2003 Specialist Mathematics
Written Examination 1 (facts, skills and applications)
Suggested Answers and Solutions

Part I (Multiple-choice) Answers

1. **D**	2. **B**	3. **E**	4. **A**	5. **E**
6. **A**	7. **D**	8. **A**	9. **A**	10. **C**
11. **D**	12. **E**	13. **B**	14. **B**	15. **B**
16. **C**	17. **C**	18. **E**	19. **A**	20. **D**
21. **D**	22. **E**	23. **C**	24. **C**	25. **B**
26. **C**	27. **E**	28. **B**	29. **D**	30. **E**

Question 1 [D]

Graph translated 4 units to the right so has the equation:

$$\frac{(x-4)^2}{a^2} - \frac{y^2}{b^2} = 1$$

at $y = 0$ and $x = 2$

$$\frac{-2^2}{a^2} - 0 = 1$$

$$a^2 = 4$$

Asymptote: $y = \pm\frac{bx}{a} = \pm\frac{3}{2}x$

$\Rightarrow b = 3$

$$\frac{(x-4)^2}{4} - \frac{y^2}{9} = 1$$

Question 2 [B]

$\sin^2(x) + \cos^2(x) = 1$

$\frac{1}{25} + \cos^2(x) = 1$

$\cos^2(x) = \frac{24}{25}$

$= \pm\frac{2\sqrt{6}}{5}$

$\cot(x) = \frac{\cos(x)}{\sin(x)}$

$= \frac{\frac{\pm 2\sqrt{6}}{5}}{\frac{1}{5}} = \pm 2\sqrt{6}$

$\cot(x)$ is **positive** when $\pi \le x \le \frac{3\pi}{2}$

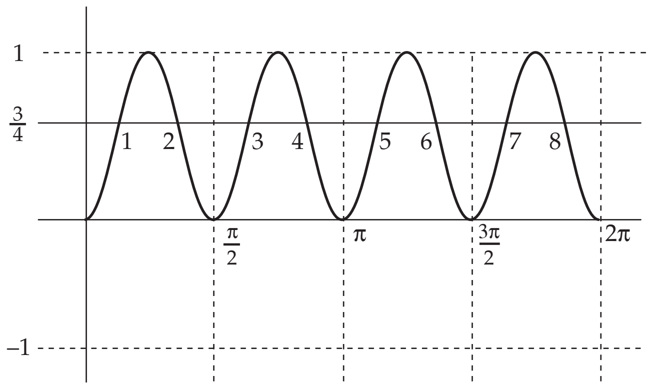
$\therefore \cot(x) = \pm 2\sqrt{6}$

Question 3 [E]

Using a graphics calculator:

$y_1 = \sin^2(2x)$

$y_2 = \frac{3}{4}$



There are 8 points of intersection.
Hence there are 8 solutions.

OR

$\sin^2 2x = \frac{3}{4}$

$\therefore \sin 2x = \pm\frac{\sqrt{3}}{2} \qquad 0 \le x \le 2\pi$

so 8 solutions.

Question 4 [A]

$y = Sin^{-1}\left(\frac{4}{x}\right)$

Let $u = 4x^{-1}$ $\quad$ $y = Sin^{-1}(u)$

$\frac{du}{dx} = -4x^{-2}$ $\quad$ $\frac{dy}{du} = \frac{1}{\sqrt{1-u^2}}$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{dy}{du}$$

$$= \frac{-4}{x^2\sqrt{1-\frac{16}{x^2}}}$$

$$= \frac{-4}{x^2\sqrt{\frac{x^2}{x^2}-\frac{16}{x^2}}}$$

$$= \frac{-4}{\frac{x^2}{x}\sqrt{x^2-16}}$$

$$= \frac{-4}{x\sqrt{x^2-16}}$$

Question 5 [E]

r cis θ

$r = \sqrt{x^2+y^2}$

Tan $\theta = \frac{y}{x}$

For r: $r = \sqrt{3+1}$
$= 2$

$\theta = Tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$= \frac{\pi}{6}$

$-\sqrt{3} - i$ is in the 3rd quadrant

$\therefore \theta = \frac{7\pi}{6}$

$-\sqrt{3} - i = 2cis\left(\frac{7\pi}{6}\right)$

Question 6 [A]

$z^2 = 4cis\left(\frac{4\pi}{3}\right)$

$z = 2cis\left(\frac{2\pi}{3} + n\pi\right)$

$z_1 = 2cis\left(\frac{2\pi}{3}\right)$

$z_2 = 2cis\left(\frac{5\pi}{3}\right)$

$z_1 = 2\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right) = 2\left(-\frac{1}{2} + \frac{\sqrt{3}i}{2}\right)$

$z_2 = 2\left(\cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right)\right) = 2\left(\frac{1}{2} - \frac{\sqrt{3}i}{2}\right)$

$z_1 = -1 + \sqrt{3}i$

$z_2 = 1 - \sqrt{3}i$

Question 7 [D]

Alternatives A, B and E can be eliminated because they are not linear factors.

For C $\quad z + 2 = 0$

Then $\quad z = (-2)$

P(–2) = –8 –8 –8 – 8 ≠ 0

∴ $z + 2$ is not a factor.

This suggests that $z + 2i$ is the correct alternative.

Just to check:

$z + 2i = 0$

when $z = -2i$

P(–2i) = 8i + 8 – 8i – 8 = 0

∴$z + 2i$ is a linear factor

OR

$$P(z) = z^3 - 2z^2 + 4z - 8$$
$$= z^2(z-2) + 4(z-2)$$
$$= (z-2)(z^2+4)$$
$$= (z-2)(z-2i)(z+2i)$$

$\therefore z + 2i$ is a linear factor.

Question 8 [A]

Equation of a circle is:

$(x+3)^2 + y^2 = 9$

$x^2 + 6x + 9 + y^2 = 9$

$x^2 + y^2 + 6x + 9 = 9$

Expanding $(z+a)(\bar{z}+b)$

$z\bar{z} + a\bar{z} + bz + ab$

$a(x - iy) + b(x + iy)$

$ax - aiy + bx + biy = 6x$

$\Rightarrow a + b = 6$

$ab = 9$

$\therefore a = b = 3$

Equation of a circle: $(z+3)(\bar{z}+3) = 9$

Note: this solution is given to allow an understanding of the mathematics.

Question 9 [A]

Alternative 1:

$|z-1| = |z+1|$

$|x + iy - 1| = |x + iy + i|$

$|(x-1) + iy| = |x + i(y+1)|$

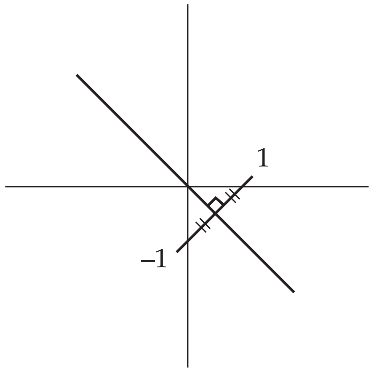
$(x-1)^2 + y^2 = x^2 + (y+1)^2$

$x^2 - 2x + 1 + y^2 = x^2 + y^2 + 2y + 1$

$-2x = 2y$
$y = -x$

Alternative 2:

$|z-1| = |z+1|$
S is the region equal distance from $z = 1$ and $z = -i$.
Plot $z = 1$ and $z = -i$ and draw a line the perpendicular bisector of the line joining $z = 1$ and $z = -i$.



Question 10 [C]

$\int_0^{\frac{\pi}{6}} \cos^3(2x)dx$

$= \int_0^{\frac{\pi}{2}} \cos 2x(1 - \sin^2(2x))dx$

Let $u = \sin(2x)$

$\frac{du}{dx} = 2\cos(2x)$

$dx = \frac{du}{2\cos(2x)}$

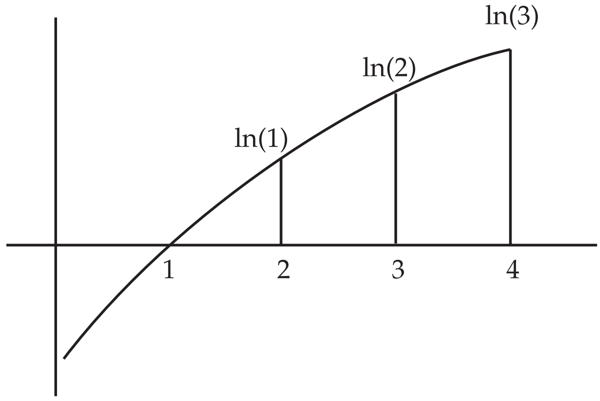
$u = \sin\left(2 \times \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

$u = \sin(0) = 0$

$\int_0^{\frac{\sqrt{3}}{2}} \cos(2x)(1 - u^2)\frac{du}{2\cos(2x)}$

$= \frac{1}{2}\int_0^{\frac{\sqrt{3}}{2}} 1 - u^2 du$

Question 11 [D]



Area 1 = $\frac{ba}{2} = \frac{1 \times \ln(2)}{2}$

Area 2 = $\frac{1}{2}$(a + b) h

= $\frac{1}{2}$ ((ln(2) + ln(3)) x 1

Area 3 = $\frac{1}{2}$ ((ln(3) + ln(4)) x 1

Total area = $\frac{\ln(2)}{2} + \frac{\ln(2) + \ln(3)}{2} + \frac{\ln(3) + \ln(4)}{2}$

= $\frac{1}{2}$ (ln(2) + ln(2) + ln(3) + ln(3) + ln(4))

= $\frac{1}{2}$ ln(2^2 x 3^2 x 4)

= ln(2 x 3 x 2)

= ln(12) $\therefore$a = 12

Question 12 [E]

$$\text{Area} = \int_0^1 2\cos\left(\frac{\pi x}{2}\right) - (x^2 - 1)dx$$

$$= \int_0^1 2\cos\left(\frac{\pi x}{2}\right) - x^2 + 1dx$$

$$= \left[\frac{4}{\pi}\sin\left(\frac{\pi x}{2}\right) - \frac{x^3}{3} + x\right]_0^1$$

$$= \left(\frac{4}{\pi}\sin\left(\frac{\pi}{2}\right) - \frac{1}{3} + 1\right) - (0)$$

$$= \frac{4}{\pi} + \frac{2}{3}$$

Question 13 [B]

$$\frac{3}{x(3-x)} = \frac{A}{x} + \frac{B}{(3-x)}$$

$$= \frac{A(3-x) + Bx}{x(3-x)}$$

$3 \equiv A(3-x) + Bx$

at $x = 3$

$3 = 3B$

$\Rightarrow B = 1$

at $x = 0$

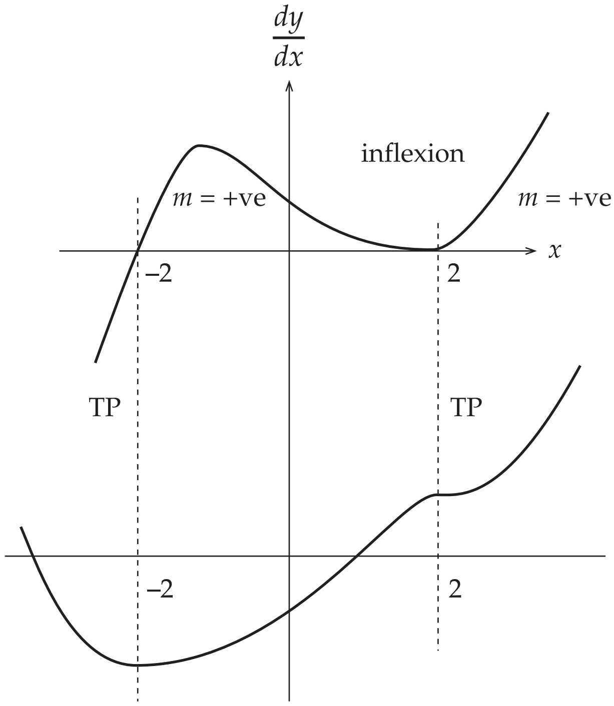
$3 = 3A$

$\Rightarrow A = 1$

$$\int \frac{3}{x(3-x)}\,dx = \int \frac{1}{x} + \frac{1}{(3-x)}\,dx$$

$= \ln x - \ln(3-x)$

Question 14 [B]



Question 15 [B]

$$f'(x) = 2\sin^2\left(\frac{x}{2}\right) - 1$$

$$= -\left(1 - \sin^2\left(\frac{x}{2}\right)\right)$$

but $\cos(2a) = 1 - \sin^2(a)$

let $a = \frac{x}{2}$

$\therefore \cos x = 1 - \sin^2\left(\frac{x}{2}\right)$

so $f'(x) = -\cos(x)$

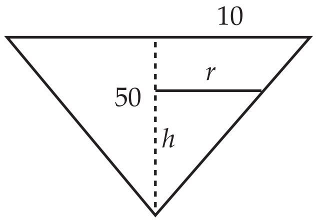
$f(x) = -\sin(x) + c$

$f\left(\frac{\pi}{2}\right) = 0 = -1 + c$

$c = 1$

$\therefore f(x) = 1 - \sin(x)$

Question 16 [C]



$r = \frac{h}{5}$

$V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi\left(\frac{h}{5}\right)^2 h$

$= \frac{\pi h^3}{3 \times 25}$

$\frac{dv}{dh} = \frac{\pi h^2}{25}$

$\frac{dh}{dt} = \frac{dh}{dv} \times \frac{dv}{dt}$

$= \frac{25}{\pi h^2} \times 600$

$= \frac{15\,000}{\pi h^2}$

Question 17 [C]

$\frac{dy}{dx} = f(x) = \cos\left(\frac{x}{2}\right)$

$x_{n+1} = x_n + h$

$y_{n+1} = y_n + hf(x_n)$

$x_0 = 0$

$y_0 = 2$

$h = 0.2$

$x_1 = 0.2$ $\quad y_1 = 2 + 0.2f(0) = 2.2$

$x_2 = 0.4$ $\quad y_2 = 2.2 + 0.2\cos\left(\frac{0.2}{2}\right)$

$= 2.2 + 0.2\cos(0.1)$

Question 18 [E]

$\overrightarrow{AC} = \underset{\sim}{p} + \underset{\sim}{q}$

$\overrightarrow{CA} = \underset{\sim}{r} + \underset{\sim}{s}$

$\Rightarrow \overrightarrow{AC} = -\underset{\sim}{r} - \underset{\sim}{s}$

$\therefore \underset{\sim}{p} + \underset{\sim}{q} = -\underset{\sim}{r} - \underset{\sim}{s}$

OR

$\underset{\sim}{p} + \underset{\sim}{q} + \underset{\sim}{r} + \underset{\sim}{s} = 0$

$\therefore \underset{\sim}{p} + \underset{\sim}{q} = -\underset{\sim}{r} - \underset{\sim}{s}$

Question 19 [A]

$\underset{\sim}{a} = \underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}$

$|\underset{\sim}{a}| = \sqrt{1+4+9}$

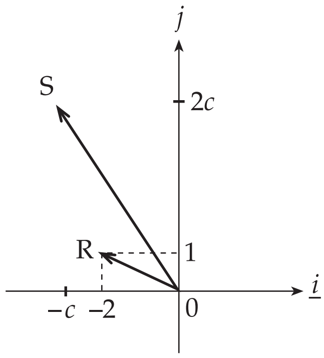
$= \sqrt{14}$

$\hat{\underset{\sim}{a}} = \frac{\underset{\sim}{a}}{|\underset{\sim}{a}|} = \frac{1}{\sqrt{14}}\left(\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}\right)$

$-\hat{\underset{\sim}{a}} = -\frac{1}{\sqrt{14}}\left(\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}\right)$

$= \frac{1}{\sqrt{14}}\left(-\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}\right)$

Question 20 [D]



$\overrightarrow{OS} = -c\underset{\sim}{i} + 2c\underset{\sim}{j}$

$\overrightarrow{OR} = -2\underset{\sim}{i} + \underset{\sim}{j}$

$\overrightarrow{RS} = -\overline{OR} + \overline{OS}$

$= 2\underset{\sim}{i} - \underset{\sim}{j} - c\underset{\sim}{i} + 2c\underset{\sim}{j}$

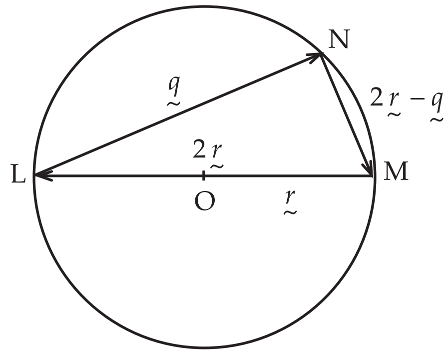
$= (2-c)\underset{\sim}{i} + (2c-1)\underset{\sim}{j}$

$\overrightarrow{OS} \cdot \overrightarrow{RS} = -c(2-c) + 2c(2c-1)$

$= -2c + c^2 + 4c^2 - 2c$

$= 5c^2 - 4c$

Question 21 [D]



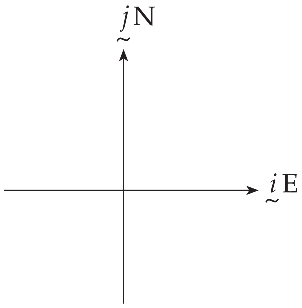
$\angle LMN$ is a right angle.

$\therefore \underset{\sim}{q} \cdot (2\underset{\sim}{r} - \underset{\sim}{q}) = 0$

$\Rightarrow 2\underset{\sim}{r} \cdot \underset{\sim}{q} - \underset{\sim}{q} \cdot \underset{\sim}{q} = 0$

$\Rightarrow 2\underset{\sim}{r} \cdot \underset{\sim}{q} = \underset{\sim}{q} \cdot \underset{\sim}{q}$

Question 22 [E]



$\underline{r} = (5t - 8)\underline{i} + (t^2 - 5t + 6)\underline{j}$

$\underline{s} = (t^2 - t)\underline{i} + (3 - t)\underline{j}$

r is north or south of *s* when:

$5t - 8 = t^2 - t$

$\Rightarrow t^2 - 6t + 8 = 0$

$(t - 2)(t - 4) = 0$

$t = 2, 4$

at $t = 2$

$L \to \underline{r} = 2\underline{i} + 0\underline{j}$

$M \to \underline{s} = 2\underline{i} + \underline{j}$

so L is south of M

at $t = 4$

$L \to \underline{r} = 12\underline{i} + 2\underline{j}$

$M \to \underline{s} = 12\underline{i} - \underline{j}$

$\therefore$ L is north of *m* when $t = 4$.

Question 23 [C]

$\underline{r}(t) = 4t\underline{i} - e^{2t}\underline{j} + 5\underline{k}$

$\dot{\underline{r}}(t) = 4\underline{i} - 2e^{2t}\underline{j}$

$\dot{\underline{r}}(0) = 4\underline{i} - 2\underline{j}$

$\left|\dot{\underline{r}}(0)\right| = \sqrt{16 + 4}$

$= \sqrt{20}$

Question 24 [C]

$\ddot{\underline{r}}(t) = \cos(t)\underline{i} - \sin(t)\underline{j}$

$\dot{\underline{r}}(t) = \sin(t)\underline{i} + \cos(t)\underline{j} + c$

$\dot{\underline{r}}(0) = \underline{i} + \underline{j}$

$\underline{i} + \underline{j} = 0\underline{i} + \underline{j} + c$

$c = \underline{i}$

$= \left(\sin(t) + 1\right)\underline{i} + \cos(t)\underline{j}$

Question 25 [B]

$F - \mu N = F_R$

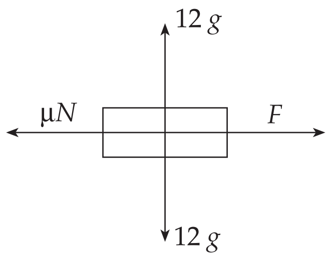
$F_R = 0.5 \times 12 = 6N$

$66 - 12g\mu = 6$

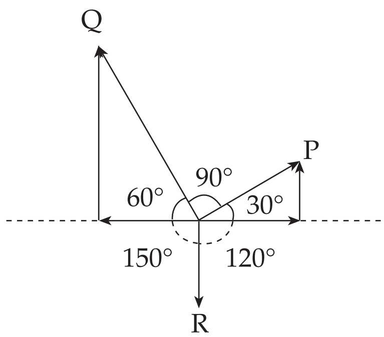
$60 = 12g\mu$

$\mu = \dfrac{60}{12g}$

$= 0.51$



Question 26 [C]



Using the components vertical and horizontal:

$\overrightarrow{OQ} + \overrightarrow{OP} = \overrightarrow{RO}$

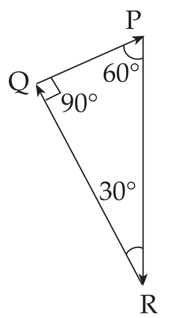
$P = R\cos 60°$

and $Q = R\cos 30°$

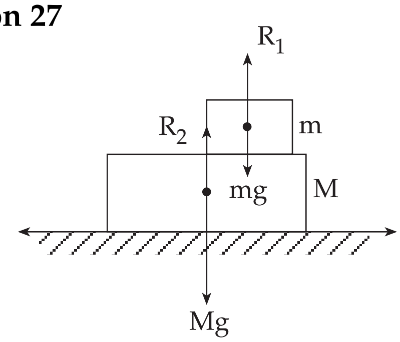
By a process of elimination:

$R = Q\cos 30°$

is not true.



Question 27 [E]



Consider m: gravity **acts on** it with a force of mg.

Block M **acts on** m to produce an equal and opposite force R_1.

Consider M: gravity **acts on** it with a force of Mg.

The ground **acts on** M to produce an equal and opposite force R_2.

Question 28 [B]

Consider each alternative:

A: $t = v - 1$

At $t = 0$, $v = 1$	possible
At $v = 0$, $t = -1$	doesn't need to be considered as $t \geq 0$

B: $t = x^2 - 1$

At $t = 0$,	$x^2 - 1 = 0$
$(x - 1)(x + 1) = 0$	
$x = +1$ and $x = -1$	impossible

This suggests that at $t = 0$ x is holding two positions simultaneously.

C: $x = t^2 - 1$

At $t = 0$, $x = -1$	possible
At $x = 0$, $t = \pm 1$,	only $t = 1$ can be considered as possible.

D: $x = v2$ -1

At $x = 0$, $v = \pm 1$	possible
At $v = 0$, $x = -1$	possible

E: $v = t - 1$

At $t = 0$, $v = -1$	possible
At $v = 0$, $t = 1$	possible

Question 29 [D]

$v = \cos(t) + \sqrt{3}\sin(t) - 1$

$\frac{dv}{dt} = -\sin(t) + \sqrt{3}\cos(t)$

Turning point at $\frac{dv}{dt} = 0$

$\sin(t) = \sqrt{3}\cos(t)$

$\tan(t) = \sqrt{3}$

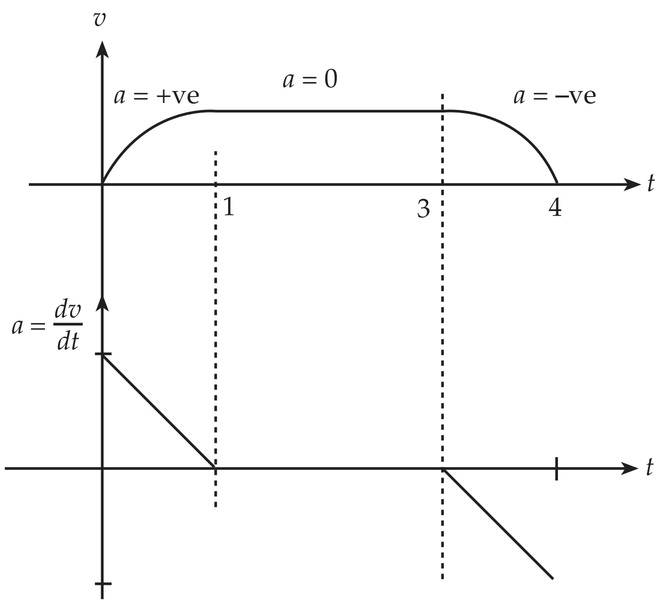
$t = \frac{\pi}{3}, \frac{4\pi}{3}, ...$

$v\left(\frac{\pi}{3}\right) = \frac{1}{2} + \frac{3}{2} - 1 = 1$

$v\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - \frac{3}{2} - 1 = -3$

Maximum speed is 3 which occurs at $\frac{4\pi}{3}$.

Question 30 [E]



Part II

Question 1

a $v = 4.5t + \cos(2t)$

$\frac{dv}{dt} = 4.5 - 2\sin(2t)$

b Max and min occurs when $\sin(2t) = \pm 1$

for $\sin(2t) = 1$

$a = 4.5 - 2 = 2.5$

for $\sin(2t) = -1$

$a = 4.5 - (-2) = 6.5$

using $a = 2.5$

$F = ma$

$= 5 \times 2.5$

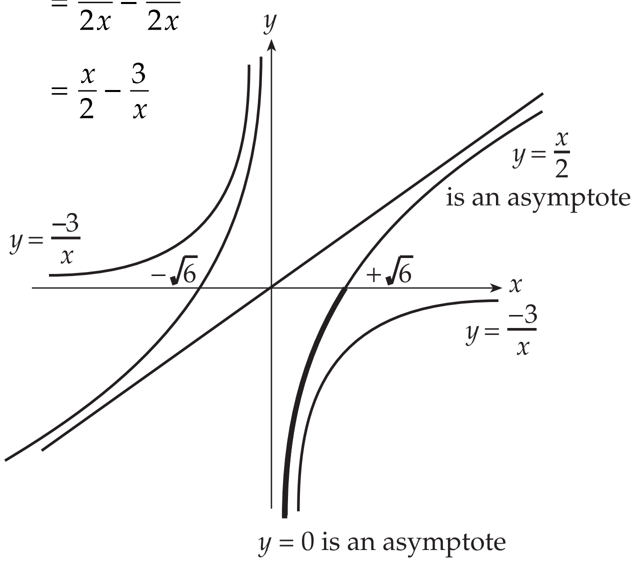
$= 12.5$ N

Question 2

$f(x) = \frac{x^2 - 6}{2x}$

$= \frac{x^2}{2x} - \frac{6}{2x}$

$= \frac{x}{2} - \frac{3}{x}$



At $y = 0$, $0 = \frac{x^2 - 6}{2x}$ $\quad x \neq 0$

$0 = (x - \sqrt{6})(x + \sqrt{6})$

x-intercepts at $x = \pm\sqrt{6}$

$f'(x) = \frac{2x \times 2x - x(x^2 - 6)}{4x^2}$

$= \frac{2x^2 + 6}{4x^2}$

$f'(x) \neq 0$ for all values of x $\therefore$ no turning point.

Question 3

$y = xe^{3x}$

$\frac{dy}{dx} = e^{3x} + 3xe^{3x}$

$= e^{3x}(1 + 3x)$

$\frac{d^2y}{dx^2} = 3e^{3x}(1 + 3x) + 3e^{3x}$

$= 3e^{3x}(1 + 3x + 1)$

$= 3e^{3x}(2 + 3x)$

Given $\frac{d^2y}{dx^2} + m\frac{dy}{dx} + ny = 0$

$3e^{3x}(2 + 3x) + me^{3x}(1 + 3x) + nxe^{3x} = 0$

$\Rightarrow 3(2 + 3x) + m(1 + 3x) + nx = 0$

$\Rightarrow 6 + 9x + m + 3mx + nx = 0$

$\Rightarrow x(3m + n) + m = -9x - 6$

$\therefore m = -6$

$3m + n = -9$

$-18 + n = -9$

$n = 9$

$m = -6$ and $n = 9$

Question 4

a $\int_2^{-2} 1 - \frac{8}{x^2 + 4}\,dx$

$= \int_2^{-2} 1 - 4 \times \frac{2}{x^2 + 2^2}\,dx$

$= \left[x - 4Tan^{-1}\left(\frac{x}{2}\right)\right]_2^{-2}$

$= \left(-2 - 4Tan^{-1}(-1)\right) - \left(2 - 4Tan^{-1}(1)\right)$

$= -2 + \pi - 2 + \pi$

$= -4 + 2\pi$

b $v = \pi\int_{-1}^{0} x^2 dy$

$y = 1 - \frac{8}{x^2 + 4}$

$\frac{8}{x^2 + 4} = 1 - y$

$x^2 = \frac{8}{1 - y} - 4$

$v = \pi\int_{-1}^{0} \frac{8}{1 - y} - 4dy$

$= -\pi\int_{-1}^{0} \frac{-8}{1 - y} + 4dy$

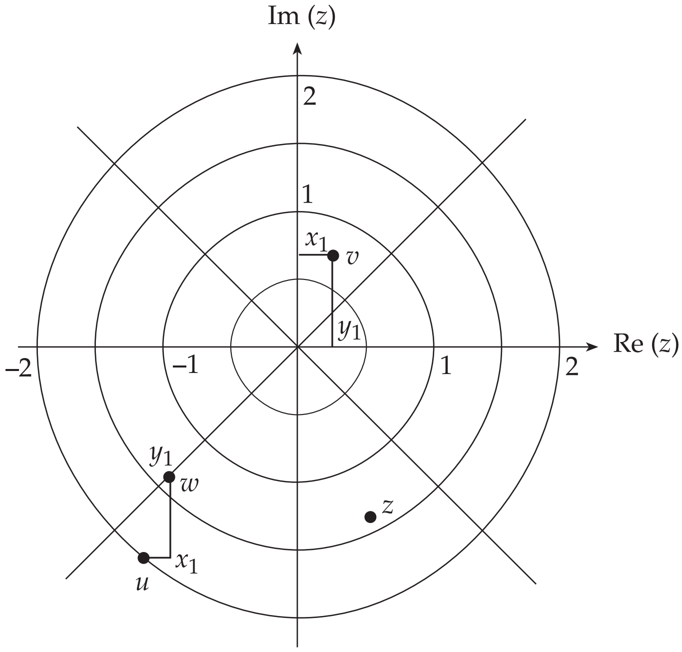
$= -\pi\left[8\ln(1 - y) + 4y\right]_{-1}^{0}$

$v = -\pi\left[\left(8\ln(1) + 0\right) - \left(8\ln(2) - 4\right)\right]$

$= \pi\left(8\ln(2) - 4\right)$

$= 4.85$ (3 significant figures)

Question 5



a $z = \sqrt{2}cis\theta$

$u = z^2 = \left(\sqrt{2}\right)^2 cis2\theta$

$= 2cis2\theta$

From diagram:

$\theta \approx -\frac{3\pi}{8}$

$\Rightarrow 2\theta = -\frac{3\pi}{4}$

$z^2 = 2cis\left(-\frac{3\pi}{4}\right)$

b $z = \sqrt{2}cis\theta$

$v = \frac{1}{z} = z^{-1} = \frac{1}{\sqrt{2}}cis(-\theta)$

$= \frac{\sqrt{2}}{2}cis(-\theta)$

using: $\theta = -\frac{3\pi}{8}$

$v = \frac{\sqrt{2}}{2}cis(\frac{3\pi}{8})$

c $w = z^2 + \frac{1}{z}$

$w = u + v$

This can be best achieved by adding x_1 to u followed by adding y_1.

Refer to the diagram.

