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Students Name:______________________________

SPECIALIST MATHEMATICS

TRIAL EXAMINATION 2

(ANALYSIS TASK)

2002

Reading Time: 15 minutes
Writing time: 90 minutes

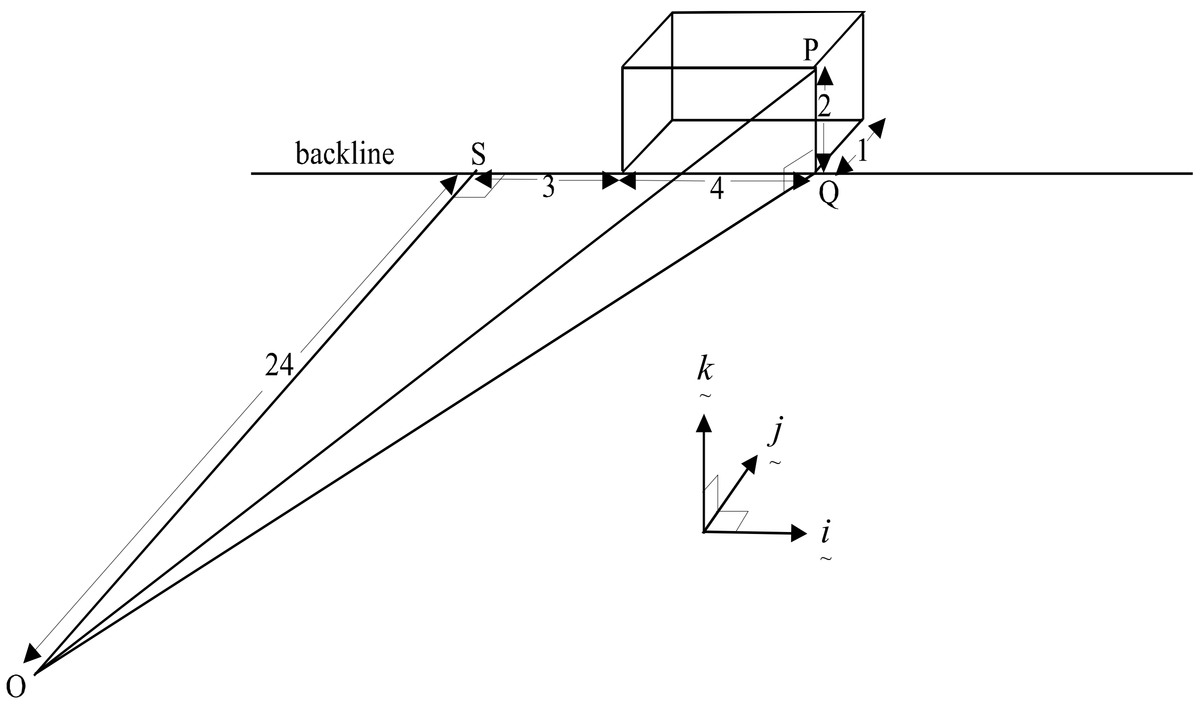
Instructions to students

This exam consists of 5 questions.
All questions should be answered.
There is a total of 60 marks available.
The marks allocated to each of the five questions are indicated throughout.
Students may bring up to two A4 pages of pre-written notes into the exam.
The acceleration due to gravity should be taken to have magnitude g m/s^2 where $g = 9.8$
Formula sheets can be found on pages 16-18 of this exam.

Question 1

Chris is practicing his goal shooting. He kicks at a goal, which is in the shape of a rectangular prism, 4 metres wide, 1 metre deep and 2 metres high. He kicks from point *O*, which is 24 metres from point *S* on the backline and 3 metres to the left of the left hand goal post. *OS* is perpendicular to the backline.

Chris kicks the ball in a straight line and hits the top right hand corner of the goal at point *P*. Point *Q* is the bottom right hand corner of the goal and is vertically below *P*.



Let $\underset{\sim}{i}$ be the unit vector which runs parallel with the backline, let $\underset{\sim}{j}$ be the unit vector which runs parallel to the length of the ground and let $\underset{\sim}{k}$ be the unit vector which runs vertically upwards. Let point *O* be the origin.

a. **i.** Express $\overrightarrow{OQ}$ in terms of $\underset{\sim}{i}$ and $\underset{\sim}{j}$.

1 mark

ii. Express $\overrightarrow{OP}$ in terms of $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$.

1 mark

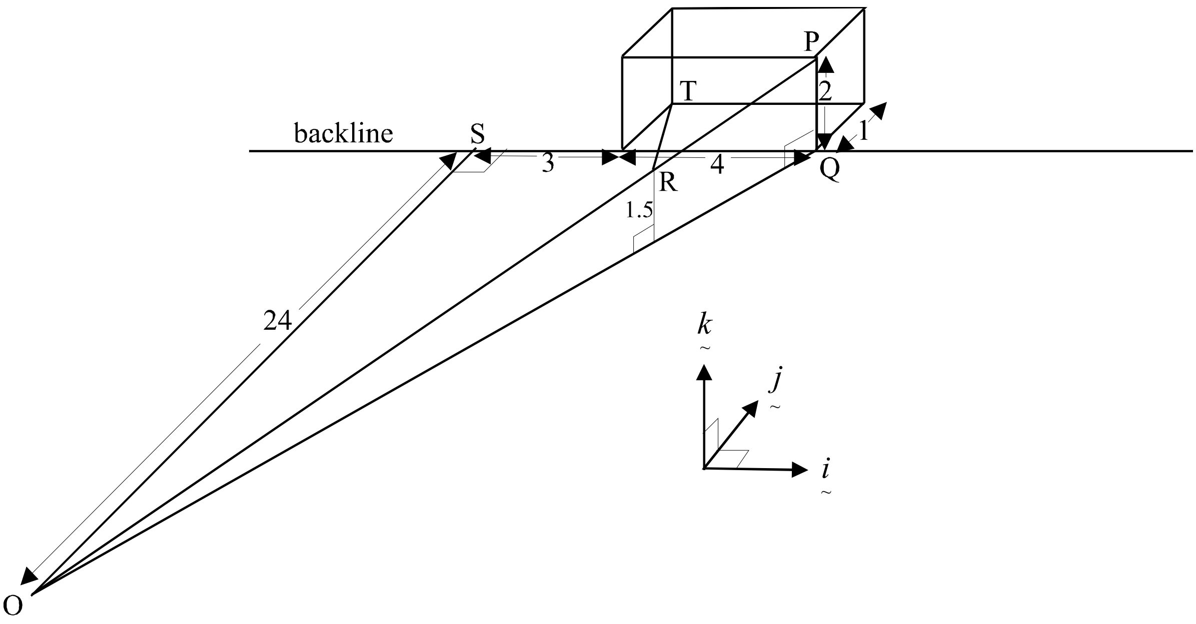
b. **i**. Show that the vector component of $\overrightarrow{OP}$, perpendicular to $\overrightarrow{OQ}$, is $2\underset{\sim}{k}$.

2 marks

ii. Hence or otherwise find $\overrightarrow{PQ}$.

2 marks

Chris has a second shot at goal from the same point *O*. This time he plays an identical shot but a team mate makes contact with the ball when it is 1.5 metres vertically above the ground at point R. With his head, this team mate deflects the ball in a straight line to point *T*, which is in the back left hand corner of the goal.



c. Show that $\overrightarrow{RT} = -2.25\,\underset{\sim}{i} + 7\,\underset{\sim}{j} - 1.5\,\underset{\sim}{k}$

4 marks

d. Hence find the angle of deflection $\angle PRT$, correct to the nearest minute.

3 marks
Total 13 marks

Question 2

A landscape gardener is paving a brick path through a park. The rate at which the path is being laid is given by

$$\frac{dl}{dt} = 2te^{\frac{-t^2}{10}}, \quad t \geq 0$$

where l is the length of the path that has been laid in metres and t is the time taken to lay the path in hours. The landscape gardener starts laying the path at 7.00am.

a. Find the rate, in metres per hour, at which the path is being laid at 10.00am. Express your answer correct to 2 decimal places.

1 mark

b. **i.** Find, using calculus, the time at which the rate at which the landscape gardener is laying the path is fastest. Express your answer to the nearest minute.

3 marks

ii. Find the fastest rate at which the landscape gardener lays the path. Express your answer correct to 1 decimal place.

1 mark

c. After 9.00am when did the rate at which the path was being laid drop below 2 metres per hour? Express your answer to the nearest minute.

1 mark

d. Find $\int 2te^{\frac{-t^2}{10}}\,dt$.

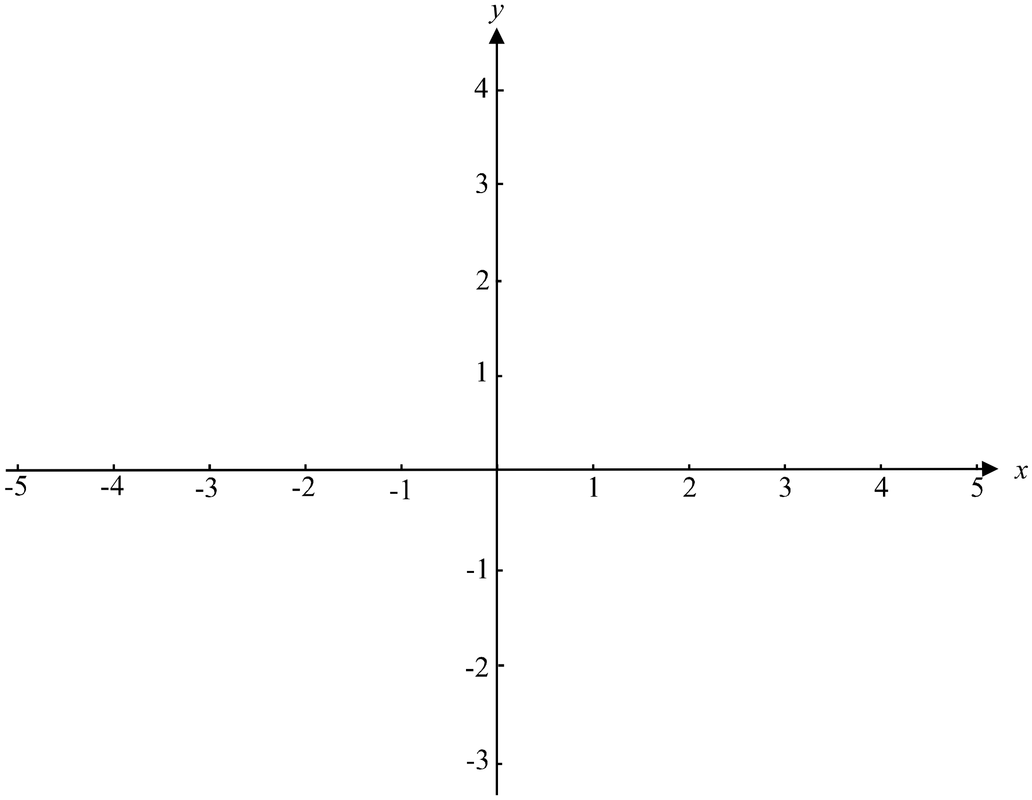
2 marks

e. Hence find the length of the path laid between 7am and 12 noon. Express your answer correct to 1 decimal place.

2 marks
Total 10 marks

Question 3

a. On the set of axes below, sketch the function of $y = \dfrac{3}{\sqrt{x(x+3)}}$.



2 marks

b. Hence write down the maximal domain of the function $y = \dfrac{3}{\sqrt{x(x+3)}}$.

____

1 mark

c. Find an approximation to $\int_1^5 \frac{3}{\sqrt{x(x+3)}}\,dx$ using the midpoint rule and 2 equal intervals. Express your answer correct to 3 significant figures.

2 marks

d. Find the area bounded by the function $y = \frac{3}{\sqrt{x(x+3)}}$ and the lines $x = -5,\ x = 2$ and $y = 1 \cdot 5$. Express your answer correct to 3 significant figures.

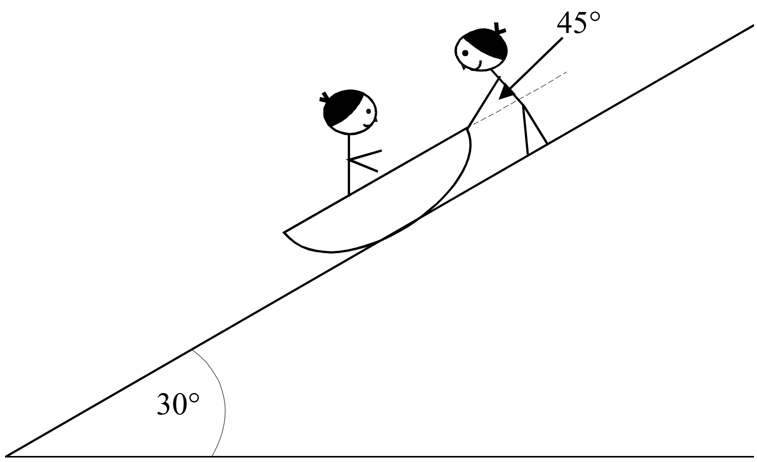
3 marks

e. The function $y = \dfrac{3}{\sqrt{x(x+3)}}$ between $x = 1$ and $x = 5$ is rotated around the x-axis to form a volume of revolution.
Find that volume and express it as an exact value.

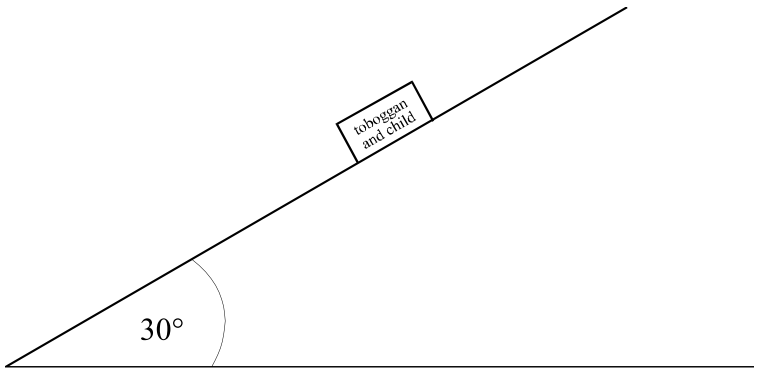
4 marks
Total 12 marks

Question 4

A father exerts a pulling force of $16\sqrt{2}\,g$ newton to pull a child in a toboggan up a rough slope at a constant speed. The slope is at an angle of 30° with the horizontal and the pulling force is acting at an angle of 45° with the slope. The child and the toboggan together, have a mass of 30kg.



a. **i.** On the diagram below show all the forces acting on the toboggan with the child in it.



2 marks

ii. Show that the coefficient of friction between the toboggan with the child in it and the slope is $\dfrac{1}{15\sqrt{3}-16}$.

4 marks

b. The father stops pulling the toboggan so that it becomes stationary. He rests for a minute and adjusts the child's hat. The father then resumes pulling, this time with a pulling force of $18\sqrt{2}\,g$ newton which he maintains for 20 seconds.

i. Find the acceleration of the toboggan, with the child in it, up the slope once the father resumes pulling. Express your answer correct to 2 decimal places.

2 marks

ii. Hence find the distance covered by the toboggan during this time.

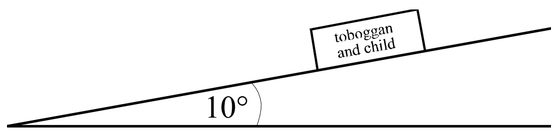
1 mark

iii. Find the speed of the toboggan with the child in it after 20 seconds.

1 mark

c. The father decides to move to a different slope, which is at an angle of 10° with the horizontal and which has a coefficient of friction of 0.2. He lets go of the toboggan on this new slope and it remains at rest.

i. Show on the diagram below the forces now acting on the toboggan with the child in it.



2 marks

ii. Explain with reasons whether or not the toboggan is on the point of sliding down the slope.

2 marks
Total 14 marks

Question 5

a. **i.** Let $z_1 = \sqrt{2} + \sqrt{2}\,i$. Express z_1 in polar form.

1 mark

ii. Find $\bar{z}_1$ in polar form.

1 mark

iii. Let $z_2 = 2cis\left(\frac{3\pi}{4}\right)$. Show that $\frac{z_1}{z_2} = -i$.

2 marks

b. z_1 and z_2 are the roots of a quadratic equation in z. Find that equation.

3 marks

c. **i.** Let $z_3 = \text{cis}\,\theta$. Explain why $(z_3)^n = \text{cis}(n\theta)$.

1 mark

ii. Hence show that $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$.

3 marks
Total 11 marks

Specialist Mathematics Formulas

Mensuration

area of a trapezium:	$\frac{1}{2}(a+b)h$
curved surface area of a cylinder:	$2\pi rh$
volume of a cylinder:	$\pi r^2 h$
volume of a cone:	$\frac{1}{3}\pi r^2 h$
volume of a pyramid:	$\frac{1}{3}Ah$
volume of a sphere:	$\frac{4}{3}\pi r^3$
area of a triangle:	$\frac{1}{2}bc\sin A$
sine rule:	$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$
cosine rule:	$c^2=a^2+b^2-2ab\cos C$

Coordinate geometry

ellipse:	$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$
hyperbola:	$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$

Circular (trigonometric) functions

$$\cos^2(x)+\sin^2(x)=1$$

$$1+\tan^2(x)=\sec^2(x) \qquad \cot^2(x)+1=\operatorname{cosec}^2(x)$$

$$\sin(x+y)=\sin(x)\cos(y)+\cos(x)\sin(y) \qquad \sin(x-y)=\sin(x)\cos(y)-\cos(x)\sin(y)$$

$$\cos(x+y)=\cos(x)\cos(y)-\sin(x)\sin(y) \qquad \cos(x-y)=\cos(x)\cos(y)+\sin(x)\sin(y)$$

$$\tan(x+y)=\frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)} \qquad \tan(x-y)=\frac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}$$

$$\cos(2x)=\cos^2(x)-\sin^2(x)=2\cos^2(x)-1=1-2\sin^2(x)$$

$$\sin(2x)=2\sin(x)\cos(x) \qquad \tan(2)x=\frac{2\tan(x)}{1-\tan^2(x)}$$

function	Sin^{-1}	Cos^{-1}	Tan^{-1}
domain	$[-1, 1]$	$[-1, 1]$	R
range	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$	$[0,\pi]$	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

(These formula sheets have been copied from the 2001 Specialist Maths Exam 1. Teachers and students are reminded that changes to formula sheets are notified in the VCE Bulletins and on the VCAA website.).

Algebra (Complex numbers)

$z = x + yi = r(\cos\theta + i\sin\theta) = r\text{cis}\,\theta$

$|z| = \sqrt{x^2 + y^2} = r$ $\qquad -\pi < \text{Arg}\,z \le \pi$

$z_1 z_2 = r_1 r_2 \text{cis}(\theta_1 + \theta_2)$ $\qquad \dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\text{cis}(\theta_1 - \theta_2)$

$z^n = r^n \text{cis}(n\theta)$ (de Moivre's theorem)

Calculus

$\dfrac{d}{dx}\left(x^n\right) = nx^{n-1}$ $\qquad \displaystyle\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c,\ n \ne -1$

$\dfrac{d}{dx}\left(e^{ax}\right) = ae^{ax}$ $\qquad \displaystyle\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$

$\dfrac{d}{dx}\left(\log_e(x)\right) = \dfrac{1}{x}$ $\qquad \displaystyle\int \frac{1}{x}\,dx = \log_e(x) + c, \text{ for } x > 0$

$\dfrac{d}{dx}\left(\sin(ax)\right) = a\cos(ax)$ $\qquad \displaystyle\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + c$

$\dfrac{d}{dx}\left(\cos(ax)\right) = -a\sin(ax)$ $\qquad \displaystyle\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + c$

$\dfrac{d}{dx}\left(\tan(ax)\right) = a\sec^2(ax)$ $\qquad \displaystyle\int \sec^2(ax)\,dx = \frac{1}{a}\tan(ax) + c$

$\dfrac{d}{dx}\left(\text{Sin}^{-1}(x)\right) = \dfrac{1}{\sqrt{1-x^2}}$ $\qquad \displaystyle\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \text{Sin}^{-1}\left(\frac{x}{a}\right) + c,\ a > 0$

$\dfrac{d}{dx}\left(\text{Cos}^{-1}(x)\right) = \dfrac{-1}{\sqrt{1-x^2}}$ $\qquad \displaystyle\int \frac{-1}{\sqrt{a^2 - x^2}}\,dx = \text{Cos}^{-1}\left(\frac{x}{a}\right) + c,\ a > 0$

$\dfrac{d}{dx}\left(\text{Tan}^{-1}(x)\right) = \dfrac{1}{1+x^2}$ $\qquad \displaystyle\int \frac{a}{a^2 + x^2}\,dx = \text{Tan}^{-1}\left(\frac{x}{a}\right) + c$

product rule: $\dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

quotient rule: $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

chain rule: $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

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mid-point rule: $\int_a^b f(x)dx \approx (b-a)f\left(\frac{a+b}{2}\right)$

trapezoidal rule: $\int_a^b f(x)dx \approx \frac{1}{2}(b-a)(f(a)+f(b))$

Euler's method: If $\frac{dy}{dx} = f(x)$, $x_0 = a$ and $y_0 = b$,
then $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + hf(x_n)$

acceleration: $a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

constant (uniform) acceleration: $v = u + at$ $\qquad s = ut + \frac{1}{2}at^2$

$v^2 = u^2 + 2as$ $\qquad s = \frac{1}{2}(u+v)t$

Vectors in two and three dimensions

$\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$

$\left|\underset{\sim}{r}\right| = \sqrt{x^2 + y^2 + z^2} = r$ $\qquad \underset{\sim}{r_1}.\underset{\sim}{r_2} = r_1 r_2 \cos\theta = x_1x_2 + y_1y_2 + z_1z_2$

$\dot{\underset{\sim}{r}} = \frac{d\underset{\sim}{r}}{dt} = \frac{dx}{dt}\underset{\sim}{i} + \frac{dy}{dt}\underset{\sim}{j} + \frac{dz}{dt}\underset{\sim}{k}$

Mechanics

momentum: $\underset{\sim}{p} = m\underset{\sim}{v}$

equation of motion: $\underset{\sim}{R} = m\underset{\sim}{a}$

friction: $F \le \mu N$

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