INSTITUTE OF MATHEMATICS VICTORIA

IMV

Mathematical Methods (CAS) U34

Written Examination 2

Accreditation Period ∼ 2023–2027

Name ______________________

Section	Time Recommended	Time Given	Marks Allocated	Marks Awarded
B	15 minutes	120 minutes	20	
C	45 minutes		60	

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers.
- Students are **NOT** permitted to bring into the examination room: notes of any kind, blank sheets of paper or correction fluid/tape.

Instructions

- Detach the formula sheet from the centre of this book during reading time.
- Write your student number in the space provided above on this page.
- All written responses must be in English.

Students are NOT permitted access to mobile phones and/or any other unauthorised electronic devices.

Instructions for Section B

- Answer all questions in pencil on the answer sheet provided for multiple-choice questions.
- Choose the response that is correct for the question.
- A correct answer scores 1, an incorrect answer scores 0.
- Marks will not be deducted for incorrect answers.
- No marks will be given if more than one answer is completed for any question.

Question 1

The period and range of the graph of $y = 2\sin(\frac{\pi}{6}(x-1)) + \tan(\frac{\pi}{5}x) - \frac{\pi}{3}$ are respectively:

A. 60π, $\mathbb{R}$

B. 60, $[-2-\frac{\pi}{3}, 2-\frac{\pi}{3}]$

C. 60π, $[-2-\frac{\pi}{3}, 2-\frac{\pi}{3}]$

D. 60, $\mathbb{R}$

Question 2

Which of the following hold true for all real values of x?

A. $\sin(\sin^2(x)) + \cos(\sin^2(x)) = 1$

B. $\sin^2(\sin(x)) + \cos^2(\cos(x)) = 1$

C. $\sin^2(\cos(x)) + \cos^2(\cos(x)) = 1$

D. $\sin^2(\sin^2(x)) + \cos^2(\cos^2(x)) = 1$

Question 3

Let $f : \mathbb{R} \to \mathbb{R}, f(x) = \log_e(p(x))$. Given that both $f(x)$ and $f^{-1}(x)$ are both defined over all real numbers, Which of the following must be true about $p(x)$?

A. $p(x) = p(-x)$

B. $-p(-x) \neq p(x)$

C. $p(x)$ is not periodic

D. $p^{-1}(x) > 0$

Question 4

Let $f : \mathbb{R} \to \mathbb{R}, f(x) = \log_e(p(x))$. Given that both f and f^{-1} are continuous over an interval D, which of the following must be true?

A. $p(x)$ is continuous and always differentiable over D

B. $p(x)$ is continuous but not necessarily differentiable over D

C. $f(x)$ has at least one turning point in D

D. $f(x)$ may have one turning point in D

Question 5

Let $f : \mathbb{R} \to \mathbb{R}, f(x) = \log_e(p(x))$. Assume that $f(x)$, $p(x)$, and their inverses are continuous and differentiable over a certain interval I. Given that $f(a) = b$, $p(a) = w$, $p'(a) = z$, $p(b) = c$, and $p'(b) = d$, what is the gradient of $f^{-1}(x)$ at the point $(b, f^{-1}(b))$, where $a, b, c, d, w, z \in I$?

A. $\frac{d}{c}$

B. $\frac{c}{d}$

C. $\frac{w}{z}$

D. $\frac{z}{w}$

Question 6

If $y = x^2(x-3)$, over which interval is y increasing?

A. $(0, 2)$

B. $(\infty, 0]$

C. $\mathbb{R}$

D. $[4, \frac{9}{2}]$

Question 7

In the IMV after-school swimming club (IMVSC), lap times are normally distributed with a mean of 35 seconds. If a quarter of the members on the team are faster than 30 seconds, find the standard deviation of swimming times, correct to two decimal places

A. 7.41

B. 2.5

C. 54.95

D. 0.67

Question 8

Zach is the star swimmer of the IMV after-school swimming club (IMVSC). He is practising his starts for his race next week. As he is eager to get the best start possible, he occasionally starts too early. The probability Zach starts too early on any individual attempt is 0.1. Find the most times Zach can start until the probability he has at least one false start is 0.9.

A. 21

B. 22

C. 23

D. 1

Question 9

Consider the lines

$$mx + 2y = 3$$
$$(m+1)x + my = 1$$

For what value(s) of m are the two lines parallel?

A. $m \in \{1-\sqrt{3}, 1+\sqrt{3}\}$

B. $m = 1-\sqrt{3}$

C. $m = 1+\sqrt{3}$

D. $m \in \mathbb{R} \setminus \{1-\sqrt{3}, 1+\sqrt{3}\}$

Question 10

What is the maximum amount of asymptotes that the graph of $y = \frac{1}{ax^2+bx+c}$ can have, where $a, b, c \in \mathbb{R}$?

A. 4

B. 3

C. 2

D. 1

Question 11

Let $h_n(x)$ be the n^{th} derivative of $h_0(x)$. That is, it has been differentiated n times, where $n = 2p$ and $p \in \mathbb{Z}^+$. If $h_0(x) = e^{-x^2}$, how many axial intercepts does $h_n(x)$ have?

A. $2n$

B. n

C. $2n-1$

D. $n+1$

Question 12

After surveying some of her friends, Emily has discovered that only 1 out of her sample of 5 friends think that caramel is the best ice cream flavour. Find, correct to 2 decimal places, a 95% confidence interval of the proportion of all her friends who think that caramel is the best ice cream flavour.

A. (0.00,0.55)

B. (-0.09,0.49)

C. (0.00,0.49)

D. (-0.14,0.55)

Question 13

If $\Pr(A) = p$ and $\Pr(B) = p^2$, which value of p, corresponds with the greatest difference between $\Pr(A \cap B)$ and $\Pr(A \cup B)$, where A and B are independent events?

A. $p = 1.0$

B. $p = 0.3$

C. $p = 0.4$

D. $p = 0.6$

Question 14

Let $f(x) = x^2 + a$. Using Newton's method, for what values of c, where $a > 0$, will an initial estimate of $x_0 = c$ lead to an infinite loop (i.e x_n alternates between two values)

A. $c = 0$

B. $c = \pm\sqrt{a}$

C. $c = \pm\sqrt{\frac{a}{3}}$

D. $c = \pm a$

Question 15

$p(x)$ is an eighth degree polynomial with eight roots. What is the minimum amount of real roots that $q(x)$ can have, where $q'(x) = p(x)$?

A. 1

B. 7

C. 8

D. 9

Question 16

If the sum and product of the solutions of $ax^2 + bx + c = 0$, what can we say about a, b and c, given that they are non-zero?

A. $b + c = 0$

B. $b = c$

C. $a = b$

D. $a = c$

Question 17

$f(x)$ is odd and periodic where the period is n domain of $\mathbb{R}$. Which of the following is necessarily true, given that $k \in \mathbb{R}$?

A. $\int_{kn}^{(k+1)n} f(x)dx + \int_{(k+1)n}^{(k+2)n} f(x)dx = 0$

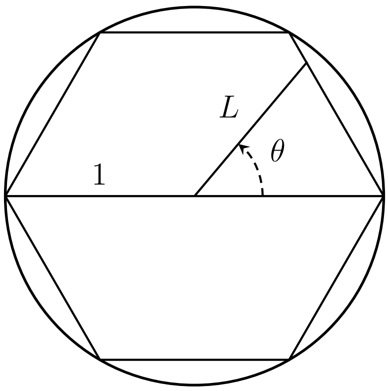
B. $\int_{kn}^{(k+1)n} f(x)dx + \int_{-kn}^{-(k+1)n} f(x)dx = 0$

C. $\int_{kn}^{(k+1)n} f(x)dx + \int_{-(k+1)n}^{-kn} f(x)dx = 0$

D. $\int_{kn}^{(k+1)n} f(x)dx = nf(k)$

Question 18

A regular hexagon is inscribed in a unit circle, as shown in the diagram below. A line, L, is connected from the centre of the circle to the edge of the hexagon. The length of the line is in terms of θ, the angle made by the line and the horizontal axes.



the average value of L in terms of θ, where $0 < \theta < 2\pi$ is:

A. $\frac{3\sqrt{3}\ln(3)}{2\pi}$

B. 0.9855

C. $\frac{\ln(27)}{2\pi}$

D. $\frac{3\sqrt{3}}{2\pi}$

Question 19

Let $f(x) : [l_1, r_1) \to \mathbb{R}, f(x) = \sqrt{x-2} + k$ and $g : [l_2, r_2) \to \mathbb{R}, g(x) = k - \log_e(2-x)$, such that the range of $(f \circ g)(x)$ and $(g \circ f)(x)$ are both $[a, \infty)$, where a is a real constant. Given that $[l_1, r_1)$ and $[l_2, r_2)$ represent each function's maximal domains for any value of k, the values of $r_1 - l_1$ and $r_2 - l_2$ are respectively:

A. $\frac{1}{e}$ and e

B. 1 and $\frac{1}{e}$

C. e and ∞

D. e and 1

Question 20

Let $f : \mathbb{R} \to \mathbb{R}$ a probability density function representing the continuous random variable X, and let $p(x) = \Pr(X < x)$. Which of the following must hold true?

A. $p(x)$ is strictly increasing

B. $p'(x) > 0$ for all $x \in \mathbb{R}$

C. The graph of $y = p(x)$ has vertical asymptotes at $y = 0$ and $y = 1$

D. If $f(b) = c$ for all $b \in D$, then $p'(b) = c$

Instructions for Section C

- Answer all questions in the space provided.
- In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.
- In questions where more than one mark is available, appropriate working must be shown
- Unless otherwise indicated, the diagrams in this book are not drawn to scale

Question 1 **(1+1+1+1+2 = 6 marks)**

Let the function $f(x)$ be defined as $(x-1)^2(x+1)^2$ where $x \in R$

a) Find the tangent of $f(x)$ at the point a in terms of a.
(1 mark)

b) Find the values of a for which the tangent has only one intersection with $f(x)$.
(1 mark)

c) Find an expression for the area bounded between $f(x)$ and the line $y = a$, in terms of a where $a \geq 1$
(1 mark)

d) The normal to $f(x)$ at b where $0 < b < 1$ intersects twice with $f(x)$. Find the minimum bounded area between the normal and $f(x)$.
(2 marks)

e) Find the sequence of transformations that transforms $y = (x^2 - 4)^2$ to $y = f(x)$.
(1 mark)

f) $f(x) + c$ can be expressed in the form $(x - i)^2(x - j)(x + k)$ where j and $k > 0$. Find the values of i,j,k and c.
(2 marks)

Question 2 **(2+1+(1+2)+1+3+2+3 = 15 marks)**

Let $f(x) = \tan(\ln(x))$

a) $f_1 : (p, q) \to \mathbb{R}, f_1(x)$ is defined such that (p, q) represents the maximal domain over which f_1 has an inverse. Find all possible values of p and q. (2 marks)

b) Find a general rule for $f^{-1}(x)$, the inverse function of $f(x)$. (1 mark)

c)

i) Find the equation of the tangent to $f(x)$ at every x intercept. (1 mark)

ii) Find the x intercepts of the tangents found in i) in the form of a general solution. (2 marks)

iii) Find the sequence of transformations that transforms one tangent found in i) to another unique tangent found in i), and state your answer as a general solution.
(1 mark)

d) Let $g : (\frac{1}{e^{\frac{\pi}{2}}}, e^{\frac{\pi}{2}}) \to \mathbb{R}, g(x) = f(x)$. Find the bounded area between $g(x)$, its inverse, the x axis and the y axis, correct to two decimal places.
(3 marks)

e) $g(x)$ is translated c units up. What is the maximum number of intersections between the translated function and its inverse.

(2 marks)

f) Find the values of c (to two decimal places) for which the translated function of $g(x)$ intersects with its inverse at the maximum number of times found in f).

(3 marks)

Question 3 **(1+1+2+2+2+2+2 = 12 marks)**

A company manufactures two types of batteries, Type A and Type B, and they produce 50% more type A batteries than type B batteries. The company claims that the lifespan of a Type A battery follows a normal distribution with a mean of 400 hours and a standard deviation of 50 hours, while the lifespan of a Type B battery follows a normal distribution with a mean of 500 hours and a standard deviation of 60 hours. If a battery lasts for under 75% of its mean duration, it is labelled defective.

a) Calculate the probability that a randomly selected Type A battery is defective. Express your answer to four decimal places.
(1 mark)

b) Calculate the probability that a randomly selected Type B battery is defective. Express your answer to four decimal places.
(1 mark)

c) A randomly chosen battery is found to be defective. What is the probability that it is a Type A battery?

(2 marks)

d) Find the minimum number of batteries to be packed into each box so that the probability that at most two batteries are defective is 0.9.

(2 marks)

e) How many boxes may be packed until it is less than 10% certain that a box contains no defective batteries?

(2 marks)

g) A worker is sorting the company inventory and comes across two unlabelled boxes. They know that one box contains only Type A batteries and the other only Type B. The worker tests one battery, and labels the box as Type A if the battery lasts less than 425 hours. Find the probability that the worker mislabels the boxes, correct to 6 decimal places.

(2 marks)

Question 4 **(2+1+1+2+3+3 = 12 marks)**

Let there be the functions $f(x) = 3x\sin(3x)$ and $g(x) = (\pi - 3x)\sin(3x)$ that are defined over R. a) Find the sequence of transformations that maps f onto g.
(2 marks)

b) Find the area bounded between $f(x)$ and $g(x)$ between $x = 0$ and $x = \frac{\pi}{6}$.
(1 mark)

c) $h(x)$ is the quotient of functions $f(x)$ and $g(x)$, where $h(x) = \frac{f(x)}{g(x)}$. Find the rule of $h(x)$.
(1 mark)

d) Find the domain and range of $h(x)$.
(2 marks)

e) Let $h_1(x) : [\pi, \infty), h_1(x) = \frac{f(x)}{g(x)}$. Show that $h_1(x)$ is strictly increasing over its entire domain.
(3 marks)

Let $p(x) = x\sin(x)$
$q(x) = x\cos(x)$

f) $p(x)$ and $q(x)$ have infinitely many positive intersections, with $x_1 = \frac{\pi}{4}$, $x_2 = \frac{5\pi}{4}$, and so on. Find the area bounded between $p(x)$, $q(x)$, the line $x = x_{2n}$, and the line $x = x_{2n+1}$, in terms of n, where $n \in \mathbb{Z}$
(3 marks)

Question 5 **(1+1+2+2+2+2+2+3 = 15 marks)**

Let $f : R \to R, f(x) = ae^{bx}$ where $a, b \in R^+$.

a) Find the equation of the tangent and normal to $f(x)$ at $x = 0$ in terms of a and b. (The normal is the line perpendicular to the tangent passing through the same point).
(1 mark)

b) Find the x intercepts of the tangent and normal in terms of a and b.
(1 mark)

c) The tangent and normal form a triangle with the x axis. Find the area of this triangle.

(2 marks)

d) Fully define the inverse of $f(x)$
(2 mark)

e) Find the values of a in terms of b such that $f(x)$ is tangential to its inverse at their single point of intersection.
(2 marks)

f) Find the equation of the tangent and normal at the x-intercept of the inverse of $f(x)$.
(2 marks)

g) Find the value(s) of a and b such that the gradient of the tangent at $f(x)$ is equal to the tangent at its inverse, and the gradient of the normal at $f(x)$ is equal to the normal of its inverse.

(2 mark)

A quadrilateral is formed by connecting:

- The tangent to $f(x)$ at its y intercept
- The normal to $f(x)$ at its y intercept
- The tangent to $f^{-1}(x)$ at its x intercept
- The normal to $f^{-1}(x)$ at its x intercept

h) Find the area of the quadrilateral, in terms of a and b where necessary. (3 marks)

End