VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victorian Certificate of Education 2004

SUPERVISOR TO ATTACH PROCESSING LABEL HERE

STUDENT NUMBER

Figures									Letter:
Words									

SPECIALIST MATHEMATICS

Written examination 2 (Analysis task)

Wednesday 3 November 2004

Reading time: 11.45 am to 12.00 noon (15 minutes)
Writing time: 12.00 noon to 1.30 pm (1 hour 30 minutes)

QUESTION AND ANSWER BOOK

Structure of book

Number of questions	*Number of questions to be answered*	*Number of marks*
5	5	60

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, up to four pages (two A4 sheets) of pre-written notes (typed or handwritten) and an approved scientific and/or graphics calculator (memory may be retained).
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Materials supplied
- Question and answer book of 15 pages with a detachable sheet of miscellaneous formulas in the centrefold.
- Working space is provided throughout the book.

Instructions
- Detach the formula sheet from the centre of this book during reading time.
- Write your **student number** in the space provided above on this page.
- All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other electronic communication devices into the examination room.

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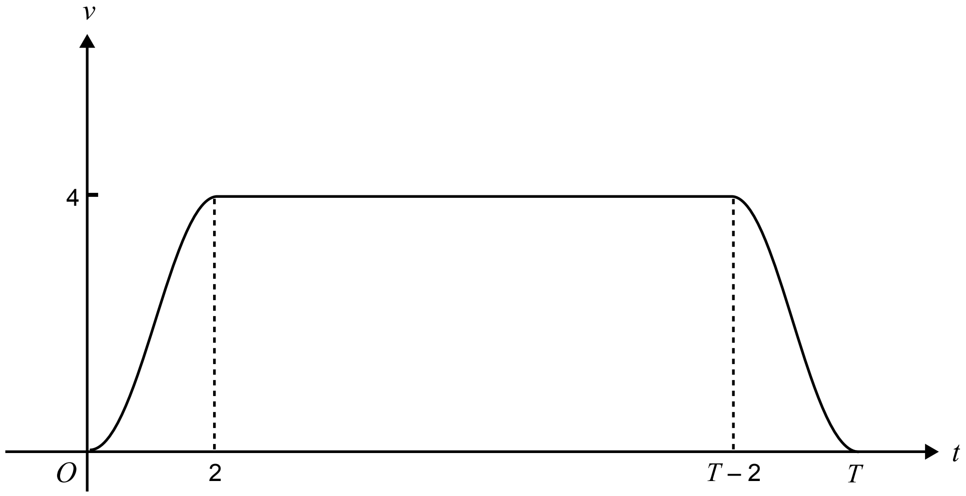
Instructions

Answer **all** questions in the spaces provided.
A decimal approximation will not be accepted if an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working must be shown.
Where an instruction to **use calculus** is stated for a question, you must show an appropriate derivative or antiderivative.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g m/s^2, where $g = 9.8$.

Working space

Question 1

The velocity-time graph below shows the velocity of a lift as it travels from the first floor to the twelfth floor of a tall building during the T seconds of its motion.



The velocity v m/s at time t s for $0 \le t \le 2$ is given by $v = t^2(3 - t)$. After the first two seconds, the lift moves with a constant velocity of 4 m/s for a time, and then decelerates to rest in the final two seconds.

The acceleration of the lift is a m/s^2 at time t s, and the velocity-time graph is symmetrical about $t = \frac{1}{2}T$.

a. **i.** Express a in terms of t for the first two seconds of the motion of the lift.

1 mark

ii. Hence find the maximum acceleration of the lift during the first two seconds of its motion.

2 marks

b. Given that the total distance travelled by the lift during its journey is 41 metres, use calculus to find the exact value of T.

3 marks

A girl of mass 29 kg is standing in the lift during its journey. The variable reaction force on the girl due to the lift is R newtons.

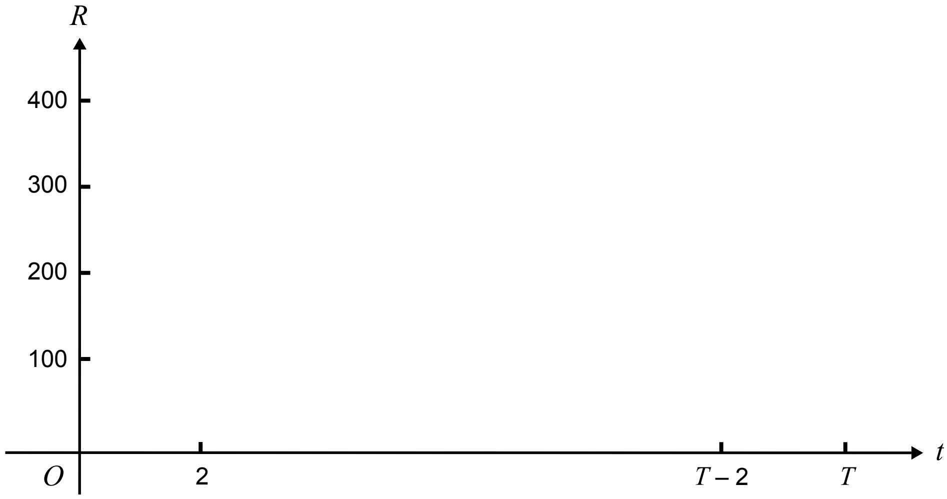
c. Show that $R = 29(a + g)$.

1 mark

d. Hence find the maximum value of R correct to the nearest integer.

1 mark

e. Sketch the graph of R versus t on the axes below.



3 marks

f. A second lift travels from the twelfth floor to the first floor of the building. It begins its journey at exactly the same time as the first lift. At all times during their motions, both lifts move with the same speed.

Exactly one second after the two lifts begin to move, the reaction force on a boy standing in the second lift is equal in magnitude to the reaction force on the girl in the first lift.

i. Find the mass of the boy in kg, correct to one decimal place.

3 marks

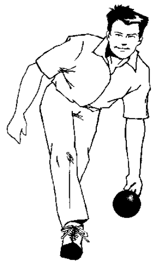
ii. Will there be another instant when the reaction forces on the girl and the boy are equal in magnitude? Explain your answer.

1 mark

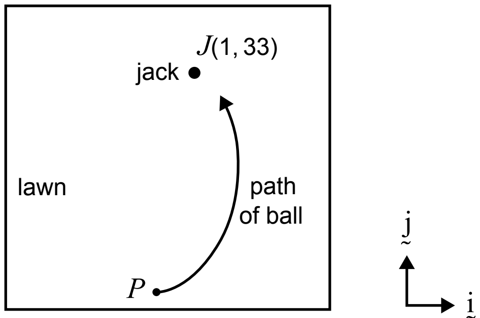
Total 15 marks

Question 2

The game of lawn bowls is played on a horizontal lawn. The aim is to roll a ball (usually called a 'bowl') to come to rest as close as possible to a target ball called the 'jack'.



Bowler



View from above

Let $\underset{\sim}{\mathrm{i}}$ be a unit vector to the right and $\underset{\sim}{\mathrm{j}}$ be a unit vector in the forward direction as shown. Displacements are measured in metres.

At one stage during a game, the jack is at the point $J(1, 33)$. The path of a particular ball in this game is modelled by

$$\underset{\sim}{\mathrm{r}}(t) = 2\sin\left(\frac{2}{15}t\right)\underset{\sim}{\mathrm{i}} + \left(2 + \frac{5}{3}t - \frac{5}{3}\sin\left(\frac{1}{3}t\right)\right)\underset{\sim}{\mathrm{j}},\ 0 \le t \le \frac{15}{2}\pi$$

where t is the time in seconds after the ball is released from the point P.

a. Write down the coordinates of P.

1 mark

b. Find an expression for the velocity, in metres per second, of the ball at time t seconds after the ball is released.

2 marks

c. At the instant the ball is released, what angle does its path make with the forward direction? Give your answer correct to the nearest tenth of a degree.

3 marks

Question 2 – continued

d. At what time, correct to the nearest tenth of a second, does the ball begin to swing left towards the jack?

2 marks

e. Determine whether the path of the ball passes through *J*.

4 marks

Total 12 marks

Question 3

The volume, V litres, of oil in an irregularly shaped tank, when the oil depth is h metres, is given by $V = 8000\,h\,\mathrm{Tan}^{-1}(h)$.

a. **i.** Find the exact volume of oil in the tank, in litres, when the oil depth is one metre.

1 mark

ii. Find the oil depth, correct to the nearest centimetre, when its volume is 10 000 litres.

1 mark

The tank is initially empty. Oil is then poured into the tank at a constant rate of 2000 litres per minute.

b. Find, in terms of h, an expression for the rate at which the oil depth is increasing, in metres per minute, when the depth is h metres.

3 marks

c. **i.** Write a definite integral, the value of which gives the time it takes in minutes for the oil depth in the tank to reach $\sqrt{3}$ metres.

1 mark

ii. Show that the exact time taken for the oil depth to reach a depth of $\sqrt{3}$ metres is $\frac{4\pi}{\sqrt{3}}$ minutes.

2 marks

Total 8 marks

Question 4

Consider the cubic equation $z^3 + az^2 + bz + c = 0$, where a, b and c are real numbers. Two of the roots of this equation are 4 and $-1 - 2i$.

a. **i.** State the third root.

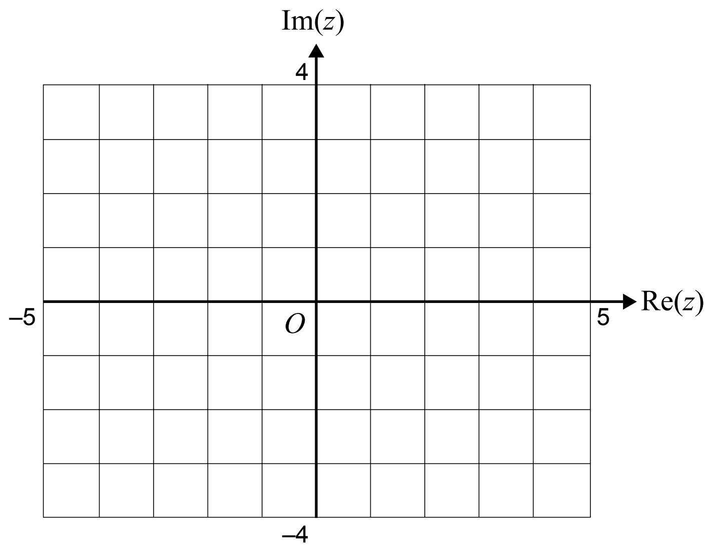
1 mark

ii. Find the values of a, b and c.

2 marks

b. Plot the three roots on the Argand diagram below.

Label the real root M, and the complex roots P and Q where P lies above the real axis.



1 mark

Let $\underset{\sim}{i}$ be a unit vector in the direction of the real axis and $\underset{\sim}{j}$ be a unit vector in the direction of the imaginary axis.

c. Express the vector $\overrightarrow{MQ}$ in terms of $\underset{\sim}{i}$ and $\underset{\sim}{j}$.

1 mark

A circle K passes through P, Q, M and $D(d, 0)$, where $d < 0$.

d. i. Express the vector $\overrightarrow{DQ}$ in terms of d, $\underset{\sim}{i}$ and $\underset{\sim}{j}$.

1 mark

ii. Use a scalar product to find d.

2 marks

The straight line that passes through Q and O can be described as the subset S of the complex plane, where $S = \{z : wz + \overline{w}\,\overline{z} = 0, z \in C\}$ and w is a complex constant.

e. Find a possible w.

3 marks

The chord of the circle K that passes through Q and O can be described as the intersection $S \cap T$, where S is the subset defined above, and T is another subset of the complex plane.

f. Find a possible T.

3 marks

Total 14 marks

Working space

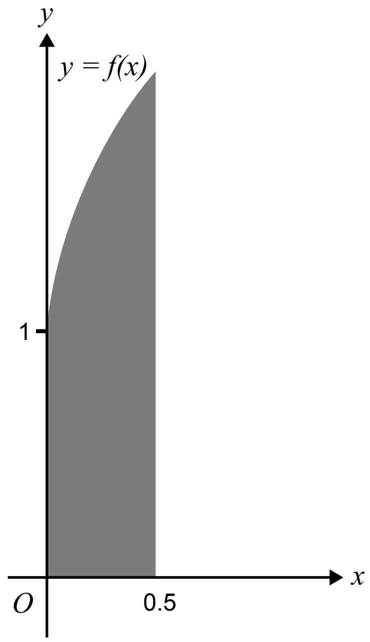
Question 5

Consider the function f with rule $f(x) = 2x^{\frac{1}{2}}\left(1-x^2\right)^{\frac{1}{4}} + \dfrac{1}{\left(1-x^2\right)^{\frac{1}{4}}}$.

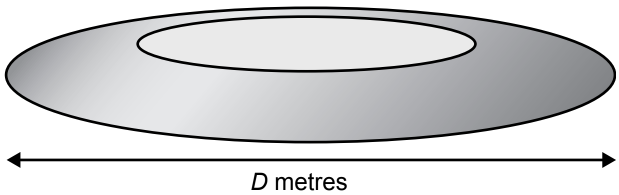
a. State the largest domain for which f is defined.

1 mark

A solid platform for a statue is constructed by rotating, about the x-axis, the region enclosed by the curve $y = f(x)$, the line $x = 0.5$, and the coordinate axes. Lengths are measured in metres.



The platform is laid flat on a horizontal surface as shown below (diagram not to scale).



b. Let D metres be the diameter of the base of the platform. Find D correct to three significant figures.

1 mark

c. Find, correct to the nearest degree, the angle of slope of the platform at any point where it meets the horizontal surface.

3 marks

d. **i.** Two of the three terms in the expansion of $(f(x))^2$ are $4x\sqrt{1-x^2}$ and $\dfrac{1}{\sqrt{1-x^2}}$.
Find the third term.

1 mark

ii. Use calculus to find the volume, in cubic metres, of the platform, correct to two significant figures.

5 marks

Total 11 marks

END OF QUESTION AND ANSWER BOOK

SPECIALIST MATHEMATICS

Written examinations 1 and 2

FORMULA SHEET

Directions to students

Detach this formula sheet during reading time.
This formula sheet is provided for your reference.

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Specialist Mathematics Formulas

Mensuration

area of a trapezium:	$\frac{1}{2}(a+b)h$
curved surface area of a cylinder:	$2\pi rh$
volume of a cylinder:	$\pi r^2 h$
volume of a cone:	$\frac{1}{3}\pi r^2 h$
volume of a pyramid:	$\frac{1}{3}Ah$
volume of a sphere:	$\frac{4}{3}\pi r^3$
area of a triangle:	$\frac{1}{2}bc\sin A$
sine rule:	$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$
cosine rule:	$c^2 = a^2 + b^2 - 2ab\cos C$

Coordinate geometry

ellipse: $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$

hyperbola: $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$

Circular (trigometric) functions

$\cos^2(x) + \sin^2(x) = 1$

$1 + \tan^2(x) = \sec^2(x)$

$\cot^2(x) + 1 = \operatorname{cosec}^2(x)$

$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$

$\sin(x - y) = \sin(x)\cos(y) - \cos(x)\sin(y)$

$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$

$\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y)$

$\tan(x+y)=\frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$

$\tan(x-y)=\frac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}$

$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$

$\sin(2x) = 2\sin(x)\cos(x)$

$\tan(2x)=\frac{2\tan(x)}{1-\tan^2(x)}$

function	Sin^{-1}	Cos^{-1}	Tan^{-1}
domain	$[-1, 1]$	$[-1, 1]$	R
range	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$	$[0, \pi]$	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

Algebra (Complex numbers)

$z = x + yi = r(\cos\theta + i\sin\theta) = r\operatorname{cis}\theta$

$|z|=\sqrt{x^2+y^2}=r$

$-\pi < \operatorname{Arg} z \le \pi$

$z_1z_2 = r_1r_2\operatorname{cis}(\theta_1 + \theta_2)$

$\frac{z_1}{z_2}=\frac{r_1}{r_2}\operatorname{cis}(\theta_1-\theta_2)$

$z^n = r^n\operatorname{cis}(n\theta)$ (de Moivre's theorem)

Calculus

$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$	$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c, n \neq -1$
$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$	$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$
$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$	$\int \frac{1}{x}\,dx = \log_e(x) + c$, for $x > 0$
$\frac{d}{dx}\left(\sin(ax)\right) = a\cos(ax)$	$\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + c$
$\frac{d}{dx}\left(\cos(ax)\right) = -a\sin(ax)$	$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + c$
$\frac{d}{dx}\left(\tan(ax)\right) = a\sec^2(ax)$	$\int \sec^2(ax)\,dx = \frac{1}{a}\tan(ax) + c$
$\frac{d}{dx}\left(\mathrm{Sin}^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \mathrm{Sin}^{-1}\left(\frac{x}{a}\right) + c, a > 0$
$\frac{d}{dx}\left(\mathrm{Cos}^{-1}(x)\right) = \frac{-1}{\sqrt{1-x^2}}$	$\int \frac{-1}{\sqrt{a^2-x^2}}\,dx = \mathrm{Cos}^{-1}\left(\frac{x}{a}\right) + c, a > 0$
$\frac{d}{dx}\left(\mathrm{Tan}^{-1}(x)\right) = \frac{1}{1+x^2}$	$\int \frac{a}{a^2+x^2}\,dx = \mathrm{Tan}^{-1}\left(\frac{x}{a}\right) + c$

product rule:	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$
quotient rule:	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
chain rule:	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$
mid-point rule:	$\int_a^b f(x)\,dx \approx (b-a)f\left(\frac{a+b}{2}\right)$
trapezoidal rule:	$\int_a^b f(x)\,dx \approx \frac{1}{2}(b-a)\left(f(a)+f(b)\right)$
Euler's method:	If $\frac{dy}{dx} = f(x)$, $x_0 = a$ and $y_0 = b$, then $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + hf(x_n)$
acceleration:	$a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$
constant (uniform) acceleration:	$v = u + at$ $\quad s = ut + \frac{1}{2}at^2$ $\quad v^2 = u^2 + 2as$ $\quad s = \frac{1}{2}(u+v)t$

Vectors in two and three dimensions

$\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$

$|\underset{\sim}{r}| = \sqrt{x^2 + y^2 + z^2} = r$

$\underset{\sim}{r}_1 \cdot \underset{\sim}{r}_2 = r_1 r_2 \cos\theta = x_1 x_2 + y_1 y_2 + z_1 z_2$

$\dot{\underset{\sim}{r}} = \frac{d\underset{\sim}{r}}{dt} = \frac{dx}{dt}\underset{\sim}{i} + \frac{dy}{dt}\underset{\sim}{j} + \frac{dz}{dt}\underset{\sim}{k}$

Mechanics

momentum: $\underset{\sim}{p} = m\underset{\sim}{v}$

equation of motion: $\underset{\sim}{R} = m\underset{\sim}{a}$

friction: $F \le \mu N$

END OF FORMULA SHEET