MAV Specialist Mathematics Examination 2 Answers & Solutions

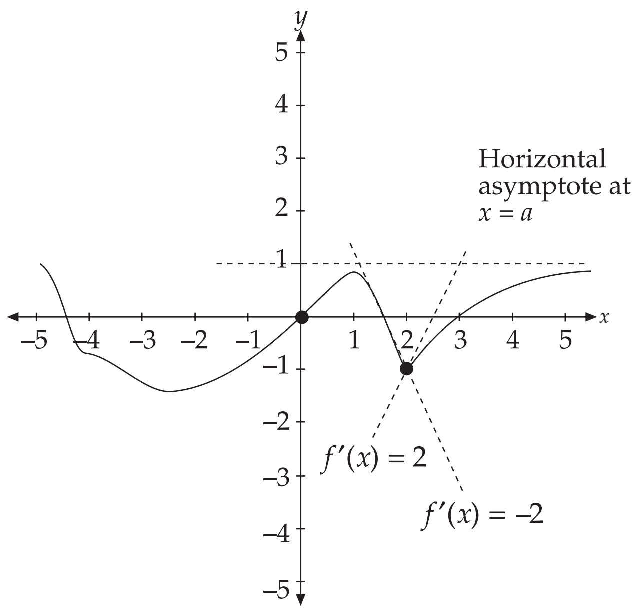
Question 1

a. At $x = -4$, $f'(x) = 0$ and $f''(x) = 0$ thus point of inflexion **A1**

At $x = -2$, $f'(x) = 0$ and $f''(x) > 0$ thus local minimum **A1**

at $x = 1$, $f'(x) = 0$ and $f''(x) < 0$ thus local maximum **A1**

b.



Shape **A1**
Correct location of turning points **A1**
Recognition of asymptotic behaviour **A1**

Question 2

a i. $\dot{\underset{\sim}{s}}(t) = (4t^3 - 9t^2 + 2)\,\underset{\sim}{i} + 2t\,\underset{\sim}{j}$

$\underset{\sim}{s}(t) = (t^4 - 3t^3 + 2t)\,\underset{\sim}{i} + t^2\,\underset{\sim}{j} + \underset{\sim}{c}$ **M1**

At $t = 0$, $\underset{\sim}{s} = 2\,\underset{\sim}{i} \Rightarrow \underset{\sim}{c} = 2\,\underset{\sim}{i}$

$\underset{\sim}{s}(t) = (t^4 - 3t^3 + 2t + 2)\,\underset{\sim}{i} + t^2\,\underset{\sim}{j}$ **A1**

ii. $\dot{\underset{\sim}{r}}(t) = -2t\,\underset{\sim}{i} + 3\,\underset{\sim}{j}$

$\underset{\sim}{r}(t) = -t^2\,\underset{\sim}{i} + 3t\,\underset{\sim}{j} + \underset{\sim}{c}$ **M1**

At $t = 0$, $\underset{\sim}{r} = 2\,\underset{\sim}{i} - 2\,\underset{\sim}{j} \Rightarrow \underset{\sim}{c} = 2\,\underset{\sim}{i} - 2\,\underset{\sim}{j}$

$\underset{\sim}{r}(t) = \left(2 - t^2\right)\underset{\sim}{i} + \left(3t - 2\right)\underset{\sim}{j}$ **A1**

b i. Boats collide if they have the same position at the same time, ie $\underset{\sim}{r}(t) = \underset{\sim}{s}(t)$ for some value of t.

$t^4 - 3t^3 + 2t + 2 = 2 - t^2$ (1)
& $t^2 = 3t - 2$ (2) **M1**

From (2)

$t^2 - 3t + 2 = 0$

$(t - 1)(t - 2) = 0$

$t = 1$ or $t = 2$ **M1**

Substituting $t = 1$ into (1)

LHS $= 1 - 3 + 2 + 2$
$= 2$
RHS $= 2 - 1$
$= 1$
LHS $\neq$ RHS **M1**

Substituting $t = 2$ into (1)

LHS $= 16 - 24 + 4 + 2$
$= -2$
RHS $= 2 - 4$
$= -2$
RHS $=$ LHS **M1**

Thus $t = 2$ satisfies both (1) and (2), which implies both Ragin' and Starin' are at the same position at the same time and hence collide. **A1**

ii. Collision at $t = 2$

$\underset{\sim}{r}(2) = (2-4)\,\underset{\sim}{i} + (6-2)\,\underset{\sim}{j}$

$= -2\,\underset{\sim}{i} + 4\,\underset{\sim}{j}$

Point of collision (–2, 4) **A1**

c. $\dot{\underset{\sim}{r}}(2) = -4\,\underset{\sim}{i} + 3\,\underset{\sim}{j}$ **M1**

$\dot{\underset{\sim}{s}}(2) = -2\,\underset{\sim}{i} + 4\,\underset{\sim}{j}$ **M1**

$$\cos\theta = \frac{\dot{\underset{\sim}{s}} \cdot \dot{\underset{\sim}{r}}}{\left|\dot{\underset{\sim}{s}}\right| \times \left|\dot{\underset{\sim}{r}}\right|}$$

$$= \frac{8+12}{\sqrt{20} \times \sqrt{25}} = \frac{20}{10\sqrt{5}}$$ **M1**

$$\theta = \text{Cos}^{-1}\left(\frac{2}{\sqrt{5}}\right)$$ **A1**

$= 26.56 \approx 27°$

Question 3

a i. $y = \text{Cos}^{-1}\left(\dfrac{x-1}{x+1}\right)$

$$u = \frac{x-1}{x+1} \quad \frac{du}{dx} = \frac{x+1-(x-1)}{(x+1)^2} = \frac{2}{(x+1)^2}$$ **M1**

$y = \text{Cos}^{-1}u$

$$\frac{dy}{du} = \frac{-1}{\sqrt{1-u^2}}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\left(\frac{x-1}{x+1}\right)^2}} \times \frac{2}{(x+1)^2}$$ **M1**

$$= \frac{-1}{\sqrt{\left(\frac{x+1}{x+1}\right)^2 - \left(\frac{x-1}{x+1}\right)^2}} \times \frac{2}{(x+1)^2}$$

$$= \frac{-1}{\sqrt{\frac{(x+1-x+1)(x+1+x-1)}{(x+1)^2}}} \times \frac{2}{(x+1)^2}$$

$$= -\frac{(x+1)}{2\sqrt{x}} \times \frac{2}{(x+1)^2}$$

$$= \frac{-1}{\sqrt{x}\;(x+1)}$$ **A1**

ii. $$\int_1^3 \frac{1}{\sqrt{x^3} + \sqrt{x}}\,dx$$

$$= -\int_1^3 \frac{-1}{\sqrt{x^3} + \sqrt{x}}\,dx$$

$$= -\left[Cos^{-1}\left(\frac{x-1}{x+1}\right)\right]_1^3$$

$$= -Cos^{-1}\left(\frac{1}{2}\right) + Cos^{-1}(0)$$ **M1**

$$= \frac{\pi}{6}$$ **A1**

b i. $$\frac{du}{dv} = \frac{-1}{\sqrt{x}(x+1)}$$

$v = \text{Tan}^{-1}\sqrt{x}$

$$\frac{dv}{dx} = \frac{1}{1+x} \times \frac{1}{2\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}(x+1)}$$ **M1**

$$\frac{du}{dx} = -2 \times \frac{dv}{dx}$$

$$\Rightarrow \frac{du}{dx} + 2 \times \frac{dv}{dx} = 0$$ **M1 A1**

$\therefore a = 2$

ii. $$\frac{du}{dx} + 2\frac{dv}{dx} = 0$$

$$\int \frac{du}{dx}dx + 2\int \frac{dv}{dx}dx = \int 0dx$$

$\Rightarrow u + 2v = c$

$$\text{Cos}^{-1}\left(\frac{x-1}{x+1}\right) + 2\text{Tan}^{-1}\sqrt{x} = c$$ **C1**

For $x = 1$

$\text{Cos}^{-1}0 + 2\text{Tan}^{-1}1$

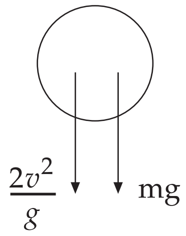
$$= \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

$\therefore c = \pi$ **A1**

Note: Answer is given

Question 4

a & b.



A1

$$F = -\left(2g + \frac{2v^2}{g}\right)$$

$$2a = -\left(2g + \frac{2v^2}{g}\right)$$ Need to see working

$$a = -g - \frac{v^2}{g}$$ Answer given **A1**

c. $a = -g - \frac{v^2}{g}$

$$v\frac{dv}{dx} = -g - \frac{v^2}{g}$$ **M1**

$$\frac{dv}{dx} = -\frac{g}{v} - \frac{v}{g}$$

$$= \frac{g^2 + v^2}{-gv}$$

$$\frac{dx}{dv} = \frac{-gv}{g^2 + v^2}$$

$$= -\frac{g}{2}\left[\frac{2v}{g^2 + v^2}\right]$$ **M1**

$$x = -\frac{g}{2}\log_e\left(v^2 + g^2\right) + c$$ **M1**

At $x = 0$ $v = 10$

$$c = \frac{g}{2}\log_e 196.04$$ **A1**

$$v = \sqrt{196.04e^{\frac{-2x}{g}} - g^2}$$ **C1**

d. $-\frac{2x}{g} = \log_e\left(\frac{v^2 + g^2}{196.04}\right)$ **M1**

At $v = 0$

$$-\frac{2x}{g} = \log_e\left(\frac{g^2}{196.04}\right)$$

$$x = \frac{g}{2}\log_e\left(\frac{g^2}{196.04}\right)$$ **M1 A1**

$$\therefore x = 3 \text{ m}$$

e. $a = -g - \frac{v^2}{g}$

$$\frac{dv}{dt} = \frac{g^2 + v^2}{-g}$$

$$\frac{dt}{dv} = \frac{-g}{g^2 + v^2}$$ **M1**

$$t = -\text{Tan}^{-1}\frac{v}{g} + c$$ **M1**

At $t = 0$, $v = 10$

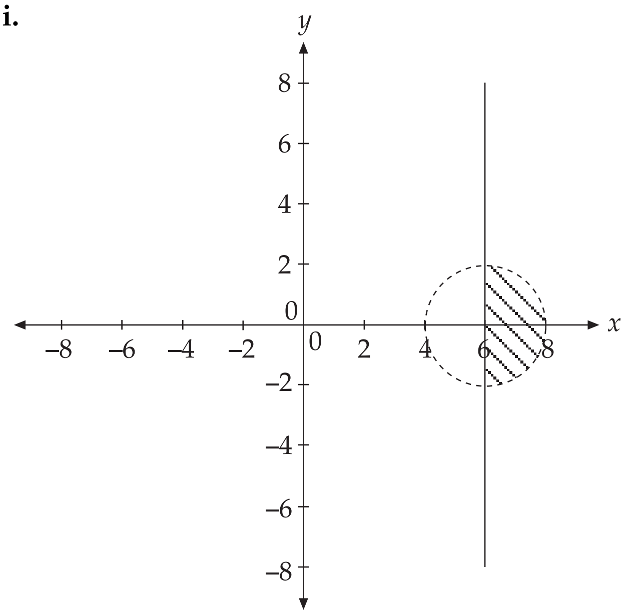
$$t = -\text{Tan}^{-1}\frac{v}{g} + \text{Tan}^{-1}\frac{10}{g}$$

Maximum height at $v = 0$

$$t = \text{Tan}^{-1}\frac{10}{g} = 0.8 \text{ seconds}$$ **A1**

Question 5

a i.



Circle of radius 2 with centre $z = 6$, Dotted line **M1**

Line parallel to Im(z) axis passing through 6, Solid line **M1**

Region Shaded **M1**

ii. $z_1 = 6 + 2i$ or $(6, 2)$ **A1**

$z_1 = 6 - 2i$ or $(6, -2)$ **A1**

b i. $r^2 = 9 + 27$ **M1**

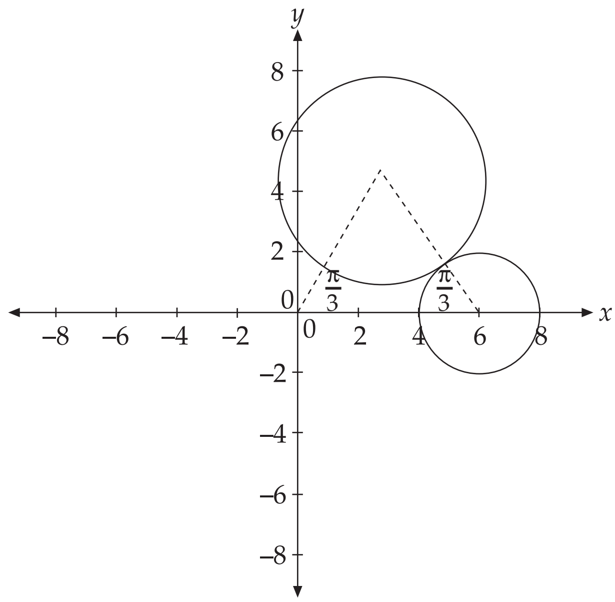
$r = 6$

$\theta = \text{Tan}^{-1} \frac{3\sqrt{3}}{3}$

$\theta = \frac{\pi}{3}$

$u = 6\text{cis}\frac{\pi}{3}$ **A1**

ii.



Circle with centre at $6\text{cis}\frac{\pi}{6}$ **M1**

One point of intersection with the radius of both circles on the same line **M1**

iii. $k + 2 = 6$ **A1**

$\therefore k = 4$

By similar triangles, point of intersection

$z = 5 + \sqrt{3}i$ or $\left(5, \sqrt{3}\right)$ **C1**

Question 6

a. $\int_0^{2\pi}\left(\sin\left(\frac{1}{2}x\right)\cos\left(\frac{1}{2}x\right)\right)^2 = \int_0^{2\pi}\left(\frac{1}{2}\sin\left(2\cdot\frac{1}{2}x\right)\right)^2 dx$ **M1**

$= \frac{1}{4}\int_0^{2\pi} \sin^2 x\, dx$

$= \frac{1}{4}\int_0^{2\pi} \frac{1}{2}\left(1 - \cos(2x)\right)dx$ **A1**

$= \frac{1}{8}\left[x - \frac{1}{2}\sin(2x)\right]_0^{2\pi} = \frac{\pi}{4}$ **A1**

b. $V = \pi\int_0^{\pi}\left(2\cos(2y) + 3\right)^2 dy$ **A1**

$= \pi\int_0^{\pi}\left(4\cos^2(2y) + 12\cos(2y) + 9\right)dy$ **M1**

$= \pi\left[\frac{1}{2}\sin(4y) + 6\sin(2y) + 11y\right]_0^{\pi}$

$= \pi(11\pi) = 11\pi^2$ **M1**

$\therefore$ the maximum volume is $11\pi^2$ cm^3. **A1**