The Mathematical Association of Victoria

SPECIALIST MATHEMATICS

Trial written examination 2
(Analysis task)

2005

Reading time: 15 minutes
Writing time: 1 hour 30 minutes

Student's Name: ____

QUESTION AND ANSWER BOOK

Structure of book

Number of questions	*Number of questions to be answered*	*Number of marks*
5	5	60

Students are NOT permitted to bring mobile phones and/or any other electronic communication devices into the examination room.

These questions have been written and published to assist students in their preparations for the 2005 Specialist Mathematics Examination 2. The questions and associated answers and solutions do not necessarily reflect the views of the Victorian Curriculum and Assessment Authority. The Association gratefully acknowledges the permission of the Authority to reproduce the formula sheet.

Published by The Mathematical Association of Victoria
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Working Space

Instructions

Answer **all** questions in the spaces sheet provided.

A decimal approximation will not be accepted if an **exact** answer is required to a question.

Where an **exact** answer is required to a question, appropriate working must be shown.

In questions where more than one mark is available, appropriate working must be shown.

Where an instruction to **use calculus** is stated for a question, you must show an appropriate derivative or anti-derivative.

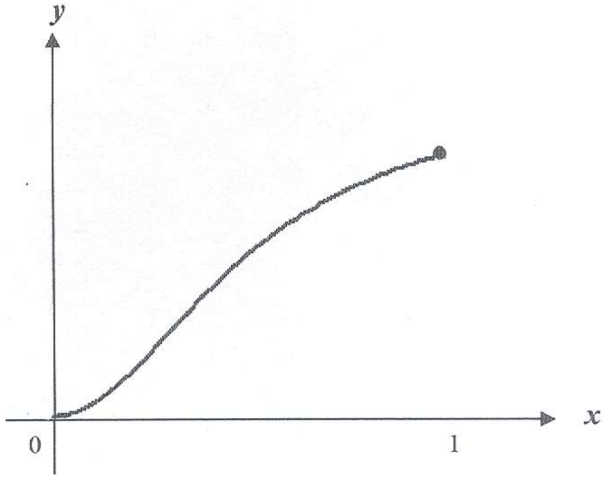
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Take the **acceleraton due to gravity** to have magnatude g m/s^2, where g = 9.8.

Working space

Question 1

The graph of $f : [0,1] \to R$ where $f(x) = 1 - \frac{1}{4x^2 + 1}$ is drawn below.



a. Write down the range of $f(x)$.

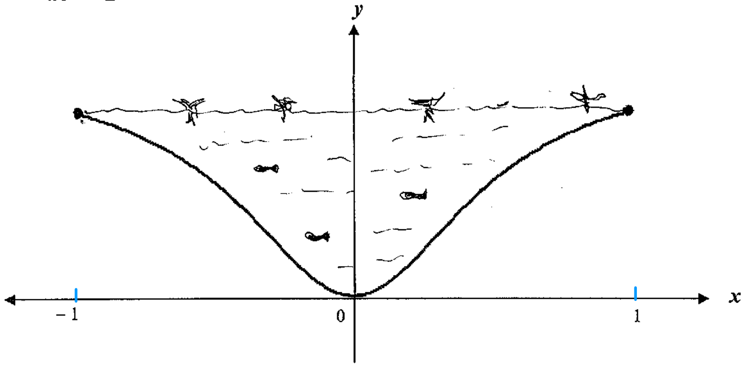
1 mark

b. Use calculus to show that $\left(\frac{\sqrt{3}}{6}, \frac{1}{4}\right)$ is the point of greatest slope of $f(x)$.

4 marks

Joshua wants to build an ornamental pond for his garden. He uses as a model, the function

$f : [0,1] \to R$ where $f(x) = 1 - \frac{1}{4x^2 + 1}$ and rotates it around the y axis. All lengths are measured in metres.



c. Find the volume of the pond when filled to capacity.
Give your answer in m^3 correct to three decimal places.

4 marks

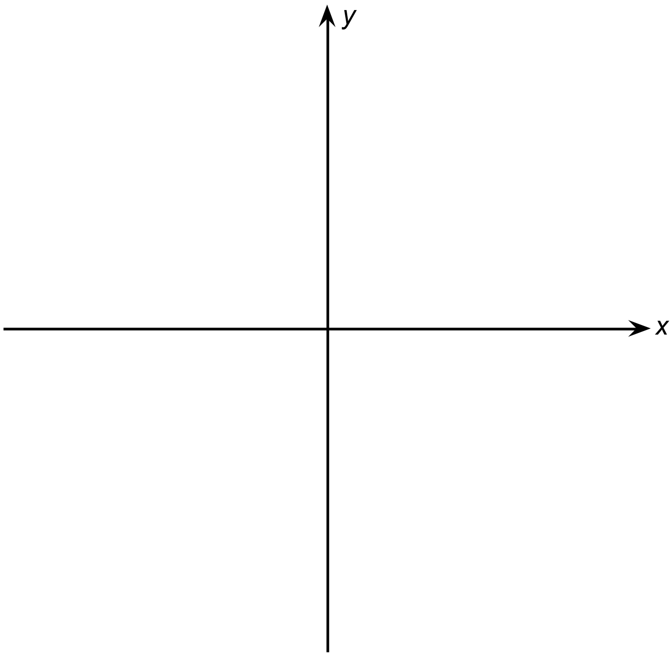
d. Once constructed, Joshua fills the pond with water at the rate of 0.012 m^3/min. Find the rate at which the water level in the pond is rising when the depth of the water reaches the point of greatest slope.

4 marks

Total 13 marks

Question 2

a. Sketch a graph of $y = 3\,\mathrm{Cos}^{-1}(2x)$ over its maximal domain.
Label the coordinates of the endpoints in exact form.



2 marks

b. Let $f(x) = x\,\mathrm{Cos}^{-1}(2x)$

i. Find $f'(x)$

ii. State the domain of $f'(x)$

2 + 1=3 marks

c. Let $g(x) = \sqrt{1 - 4x^2}$
Find $g'(x)$

2 marks

d. **i.** Express $3\,\mathrm{Cos}^{-1}(2x)$ in terms of $f'(x)$ and $g'(x)$

ii. Hence show that $\int_0^{\frac{1}{4}} 3\,\mathrm{Cos}^{-1}(2x)\,dx = \frac{\pi - 3\sqrt{3} + 6}{4}$

2 + 3 = 5 marks
Total 12 marks

continued

Question 3

$u = \sqrt{3} + i$ and $v = 1 - \sqrt{3}i$

a. Express u and v in polar form.

2 marks

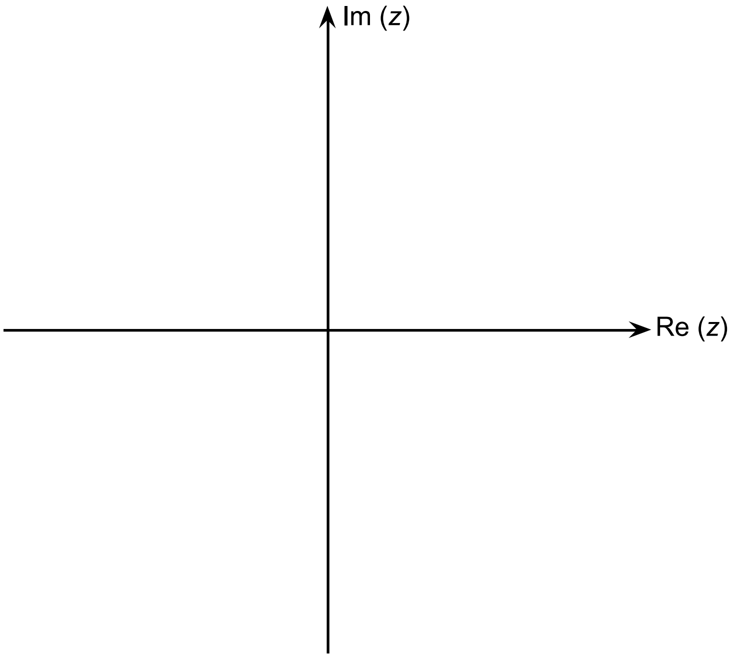
b. Find uv in Cartesian form.

1 mark

c. Find i^2u in polar form.

1 mark

d. Plot u, v and i^2u on an Argand diagram.



2 marks

Question 3– continued
TURN OVER

e. Show $i^2u = i^3v$. Explain your result geometrically.

2 marks

Total 8 marks

Question 4

A researcher studying a particular species of penguin on an island noticed that over time the penguin population was declining. She estimated that the rate at which the penguin population was decreasing was proportional to the number of penguins on the island at the time.

Let N be the number of penguins on the island t years after the study began.
A differential equation that models the decreasing penguin population over time is

given by $\frac{dN}{dt} = kN$ where $k < 0$.

a. At the time the study began there were 700 penguins on the island.
Solve the differential equation to show that $N = 700e^{kt}$.

2 marks

b. Two years after the study began, the penguin population had fallen to 550.
Find the value of k correct to two decimal places.

1 mark

Question 4 – continued

c. Assume this trend continues. How many years after the study began will there be fewer than 50 penguins on the island?

1 mark

By the start of the third year, penguin numbers on the island had fallen to 488. The researcher wanted to arrest this decline. At the start of the third year and each year thereafter, she relocated P penguins of the same species to the island from another location.

A differential equation that models the changing penguin population on the island over time is given by

given by $\frac{dN}{dt} = P + mN$ where $m < 0, t \geq 3$

d. Solve the differential equation to show that $N = \frac{(P + 488m)e^{m(t-3)} - P}{m}$

3 marks

e. **i** If $P = 60$ and $m = -0.05$, find the number of penguins on the island eight years after the study began.

ii. Determine an upper limit to the number of penguins the island can sustain in the longer term.

1 + 1 = 2 marks

f. Suppose that in the longer term the island can sustain fewer than 800 penguins.
Determine a relationship between P and m so that this may occur.

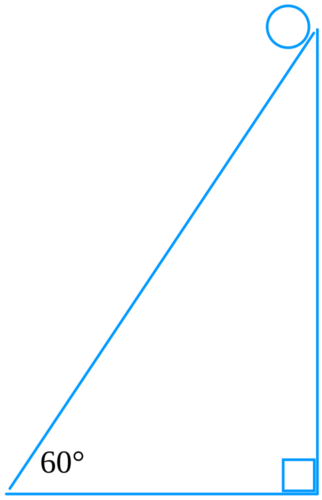
2 marks

Total 11 marks

Question 5

Bushwalking in the Victorian Alps, Scott is walking along a narrow track high above the Howqua River, carrying a 35 kg backpack. Scott stops on the track and removes his backpack to take a short rest.

Unfortunately, Scott slips and his backpack slides down an embankment inclined at an angle of 60° to the horizontal. Assume the acceleration due to gravity, g = 9.8 m/s^2.



a. Find, correct to one decimal place, the Normal reaction force, *N*, that the ground exerts on the backpack, when it is sliding down the embankment.

1 mark

Five metres down the embankment, the backpack collides with a small bush of mass 3 kg, which the sliding backpack rips out of the ground.

b. **i** If the backpack is travelling at a speed of 8 m/s when it collides with the bush, find the acceleration down the slope of the backpack.

1 mark

ii Find, correct to two decimal places, the coefficient of friction, μ, between the backpack and the ground.

2 marks

c. **i** Find the momentum of the backpack the instant before it collides with the bush.

1 mark

ii If the momentum of the tangled bush and backpack after their collision is equal to the momentum of the backpack the instant before the collision, show that the speed of the tangled backpack and bush is $\frac{140}{19}$ m/s.

1 mark

d. If it takes a further 1.4 seconds for the combined backpack and bush to travel the remaining 16 metres to the edge of the cliff, show that the coefficient of friction between the ground and the tangled backpack and bush, correct to two decimal places, is 0.55.

3 marks

e. Find, correct to one decimal place, the speed with which the tangled backpack and bush fall over the cliff.

1 mark

f. At the edge of the cliff, let $\boldsymbol{i}$ be a unit vector in the horizontal x-direction and $\boldsymbol{j}$ be a unit vector in the vertical y-direction as shown in the diagram below. Express the velocity of the tangled backpack and bush, in the form $x\boldsymbol{i} + y\boldsymbol{j}$, at the instant it falls over the cliff, correct to two decimal places.

2 marks

g. The tangled backpack and bush fall over the edge of the cliff with an acceleration given by $\boldsymbol{a} = -g\boldsymbol{j}$. Find, in unit vector form, the velocity of the tangled backpack and bush at any time t, correct to two decimal places.

2 marks

The tangled backpack and bush take 1.5 seconds to fall from the edge of the cliff to the river below.

h. **i** Find, correct to one decimal place, the speed at which they strike the water.

1 mark

ii Find, correct to one decimal place, how far out from the cliff wall the tangled bush and backpack hits the water.

1 mark

END OF QUESTION AND ANSWER BOOK

SPECIALIST MATHEMATICS

Written examinations 1 and 2

FORMULA SHEET

Directions to students

This formula sheet is provided for your reference.

Specialist Mathematics Formulas

Mensuration

area of a trapezium:	$\frac{1}{2}(a+b)h$
curved surface area of a cylinder:	$2\pi rh$
volume of a cylinder:	$\pi r^2 h$
volume of a cone:	$\frac{1}{3}\pi r^2 h$
volume of a pyramid:	$\frac{1}{3}Ah$
volume of a sphere:	$\frac{4}{3}\pi r^3$
area of a triangle:	$\frac{1}{2}bc\sin A$
sine rule:	$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$
cosine rule:	$c^2 = a^2 + b^2 - 2ab\cos C$

Coordinate geometry

ellipse: $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$

hyperbola: $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$

Circular (trigometric) functions

$\cos^2(x)+\sin^2(x)=1$

$1+\tan^2(x)=\sec^2(x)$

$\cot^2(x)+1=\mathrm{cosec}^2(x)$

$\sin(x+y)=\sin(x)\cos(y)+\cos(x)\sin(y)$

$\sin(x-y)=\sin(x)\cos(y)-\cos(x)\sin(y)$

$\cos(x+y)=\cos(x)\cos(y)-\sin(x)\sin(y)$

$\cos(x-y)=\cos(x)\cos(y)+\sin(x)\sin(y)$

$\tan(x+y)=\frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$

$\tan(x-y)=\frac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}$

$\cos(2x)=\cos^2(x)-\sin^2(x)=2\cos^2(x)-1=1-2\sin^2(x)$

$\sin(2x)=2\sin(x)\cos(x)$

$\tan(2x)=\frac{2\tan(x)}{1-\tan^2(x)}$

function	Sin^{-1}	Cos^{-1}	Tan^{-1}
domain	$[-1, 1]$	$[-1, 1]$	R
range	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$	$[0, \pi]$	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

Algebra (Complex numbers)

$z = x + yi = r(\cos\theta + i\sin\theta) = r\,\mathrm{cis}\,\theta$

$|z|=\sqrt{x^2+y^2}=r$

$-\pi < \mathrm{Arg}\, z \le \pi$

$z_1 z_2 = r_1 r_2\,\mathrm{cis}(\theta_1+\theta_2)$

$\frac{z_1}{z_2}=\frac{r_1}{r_2}\mathrm{cis}(\theta_1-\theta_2)$

$z^n = r^n\,\mathrm{cis}(n\theta)$ (de Moivre's theorem)

Calculus

$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$

$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c, n \neq -1$

$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$

$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$

$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$

$\int \frac{1}{x}\,dx = \log_e(x) + c$, for $x > 0$

$\frac{d}{dx}\left(\sin(ax)\right) = a\cos(ax)$

$\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + c$

$\frac{d}{dx}\left(\cos(ax)\right) = -a\sin(ax)$

$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + c$

$\frac{d}{dx}\left(\tan(ax)\right) = a\sec^2(ax)$

$\int \sec^2(ax)\,dx = \frac{1}{a}\tan(ax) + c$

$\frac{d}{dx}\left(\text{Sin}^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}}$

$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \text{Sin}^{-1}\left(\frac{x}{a}\right) + c, a > 0$

$\frac{d}{dx}\left(\text{Cos}^{-1}(x)\right) = \frac{-1}{\sqrt{1-x^2}}$

$\int \frac{-1}{\sqrt{a^2-x^2}}\,dx = \text{Cos}^{-1}\left(\frac{x}{a}\right) + c, a > 0$

$\frac{d}{dx}\left(\text{Tan}^{-1}(x)\right) = \frac{1}{1+x^2}$

$\int \frac{a}{a^2+x^2}\,dx = \text{Tan}^{-1}\left(\frac{x}{a}\right) + c$

product rule: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

chain rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

mid-point rule: $\int_a^b f(x)\,dx \approx (b-a)f\left(\frac{a+b}{2}\right)$

trapezoidal rule: $\int_a^b f(x)\,dx \approx \frac{1}{2}(b-a)\left(f(a) + f(b)\right)$

Euler's method: If $\frac{dy}{dx} = f(x)$, $x_0 = a$ and $y_0 = b$, then $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + hf(x_n)$

acceleration: $a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

constant (uniform) acceleration: $v = u + at$ $\quad s = ut + \frac{1}{2}at^2$ $\quad v^2 = u^2 + 2as$ $\quad s = \frac{1}{2}(u+v)t$

Vectors in two and three dimensions

$\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$

$|\underset{\sim}{r}| = \sqrt{x^2 + y^2 + z^2} = r$

$\underset{\sim}{r}_1 \cdot \underset{\sim}{r}_2 = r_1 r_2 \cos\theta = x_1 x_2 + y_1 y_2 + z_1 z_2$

$\dot{\underset{\sim}{r}} = \frac{d\underset{\sim}{r}}{dt} = \frac{dx}{dt}\underset{\sim}{i} + \frac{dy}{dt}\underset{\sim}{j} + \frac{dz}{dt}\underset{\sim}{k}$

Mechanics

momentum: $\underset{\sim}{p} = m\underset{\sim}{v}$

equation of motion: $\underset{\sim}{R} = m\underset{\sim}{a}$

friction: $F \le \mu N$

END OF FORMULA SHEET