MAV Specialist Mathematics Examination 2
Answers & Solutions

Question 1

a $4x^2 + y^2 - 8y = 0$

$4x^2 + y^2 - 8y + 16 = 16$

$4x^2 + (y-4)^2 = 16$ **[M1]**

$\frac{x^2}{4} + \frac{(y-4)^2}{16} = 1$ **[A1]**

$\frac{x^2}{2^2} + \frac{(y-4)}{4^2} = 1$

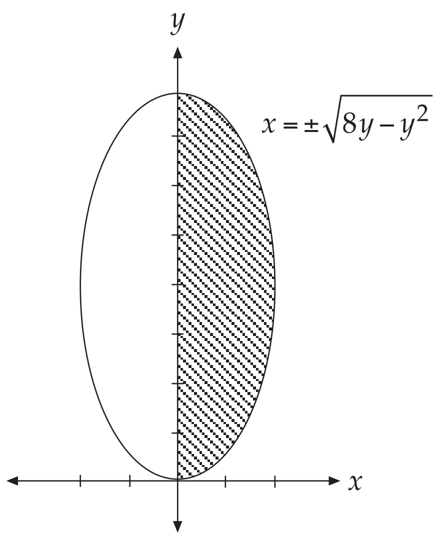
b i $4x^2 = 8y - y^2$

$x^2 = \frac{8y - y^2}{4}$

$x = \pm\frac{1}{2}\sqrt{8y - y^2}$ **[A1]**

The area of the ellipse will be double the shaded area.

$\therefore A = 2 \times \int_{y=0}^{y=8} \frac{1}{2}\sqrt{8y - y^2}\,dy$ **[M1]**



$= \int_0^8 \sqrt{8y - y^2}\,dy$ use integration function on a graphics calculator

= 25.13 square units **[A1]**

Note: Area of an ellipse given by $A = \pi ab$ is also acceptable.

b ii Volume formed by rotating about the y-axis is given by $V = \int \pi x^2 dy$

$V = \int_0^8 \pi\left(\frac{8y - y^2}{4}\right)dy$ **[M1]**

$= \frac{\pi}{4}\int_0^8 \left(8y - y^2\right)dy$

$= \frac{\pi}{4}\left[4y^2 - \frac{1}{3}y^3\right]_0^8$ **[A1]**

$= \frac{\pi}{4}\left[\left(256 - \frac{512}{3}\right) - 0\right]$

$= \frac{64\pi}{3}$ **[A1]**

c i Being released from the balloon, the tennis ball will initially also be moving upwards at 3m/s. Considering upwards as the positive direction: $u = 3$ m/s, $a = -9.8$ m/s^2, $s = -75$m.

$s = ut + \frac{1}{2}at^2$ **[M1]**

$-75 = 3t + \frac{1}{2}(-9.8)t^2$

$9.8t^2 - 6t - 150 = 0$ using the quadratic formula

$t = 4.230, -3.618$

$\therefore$ $t = 4.230$, since $t > 0$ **[A1]**

c ii $u = 3$ m/s, $a = -9.8$ m/s^2, $s = -75$ m, $v = ?$

$v^2 - u^2 = 2as$ **[M1]**

$v^2 = 2as + u^2$

$v = 38.458$ m/s **[A1]**

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Question 2

a i $z^2 - 4z + 6 = 0$

$z = \dfrac{4 \pm \sqrt{16 - 4(1)(6)}}{2}$

$= \dfrac{4 \pm \sqrt{-8}}{2}$

$= \dfrac{4 \pm \sqrt{8i^2}}{2}$

$= \dfrac{4 \pm 2i\sqrt{2}}{2}$

$= 2 \pm i\sqrt{2}$ **[A1]**

a ii $\tan\theta = \dfrac{\sqrt{2}}{2}$

$\theta = 35.26°$ **[A1]**

$z = \sqrt{6}cis35.26°, \sqrt{6}cis(-35.26°)$ **[A1]**

b i **[M1]**

$$\begin{array}{r|l} & z + (a+4) \\ \hline z^2 - 4z + 6 & z^3 + az^2 + bz + 6 \\ & z^3 - 4z^2 + 6z \\ \hline & (a+4)z^2 + (b-6)z + 6 \\ & (a+4)z^2 + (-4a-16)z + (6a+24) \\ \hline & (b+4a+10)z + (-6a-18) \end{array}$$

[A1]

Since $z^2 - 4z + 6$ is a solution, then remainder = 0.

$\therefore b + 4a + 10 = 0$ **[M1]**

$-6a - 18 = 0 \Rightarrow a = -3, b = 2$ **[A1]**

b ii Since $a = -3$ then the third factor is $z + 1$, hence third solution is $z = -1$. **[A1]**

Question 3

a $\dfrac{dN}{dt} = kN$

$\dfrac{dt}{dN} = \dfrac{1}{kN}$

$t = \dfrac{1}{k}\log_e N + c$ **[M1]**

$t = 0, N = 10 \Rightarrow c = -\dfrac{1}{k}\log_e 10$

$\therefore t = \dfrac{1}{k}\log_e N - \dfrac{1}{k}\log_e 10$

$t = \dfrac{1}{k}\log_e\left(\dfrac{N}{10}\right)$ **[M1]**

$tk = \log_e\left(\dfrac{N}{10}\right)$

$e^{kt} = \dfrac{N}{10}$ **[M1]**

$N = 10e^{kt}$

b $N = 10e^{kt}$

$t = 2, N = 70$

$70 = 10e^{2k}$

$\log_e 7 = \log_e e^{2k}$ **[M1]**

$k = \dfrac{1}{2}\log_e 7$

c $N = 10e^{\left(\frac{1}{2}\log_e 7\right)5}$

N = 1296 birds **[A1]**

d $10000 = 10e^{\left(\frac{1}{2}\log_e 7\right)t}$

$\log_e 1000 = \left(\dfrac{1}{2}\log_e 7\right)t$

$t = \dfrac{2\log_e 1000}{\log_e 7}$ **[M1]**

$t = 7.099$, therefore it takes 8 years to exceed 10 000. **[A1]**

e i $\frac{dN}{dt} = kN(6000-N)$

$\frac{dt}{dN} = \frac{1}{kN(6000-N)}$

$t = \frac{1}{k}\int \frac{1}{N(6000-N)}dN$

$kt = \int \frac{1}{N(6000-N)}dN$

$\frac{1}{N(6000-N)} \equiv \frac{a}{N} + \frac{b}{6000-N}$

$\equiv \frac{a(6000-N)+bN}{N(6000-N)}$ **[M1]**

$1 \equiv a(6000-N)+bN$

$N = 0, a = \frac{1}{6000}$

$N = 6000, b = \frac{1}{6000}$

$\therefore kt = \frac{1}{6000}\int\left(\frac{1}{N} + \frac{1}{6000-N}\right)dN$

$6000kt = \log_e N - \log_e(6000-N) + c$ **[A1]**

$6000kt - c = \log_e\left(\frac{N}{6000-N}\right)$

$e^{6000kt}.e^{-c} = \frac{N}{6000-N}$

$Ae^{6000kt} = \frac{N}{6000-N}$, where $A = e^{-c}$

$N = (6000-N)Ae^{6000kt}$ **[M1]**

$N = 6000Ae^{6000kt} - NAe^{6000kt}$

$N + NAe^{6000kt} = 6000Ae^{6000kt}$

$N\left(1+Ae^{6000kt}\right) = 6000Ae^{6000kt}$ **[A1]**

$N = \frac{6000Ae^{6000kt}}{1+Ae^{6000kt}}$

e ii $t = 0, N = 10 \Rightarrow 10 = \frac{6000A}{1+A}$

$10 + 10A = 6000A$

$10 = 5990A$

$A = \frac{1}{599}$ **[A1]**

Question 4

a $\overrightarrow{OA} = 60\underset{\sim}{j}$

$\overrightarrow{OB} = 50\underset{\sim}{i} + 90\underset{\sim}{j} + 5\underset{\sim}{k}$

$\overrightarrow{OC} = 80\underset{\sim}{i} + 65\underset{\sim}{j} + 17\underset{\sim}{k}$

$\overrightarrow{OD} = 15\underset{\sim}{i} + 45\underset{\sim}{j} + 20\underset{\sim}{k}$ **[A2]**

b i $\overrightarrow{BC} = 30\underset{\sim}{i} - 25\underset{\sim}{j} + 12\underset{\sim}{k}$

$\left|\overrightarrow{BC}\right| = \sqrt{30^2 + 25^2 + 12^2}$

$\simeq 41$ metres **[A1]**

b ii $\cos\theta = \frac{\overrightarrow{AB}.\overrightarrow{BC}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{BC}\right|}$ **[M1]**

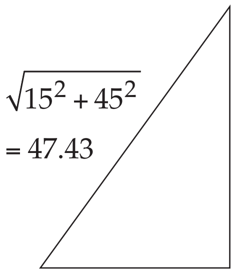
$\cos\theta = \frac{50(30)+30(-25)+5(12)}{\sqrt{50^2+30^2+5^2}.\sqrt{30^2+25^2+12^2}}$

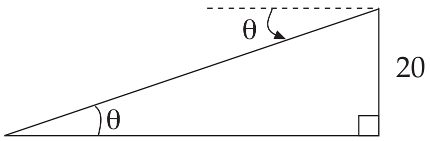
$\cos\theta = 0.3388$

$\theta = 70°$ **[A1]**

c $\overrightarrow{OD} = 15\underset{\sim}{i} + 45\underset{\sim}{j} + 20\underset{\sim}{k}$

$\sqrt{15^2 + 45^2} = 47.43$ **[A1]**





[M1]

$\tan\theta = \dfrac{20}{47.43}$

$\theta = 23°$ **[A1]**

d Want scalar resolute of $\overrightarrow{AB}$ in the direction of $\overrightarrow{BC}$.

$\overrightarrow{AB}.\overrightarrow{BC}$ **[M1]**

$= \left(50\underset{\sim}{i} + 30\underset{\sim}{j} + 5\underset{\sim}{k}\right) . \dfrac{\left(30\underset{\sim}{i} - 25\underset{\sim}{j} + 12\underset{\sim}{k}\right)}{\sqrt{30^2 + 25^2 + 12^2}}$

$= 19.83\text{m}$

$\approx 20\text{m}$ **[A1]**

Question 5

a $[1, \infty)$ **[A1]**

b $y = 4\text{Cos}^{-1}\left(\dfrac{1}{\sqrt{x}}\right)$ Let $u = \dfrac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$ **[M1]**

$y = 4\text{Cos}^{-1}u$ $\qquad \dfrac{du}{dx} = -\dfrac{1}{2}x^{-\frac{3}{2}}$

$\dfrac{dy}{dx} = \dfrac{dy}{du} . \dfrac{du}{dx}$

$f'(x) = \dfrac{-4}{\sqrt{1-u^2}} \times -\dfrac{1}{2}x^{-\frac{3}{2}}$ [M1]

$= \dfrac{2}{\sqrt{1-\dfrac{1}{x}}} \times \dfrac{1}{x\sqrt{x}}$ [A1]

$= \dfrac{2}{x\sqrt{x-1}}$

c $\displaystyle\int_2^4 \frac{1}{x\sqrt{x-1}}\,dx$

$= \left[2\text{Cos}^{-1}\left(\dfrac{1}{\sqrt{x}}\right)\right]_2^4$ **[A1]**

$= 2\left(\text{Cos}^{-1}\dfrac{1}{2} - \text{Cos}^{-1}\left(\dfrac{1}{\sqrt{2}}\right)\right)$

$= 2\left(\dfrac{\pi}{3} - \dfrac{\pi}{4}\right)$ **[A1]**

$= \dfrac{\pi}{6}$

d i A $= f(2.5) + f(3.5)$ **[M1]**

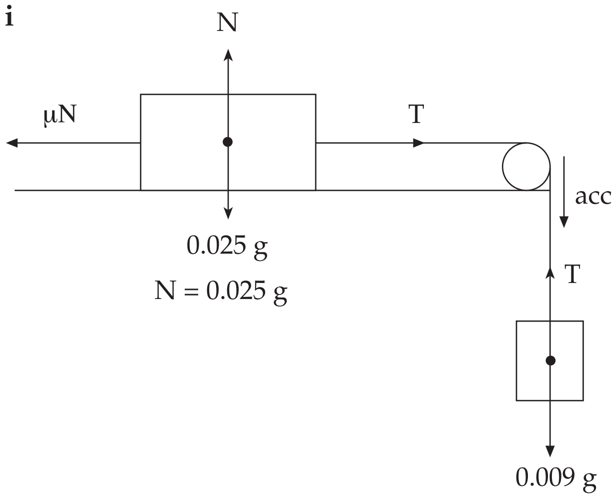
$= 0.3266 + 0.1807$

$\simeq 0.507$ **[A1]**

ii $\dfrac{\pi}{6} \simeq 0.507$ Inaccuracy results from the steep slope approaching $x = 2$. **[A1]**

Question 6

a **i**



$T - \mu N = m(0.025)$

$T - 0.2(0.025g) = 0.025a \quad (1)$

$0.009g - T = 0.009a \quad (2)$ **[M1]**

$(1) + (2)$

$0.0392 = 0.034a$

$a = 1.153 \text{m/s}^2$ **[A1]**

ii substituting $a = 1.153 \text{m/s}^2$ into (2) gives

$T = 0.078$ Newtons **[A1]**

b $T - \mu N = 0.025a \quad (1)$

$0.009g - T = 0.009a \quad (2)$

$(1) + (2)$

$0.009g - 0.025g\mu = 0.034a$ **[A1]**

$(0.009 - 0.025\mu)g = 0.034a$

$a = \frac{(0.009 - 0.025\mu)g}{0.034}$

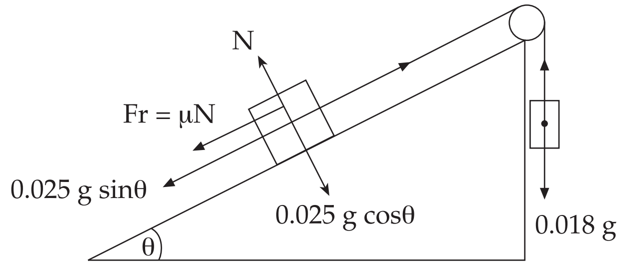
since acceleration is constant

$s = ut + \frac{1}{2}at^2$ **[M1]**

$0.5 = 0(2) + \frac{1}{2}\left[\frac{(0.009 - 0.025\mu)g}{0.034}\right]4$

$\mu = 0.325$ **[A1]**

c



$N = 0.025g\cos\theta$

$T = 0.018g \quad (1)$

$T = 0.025g\sin\theta + \mu N$

$T = 0.025g\sin\theta + 0.40 \times 0.025g\cos\theta \quad (2)$ **[M1]**

equating (1) and (2)

$0.018g = 0.025g\sin\theta + 0.01g\cos\theta$ **[A1]**

$0.018 = 0.025\sin\theta + 0.01\cos\theta$

$18 = 25\sin\theta + 10\cos\theta$

using a graphics calculator to solve:

$\theta = 20.15°$ **[A1]**

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