THE HEFFERNAN GROUP

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Student Name……………………………………

SPECIALIST MATHEMATICS

TRIAL EXAMINATION 2

(ANALYSIS TASK)

2001

Reading Time: 15 minutes
Writing time: 90 minutes

Instructions to students

This exam consists of 6 questions.
All questions should be answered.
There is a total of 60 marks available.
The marks allocated to each of the six questions are indicated throughout.
Students may bring up to two A4 pages of pre-written notes into the exam.
The acceleration due to gravity should be taken to have magnitude g m/s^2 where $g = 9.8$

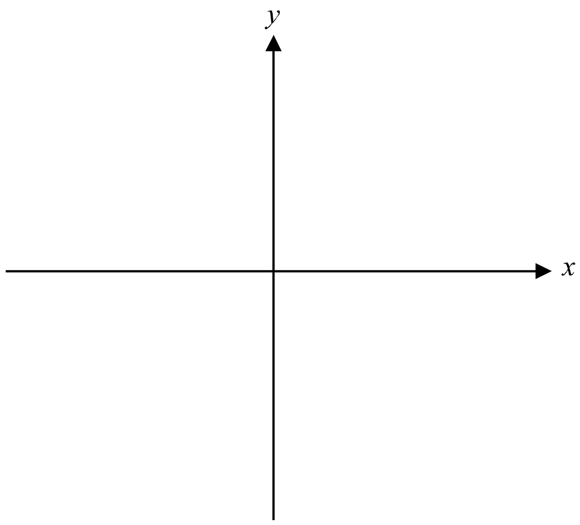
Question 1

Consider the function $f:(-\frac{\pi}{2},\frac{\pi}{2}) \rightarrow R$ where $f(x) = x\sec x$.

a. Find the y-intercept of the graph of $y = f(x)$.

1 mark

b. Sketch the graph of $y = f(x)$ on the set of axes below.



2 marks

c. Find $f'(x)$

1 mark

d. Explain whether or not the graph of $y = f(x)$ has a stationary point at the point where $x = 0$.

1 mark

e. Use the fact that $f''(x) = \dfrac{x + x\sin^2 x + 2\sin x\cos x}{\cos^3 x}$ to find the coordinates of the point on the graph of $y = f(x)$ where the gradient is a minimum.

2 marks

f. Verify that $f(x) = x\sec x$ is a solution to the differential equation

$$\frac{f''(x)}{\sec^3 x} - \frac{\text{cosx}}{\text{cosec}^2 x} f(x) = x + \sin(2x)$$

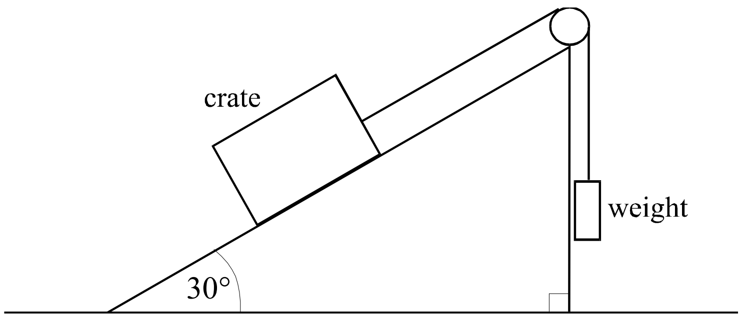
2 marks

Total 9 marks

Question 2

A crate of mass 20kg rests on a rough inclined plane and is attached to a light, inextensible string that passes over a smooth pulley and is attached to a weight of mass 15kg.
The crate is on the point of moving up the incline. Let T be the tension force in the string and let N be the normal force.

a. On the diagram below, mark in the forces acting on the crate and on the weight.



1 mark

b. Find the coefficient of friction between the crate and the plane.

3 marks

A force of 100 Newton acts parallel to the inclined plane and pulls the crate down the slope with the 15 kg weight still attached.

c. Find the acceleration of the 15kg weight. Express your answer correct to 2 decimal places.

4marks

Total 8 marks

Question 3

Let $u = 5i$.

a. Express u in polar form.

1 mark

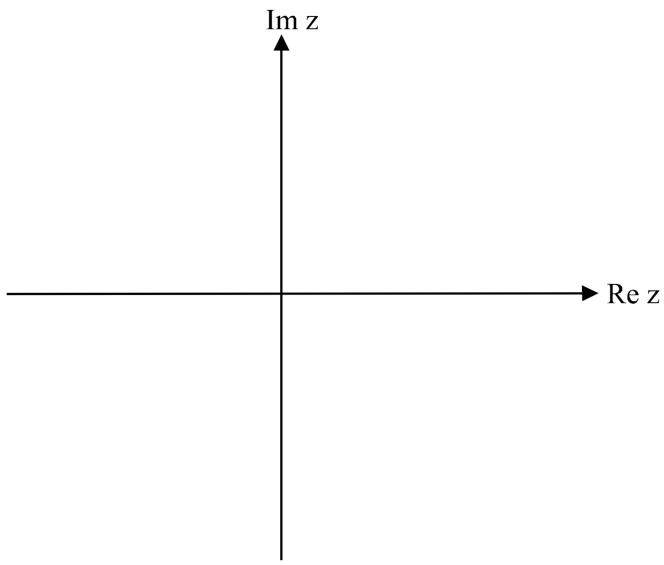
b. If $v = \overline{u} + |u| - 1 + 6i + \text{Re}\,u$, find v in Cartesian form.

1 mark

c. If $z = x + yi$, where x and y are real and $|z - 5i| = |z - 4 - i|$, express y in terms of x.

3 marks

d. Hence or otherwise, sketch on the Argand diagram below, *S*, where $S = \{z : |z - 5i| < |z - 4 - i|\}$ and $z \in C$. Indicate clearly whether or not the boundary is included.



2 marks

e. Write down, in Cartesian form, the cube root(s) of 8 that lie in *S*.

2 marks

Total 9 marks

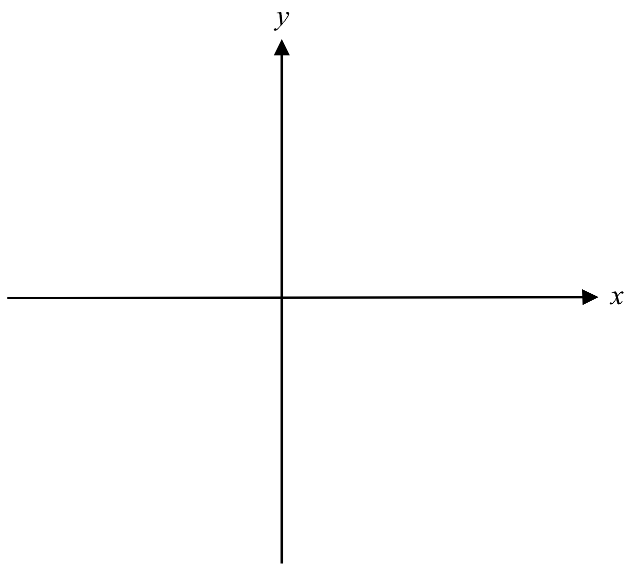
Question 4

Consider the function $f:[-a,\ a] \to R$ where $f(x) = \text{Sin}^{-1}(\frac{x}{2})$.

a. Write down the value of a given that f has a maximal domain.

1 mark

b. **i.** Sketch the graph of $y = f(x)$ on the set of axes below.



1 mark

ii. Show algebraically that the graph in part **i.** has no stationary points.

2 marks

c. Find $\frac{d}{dx}\left(x \text{ Sin}^{-1} \frac{x}{2}\right)$

1 mark

d. **i.** Hence, use calculus to find an antiderivative of $\text{Sin}^{-1} \frac{x}{2}$.

2 marks

ii. Hence find the area enclosed by the graph of $y = f(x)$, the x-axis and the line with equation $x = 1$. Express your answer as an exact value.

2 marks

e. Use calculus to find the volume of the solid of rotation formed when the graph of $f(x)$ between $y = 0$ and $y = \frac{\pi}{2}$, is rotated about the y-axis. Express your answer as an exact value.

3 marks

Total 12 marks

Question 5

A particle moves so that its position vector at time *t*, $t \geq 1$, is given by

$$\underset{\sim}{r}(t) = (t + \frac{1}{t})\underset{\sim}{i} + (t - \frac{1}{t})\underset{\sim}{j}$$

a. Find an expression for the distance from the origin to the particle at time *t*.

1 mark

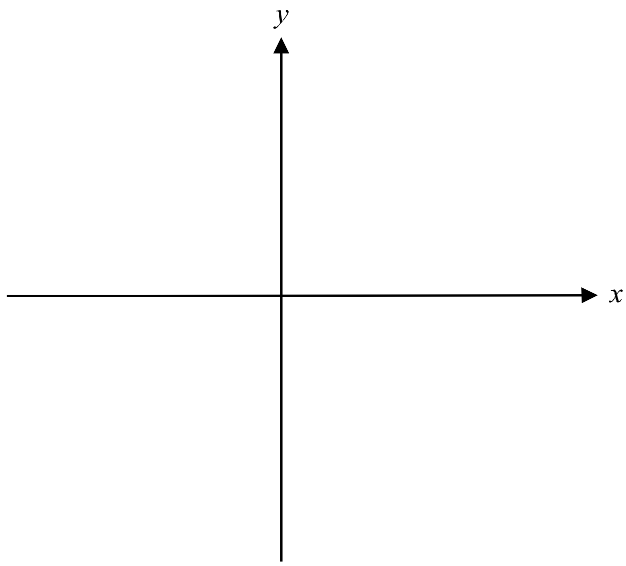
b. Find the speed of the particle at time $t = 5$.

2 marks

c. **i.** Find the Cartesian equation of the path of the particle and state the domain and range.

4 marks

ii. Sketch the path of the particle on the diagram below.



1 mark

A second particle has a velocity vector given by

$$\underset{\sim}{v}_B(t) = (2 - \frac{2}{t^2})\underset{\sim}{i} + (2 + \frac{2}{t^2})\underset{\sim}{j}, \quad t > 0$$

When $t = 1$, the position vector is given by $\underset{\sim}{r}_B(t) = 4\underset{\sim}{i}$.

d. Show that the position vectors of this second particle and of the first particle at time t are parallel.

3 marks

Total 11 marks

Question 6

Julie gets onto a straight stretch of a freeway at time $t = 0$ and has a velocity in m/s at time t seconds of

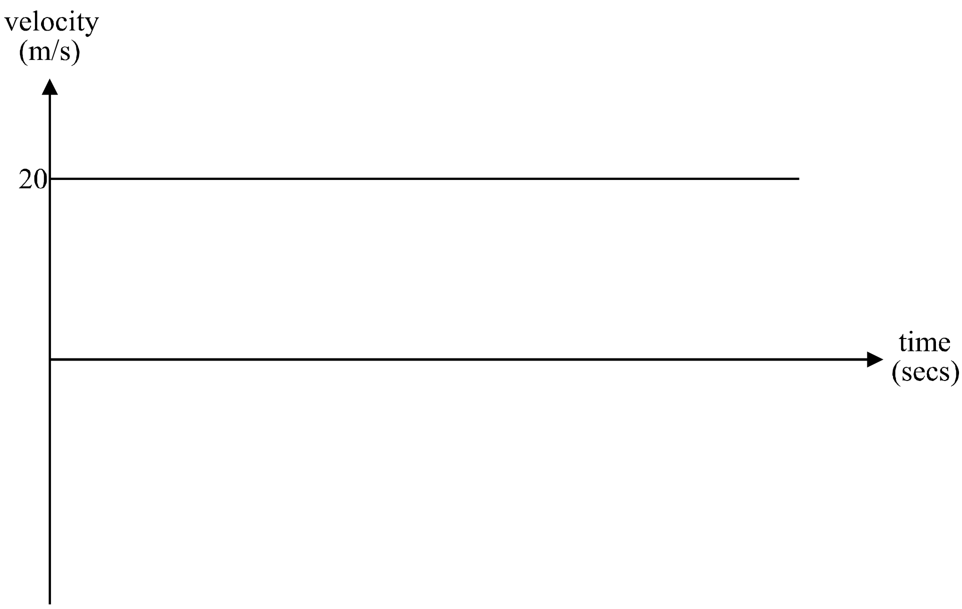
$$v(t) = 0.03t^2 + 15 \qquad t \in [0, 25]$$

a. What is Julie's entry speed onto the freeway?

1 mark

Julie's friend Tom is also traveling on the freeway in the same direction as Julie. Tom passes Julie the instant she gets onto the freeway. Tom is travelling at a constant speed of 20m/s.

The velocity-time graph for Tom's travel is shown below.



b. Sketch on the same set of axes the velocity-time graph for Julie.

1 mark

c. Find, correct to 2 decimal places, the time when Julie catches up to Tom.

2 marks

Further down the freeway, Julie spots what she thinks might be a police speed camera. From that point on, her acceleration may be described by the equation

$$\frac{dv}{dt} = -0.02(v^2 - 625), \qquad t \geq 0$$

d. **i.** Solve this differential equation to obtain t in terms of v given that at the start of this period of acceleration, $v = 35$.

3 marks

ii. Hence show that $v = \dfrac{25(6e^{t}+1)}{6e^{t}-1}$

2 marks

f. If Julie were to maintain this pattern of deceleration, find her limiting velocity. Explain your answer.

2 marks

Total 11 marks