 1 (Facts, skills, and applications)
Suggested answers and solutions

Part 1 (Multiple-choice) Answers

1.	**E**	2.	**D**	3.	**C**	4.	**D**	5.	**E**
6	**D**	7.	**A**	8.	**C**	9.	**B**	10.	**B**
11.	**A**	12.	**C**	13.	**A**	14.	**A**	15.	**D**
16.	**E**	17.	**D**	18.	**A**	19.	**B**	20.	**C**
21.	**B**	22.	**B**	23.	**C**	24.	**E**	25.	**A**
26.	**C**	27.	**E**	28.	**A**	29.	**C**	30.	**B**

Part 1 (Multiple-choice) Solutions

Question 1 [E]

$$\frac{3(x+3)^2}{5} + \frac{(y-4)^2}{6} = 3$$

$$\frac{(x+3)^2}{5} + \frac{(y-4)^2}{18} = 1$$

Maximum y-value occurs when $x + 3 = 0$

$$(y-4)^2 = 18$$

$$y - 4 = 3\sqrt{2}$$

$$y = 4 + 3\sqrt{2}$$

Question 2 [D]

For there to be no vertical asymptote

$x^2 + mx - n \neq 0$

This means that $b^2 - 4ac < 0$, where $a = 1$, $b = m$ and $c = -n$

$$\therefore\ m^2 + 4n < 0$$

$$\Rightarrow \quad m^2 < -4n$$

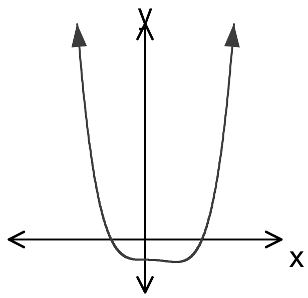
Question 3 [C]

$$x^4 - x^3 = \operatorname{cosec}^2 x - \cot^2 x$$

$$x^4 - x^3 = 1$$

$$x^4 - x^3 - 1 = 0$$

Use a graphics calculator to sketch the graph.



We have two points of intersection

Question 4 [A]

The graph has the same shape as $y = \cot(x)$

$y = \cot(x)$

Period length is $\frac{\pi}{2}$

$y = -\cot(2x)$

Phase shift is $\frac{\pi}{12}$

$$y = -\cot\left(2\left(x - \frac{\pi}{12}\right)\right)$$

$$= -\cot\left(2x - \frac{\pi}{6}\right)$$

Question 5 [E]

$$y = \text{Tan}^{-1}(\sqrt{3x})$$

Let $u = \sqrt{3x}$ and $y = \text{Tan}^{-1}(u)$

$$\frac{du}{dx} = \frac{\sqrt{3}}{2\sqrt{x}} \text{ and } \frac{dy}{du} = \frac{1}{1 + u^2}$$

$$\frac{dy}{dx} = \frac{1}{1 + 3x} \times \frac{\sqrt{3}}{2\sqrt{x}}$$

$$= \frac{\sqrt{3}}{2\sqrt{x}(1 + 3x)}$$

Question 6 [D]

$$z = \frac{3 - 6i}{2 + i} \times \frac{2 - i}{2 - i}$$

$$= -\frac{15i}{5}$$

$$= -3i$$

$$|z| = 3 \text{ and } \text{Arg}(z) = -\frac{\pi}{2}$$

Question 7 [A]

$$u = 7\text{cis}\left(\frac{\pi}{4}\right) \text{ and } v = a\text{cis}(b)$$

$$uv = 7a\text{cis}\left(\frac{\pi}{4} + b\right)$$

$$7a = 42 \quad \text{and} \quad \frac{\pi}{4} + b = \frac{\pi}{20}$$

$$a = 6 \quad \text{and} \quad b = \frac{\pi}{20} - \frac{5\pi}{20} = -\frac{\pi}{5}$$

Question 8 [C]

$$\Delta = b^2 - 4ac$$

$$a = 1 + i,\ b = 4i \text{ and } c = -2(1 - i)$$

$$\Delta = 16i^2 + 8(1 - i)(1 + i)$$

$$= -16 + 16$$

$$= 0$$

Question 9 [B]

$$u = \sqrt{2}\,\text{cis}\left(\frac{\pi}{16}\right)$$

$$z = u^4$$

$$z = (\sqrt{2})^4\text{cis}\left(\frac{4\pi}{16}\right)$$

$$= 4\text{cis}\left(\frac{\pi}{4}\right)$$

$$z^{-1} = \tfrac{1}{4}\,\text{cis}\left(-\frac{\pi}{4}\right)$$

Question 10 [B]

$$|z - 1| + |z + 1| = 3$$

This locus is an ellipse with foci at (1,0) and (-1,0). This sum of the distances from these points is 3.

Question 11 [A]

$$\int \frac{6}{\sqrt{1 - 4x^2}}\,dx$$

$$= \int \frac{6}{2\sqrt{\tfrac{1}{4} - x^2}}\,dx$$

$$= \int \frac{3}{\sqrt{\tfrac{1}{4} - x^2}}\,dx$$

$$= 3\text{Sin}^{-1}(2x)$$

Question 12 **[C]**

$$\int_{\frac{\pi}{2}}^{\pi} \sin^2(2x)\sin(2x)\,dx$$

$$= \int_{\frac{\pi}{2}}^{\pi} \left(1 - \cos^2(2x)\right)\sin(2x)\,dx$$

Let $u = \cos(2x)$

$$\frac{du}{dx} = -2\sin(2x)$$

$$dx = \frac{du}{-2\sin(2x)}$$

For the terminals

$u = \cos(2x)$

When $x = \pi$, $\cos(2\pi) = 1$

When $x = \frac{\pi}{2}$, $\cos(\pi) = -1$

Substituting appropriate values we gain

$$\int_{-1}^{1} \left(1 - u^2\right)\sin(2x) \times \frac{du}{-2\sin(2x)}$$

$$= -\frac{1}{2}\int_{-1}^{1} \left(1 - u^2\right)du$$

Question 13 **[A]**

$$V = \pi \int y^2 dx$$

For $y = \frac{5}{x^2 + 1}$

$$V_1 = \pi \int_0^2 \left(\frac{5}{x^2 + 1}\right)^2 dx$$

For $y = 1$

$$V_2 = \pi \int_0^2 1^2\, dx$$

$$V = V_1 - V_2$$

$$= \pi \int_0^2 \left(\frac{5}{x^2 + 1}\right)^2 - 1\, dx$$

Question 14 **[A]**

Using a graphics calculator for example **fnInt((x+3)/(2sin(x)),x,4,5)** we gain the result -4.014 correct to three decimals places.

Question 15 **[D]**

$$\int x\sqrt{3 - x}\;dx$$

Let $u = 3 - x$

$\Rightarrow x = 3 - u$

$$\frac{du}{dx} = -1$$

$$dx = -du$$

Substituting appropriate values into original equation

$$\int x\sqrt{3 - x}\;dx$$

$$= \int (3 - u)\sqrt{u}\;\; -du$$

$$= -\int 3\sqrt{u} - u^{\frac{3}{2}}\,du$$

$$= -\left[2u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}\right]$$

$$= -2u^{\frac{3}{2}} + \frac{2}{5}u^{\frac{5}{2}}$$

Substituting $u = 3 - x$

$$-2(3 - x)^{\frac{3}{2}} + \frac{2}{5}(3 - x)^{\frac{5}{2}}$$

Question 16 **[E]**

$$\int_0^{\frac{\pi}{8}} \sec^2(2x)\, e^{2\tan(2x)}$$

Let $u = 2\tan(2x)$

$$\frac{du}{dx} = 4\sec^2(2x)$$

$$dx = \frac{du}{4\sec^2(2x)}$$

Evaulating terminals

$$u_1 = 2\tan\left(\frac{\pi}{4}\right) = 2$$

$$u_2 = 2\tan(0) = 0$$

Substituting into the original equation to gain

$$\int_0^2 \sec^2(2x)\, e^u \times \frac{du}{4\sec^2(2x)}$$

$$= \frac{1}{4}\int_0^2 e^u\, du$$

$$= \frac{1}{4}\left[e^u\right]_0^2$$

$$= \frac{1}{4}\left(e^2 - e^0\right)$$

$$= \frac{1}{4}\left(e^2 - 1\right)$$

Question 17 **[D]**

$$\frac{dy}{dx} = f(x) = e^{-x}$$

$x_0 = 2$, $y_0 = 1$, and $h = 0.1$

$x_1 = 2.1$ $\quad y_1 = 1 + 0.1e^{-2} = 1.01353$

$x_2 = 2.2$ $\quad y_2 = 1.01353 + 0.1e^{-2.1} = 1.02578$

$y_2 \approx 1.0258$

Question 18 **[A]**

$$\frac{dS}{dt} = 10$$

$$S = \pi r^2$$

$$\frac{dS}{dt} = 2\pi r$$

$$\frac{dr}{dt} = \frac{dr}{ds} \times \frac{ds}{dt}$$

$$= \frac{1}{2\pi r} \times 10$$

$$= \frac{5}{\pi r}$$

Question 19 **[B]**

$$\frac{dy}{dx} = y^2 + 1$$

$$\frac{dx}{dy} = \frac{1}{y^2 + 1}$$

$$x = \text{Tan}^{-1} y + c$$

At $y = 1$, $x = 0$

$$0 = \text{Tan}^{-1} 1 + c$$

$$0 = \frac{\pi}{4} + c$$

$$c = -\frac{\pi}{4}$$

$$x = \text{Tan}^{-1}(y) - \frac{\pi}{4}$$

$$x + \frac{\pi}{4} = \text{Tan}^{-1}(y)$$

$$y = \tan\left(x + \frac{\pi}{4}\right)$$

Question 20 **[C]**

$$a = v\frac{dv}{dx}$$

$$\frac{dv}{dx} = -\frac{1}{\left(1 - x^2\right)^{\frac{3}{2}}} \times -\frac{2x}{1} = \frac{2x}{\left(1 - x^2\right)^{\frac{3}{2}}}$$

$$a = \frac{2}{\left(1 - x^2\right)^{\frac{1}{2}}} \times \frac{2x}{\left(1 - x^2\right)^{\frac{3}{2}}}$$

$$= \frac{4x}{\left(1 - x^2\right)^2}$$

Alternatively

$$a = \frac{d\left(\frac{1}{2}v^2\right)}{dx}$$

$$v = \frac{2}{\sqrt{1 - x^2}}$$

$$v^2 = \frac{4}{1 - x^2}$$

$$\frac{v^2}{2} = \frac{2}{1 - x^2}$$

$$\frac{d\left(\frac{1}{2}v^2\right)}{dx} = \frac{4x}{\left(1 - x^2\right)^2}$$

Question 21 **[B]**

$$\frac{dv}{dt} = \frac{3}{v^2 - 9}$$

$$dt = \frac{v^2 - 9}{3}\,dv$$

$$t = \int_2^1 \frac{v^2 - 9}{3}\,dv$$

The terminals are reversed because the curve is below the x-axis for the interval [1, 2]

Question 22 **[B]**

For *PQRS* to be a parallelogram

$$\overrightarrow{PO} = \overrightarrow{SR}$$

$$\overrightarrow{PQ} = \underset{\sim}{q} - \underset{\sim}{p}$$
$$= \underset{\sim}{i} + y\underset{\sim}{j} - -3\underset{\sim}{k}$$
$$= \underset{\sim}{i} + y\underset{\sim}{j} + 3\underset{\sim}{k}$$

$$\overrightarrow{SR} = \underset{\sim}{r} - \underset{\sim}{s}$$
$$= 5\underset{\sim}{i} + 2x\underset{\sim}{j} + \underset{\sim}{k} - y\underset{\sim}{i} + 2\underset{\sim}{k}$$
$$= (5 - y)\underset{\sim}{i} + 2x\underset{\sim}{j} + 3\underset{\sim}{k}$$

$$5 - y = 1$$
$$\Rightarrow\ y = 4$$
$$2x = y$$
$$\Rightarrow\ x = 2$$

Question 23 **[C]**

Let the point (2, 2, -1) be represented by the vector $\underset{\sim}{a} = 2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$ and the point (-4, 0, -3) be represented by the vector $\underset{\sim}{b} = -4\underset{\sim}{i} - 3\underset{\sim}{k}$

$$\underset{\sim}{a} \times \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$$

$$\underset{\sim}{a} \times \underset{\sim}{b} = -8 + 0 + 3$$

$$|\underset{\sim}{a}| = \sqrt{4 + 4 + 1} = 3$$

$$|\underset{\sim}{b}| = \sqrt{16 + 9} = 5$$

$$\cos\theta = \frac{\underset{\sim}{a} \times \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$$
$$= \frac{-5}{3 \times 5}$$
$$= -\frac{1}{3}$$

Question 24 **[E]**

$(x + 1)^2 + y^2 + 1$ is a circle with centre (-1,0) and radius 1

Let $x + 1 = \cos t$

$\Rightarrow \; x = \cos t - 1$

This possibility is not evident in the answers.

Let $x + 1 = \sin t$

$\Rightarrow \; x = \sin t - 1$

and $y = \cos t$

$r(t) = (\sin(t) - 1)\,i + \cos(t)\,j$

Question 25 **[A]**

$\underset{\sim}{r}(t) = \left(3t^2 - 2\right)\underset{\sim}{i} - (7 - 5t)\underset{\sim}{j} - 4\underset{\sim}{k}$

$\dot{\underset{\sim}{r}}(t) = 6t\underset{\sim}{i} - -5\underset{\sim}{j}$

$= 6t\,i + 5\,j$

Question 26 **[C]**

$\underset{\sim}{a} = \dfrac{d\underset{\sim}{v}}{dt} = e^{-0.1t}\underset{\sim}{i} + (6t)\underset{\sim}{j}$

$v = 10e^{-0.1t}\,i + 3t^2\,j + c$

At $t = 0$, $v = 0$

$0 = -10\underset{\sim}{i} + \underset{\sim}{c}$

$\underset{\sim}{c} = 10\underset{\sim}{i}$

$\underset{\sim}{v} = -10e^{-0.1t}\underset{\sim}{i} + 3t^2\underset{\sim}{j} + 10\underset{\sim}{i}$

$= 10\left(1 - e^{-0.1t}\right)i + 3t^2\,j$

Question 27 **[E]**

$F = ma$

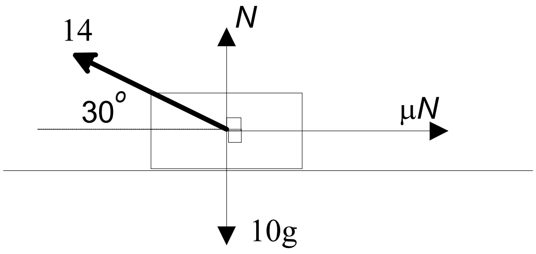
$= 5(20 - 10\cos(2t))$

Max value occurs when $\cos(2t) = -1$

$F = 5(20 + 10)$

$= 150$

Question 28 **[A]**



$7\sqrt{3} = \mu N$

$\mu = \dfrac{7\sqrt{3}}{N}$

$7 + N = 10g$

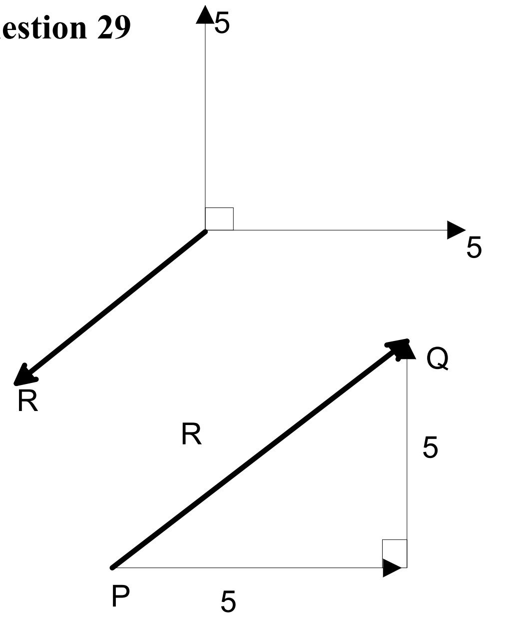
$N = 10g - 7$

$= 98 - 7 = 91$

$\mu = \dfrac{7\sqrt{3}}{91}$

$= \dfrac{\sqrt{3}}{13}$

Question 29 **[C]**



$\underset{\sim}{Q} + \underset{\sim}{P} = 5\sqrt{2}$ North East

$\underset{\sim}{R}$ must act with equal and opposite force, which is $5\sqrt{2}$ Southwest

Question 30 **[B]**

$$F_r = 200a = 1000 - 200g$$

$$a = \tfrac{1000}{200} - \frac{200g}{200}$$

$$= 5 - 9.8$$

$$= -4.8$$

Part II (Short-answer) Solutions

Question 1

a

Equation of line: $y - y_1 = m(x - x_1)$

$$m = \tfrac{2}{3}$$

Point on straight line: (c, 3)

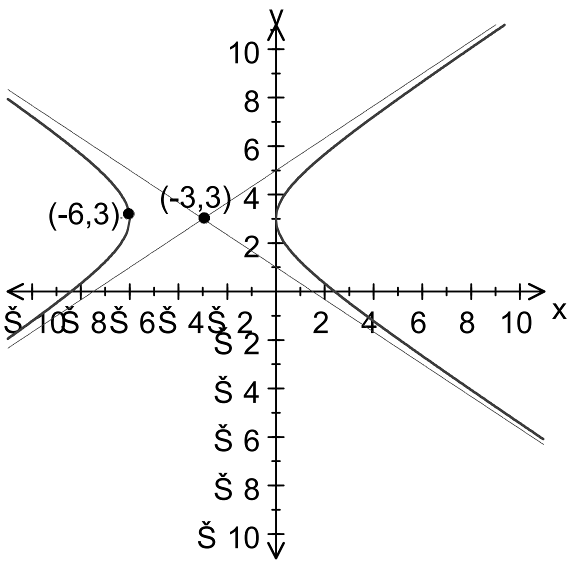
$$y - 3 = \tfrac{2}{3}(x - c)$$

$$y = \tfrac{2}{3}x - \tfrac{2}{3}c + 3$$

$$-\tfrac{2}{3}c + 3 = 5$$

$$c = -3$$

b



Question 2

$$y = e^{2x}\cos(x)$$

$$\frac{dy}{dx} = 2e^{2x}\cos(x) - e^{2x}\sin(x)$$

$$= e^{2x}(2\cos(x) - \sin(x))$$

$$\frac{d^2y}{dx^2} = 2e^{2x}(2\cos(x) - \sin(x)) + e^{2x}(-2\sin(x) - \cos(x)$$

$$= e^{2x}(4\cos(x) - 2\sin(x) - 2\sin(x) - \cos(x)$$

$$= e^{2x}(3\cos(x) - 4\sin(x))$$

$$\frac{d^2y}{dx^2} + k\frac{dy}{dx} + y = -2e^{2x}\sin(x)$$

$$LHS = e^{2x}(3\cos(x) - 4\sin(x)) + ke^{2x}(2\cos(x) - \sin(x)$$

$$+ e^{2x}\cos(x)$$

$$= e^{2x}(4\cos(x) + 2k\cos(x) - 4\sin(x) - k\sin(x))$$

$$\cos(x):\quad 3 + 2k + 1 = 0$$

$$k = -2$$

$$\sin(x):\quad -4 - k = -2$$

$$k = -2$$

Question 3

a

Let $a = 3i - 2i + k$ and $b = 2i + xi + 2k$

We know that the two vectors are perpendicular.

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = 6 - 2x + 2 = 0$$
$$2x = 8$$
$$x = 4$$

b

$$\begin{aligned} \underset{\sim}{u} &= \underset{\sim}{a} + \underset{\sim}{b} \\ &= 3\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k} + 2\underset{\sim}{i} + 4\underset{\sim}{j} + 2\underset{\sim}{k} \\ &= 5\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k} \end{aligned}$$

Question 4

a

$$v = (t - 4)\tan\left(\frac{\pi t}{48}\right)$$

at $t = 12$

$$v = (12 - 4)\tan\left(\frac{12\pi}{48}\right)$$
$$= 8\tan\left(\frac{\pi}{4}\right)$$
$$= 8$$

b

Cyclist B passes Cyclist A when they are the same distance from the starting point.

At $t = 12$

Cyclist A: Dist = 72 m

Cyclist B:

$$\text{Dist} = \int_4^{12} (t - 4)\tan\left(\frac{\pi t}{48}\right) = 22.8922$$

Cyclist B remains 72 – 22.89222 = 49.1078

Cyclist B is traveling 2 m/s faster than Cyclist A

He will now catch Cyclist A in 24.55389 seconds

Therefore it takes him 12 + 24.55389 seconds to catch Cyclist A, which is 36.6 seconds to the nearest tenth of a second.

Question 5

$$135 - \mu N = 0.5(12 + 18)$$
$$N = 30g$$

$$135 - \mu\, 30g = 15$$
$$120 = \mu\, 30g$$
$$\mu = \frac{120}{30g}$$
$$= 0.4081$$
$$\approx 0.41$$

b

$$T - 0.41 \times 18g = 0.5 \times 18$$
$$T = 9 + 0.41 \times 18g$$
$$= 81.324$$

