

## VCE Mathematical Methods Units 3&4

### Written Examination 2

### Suggested Solutions

#### SECTION A – MULTIPLE-CHOICE QUESTIONS

1	A	B	C	D	E
2	A	B	C	D	E
3	A	B	C	D	E
4	A	B	C	D	E
5	A	B	C	D	E
6	A	B	C	D	E
7	A	B	C	D	E
8	A	B	C	D	E
9	A	B	C	D	E
10	A	B	C	D	E
11	A	B	C	D	E
12	A	B	C	D	E
13	A	B	C	D	E
14	A	B	C	D	E
15	A	B	C	D	E
16	A	B	C	D	E
17	A	B	C	D	E
18	A	B	C	D	E
19	A	B	C	D	E
20	A	B	C	D	E

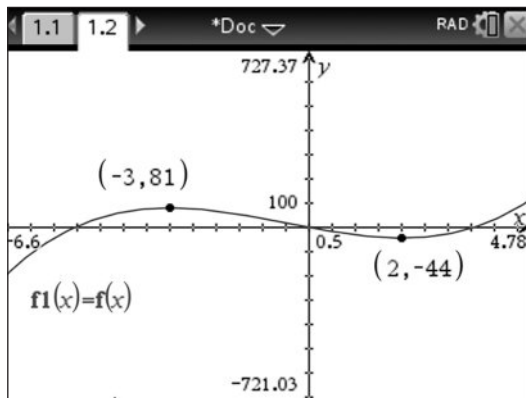
**Question 1 B**

$$\text{period} = \frac{2\pi}{\pi} = 2$$

$$\text{range} = [2 - 3, 2 + 3] = [-1, 5]$$

**Question 2 C**

Using a CAS calculator gives:



**C** is not a true statement and is therefore the required response. The graph does have a point of inflection.

**A**, **B** and **D** are true statements and are therefore not the required response.

**E** is a true statement and is therefore not the required response. The function does not have an inverse since it is not monotonic.

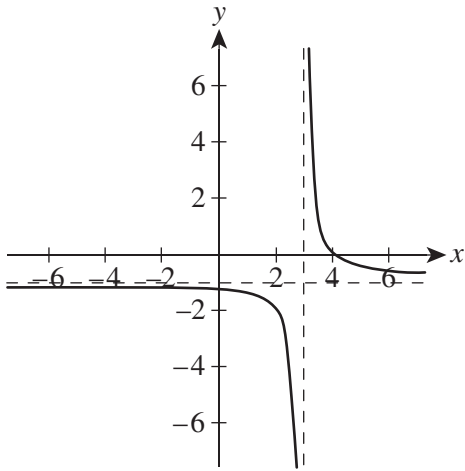
**Question 3 E**

Using a CAS calculator to consider three cases where the second ball is a different colour to the first ball gives:

$$\frac{4}{10} \cdot \frac{6}{9} + \frac{5}{10} \cdot \frac{5}{9} + \frac{1}{10} \cdot \frac{9}{9} = \frac{29}{45}$$

**Question 4 C**

C is correct. The graph of  $f(x) = -1 + \frac{1}{x-3}$  is as follows.



A is incorrect. This option has the asymptote  $x = 3$  only.

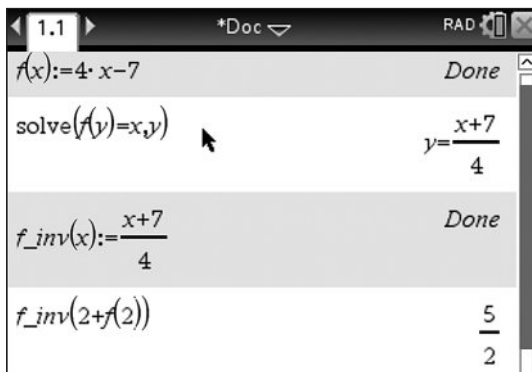
B is incorrect. This option has the asymptote  $y = -1$  only.

D is incorrect. This option has the asymptotes  $x = -1$  and  $y = 3$ .

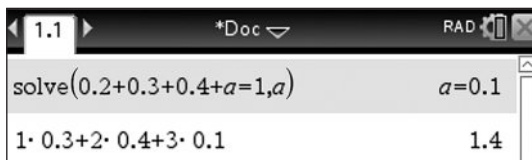
E is incorrect. This option has the asymptotes  $x = 3$  and  $y = 0$ .

**Question 5 C**

Using a CAS calculator gives:

**Question 6 A**

Using a CAS calculator gives:



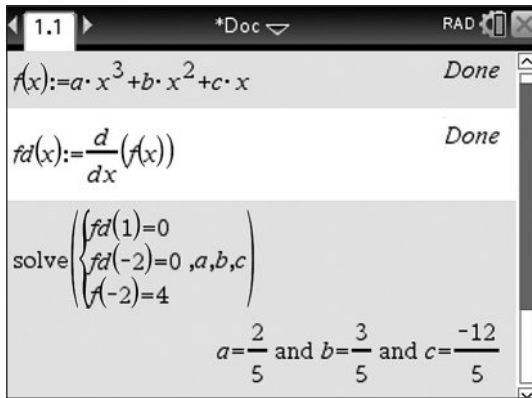
**Question 7 B**

Determining the average value gives:

$$\begin{aligned} \frac{1}{4-0} \int_0^4 f(x) dx &= \frac{1}{4} \text{ of area under } f(x) \\ &= \frac{1}{4} \left( \frac{3}{2}(4+1) + \frac{1}{2}(1+3) \right) \\ &= \frac{19}{8} \end{aligned}$$

**Question 8 A**

Using a CAS calculator gives:

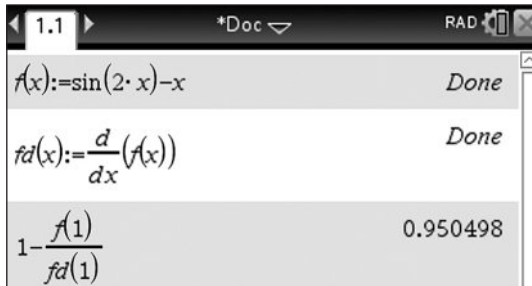


**Question 9 C**

The algorithm returns an angle multiplied by  $\frac{180}{\pi}$  or  $\frac{\pi}{180}$  depending on the initial unit. This is used to convert a given angle between degrees and radians.

**Question 10 D**

Using a CAS calculator gives:

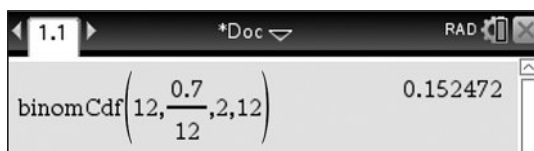


**Question 11 B**

$$\begin{aligned}\Pr(17.2 \leq X \leq 18.4) &= \Pr\left(\frac{17.2-18}{0.4} \leq Z \leq \frac{18.4-18}{0.4}\right) \\ &= \Pr(-2 \leq Z \leq 1) \\ &= \Pr(-1 \leq Z \leq 2)\end{aligned}$$

**Question 12 C**

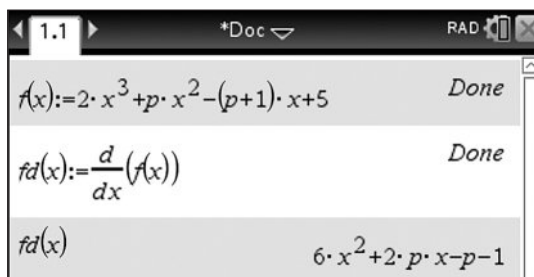
Using a CAS calculator gives:



A screenshot of a CAS calculator interface. The top bar shows '1.1', '\*Doc', and 'RAD'. The main display area shows the function  $\text{binomCdf}\left(12, \frac{0.7}{12}, 2, 12\right)$  on the left and the result  $0.152472$  on the right.

**Question 13 A**

Using a CAS calculator gives:



A screenshot of a CAS calculator interface. The top bar shows '1.1', '\*Doc', and 'RAD'. The main display area shows three lines of input and output:
   
1.  $f(x) := 2 \cdot x^3 + p \cdot x^2 - (p+1) \cdot x + 5$  with 'Done' on the right.
   
2.  $f'(x) := \frac{d}{dx}(f(x))$  with 'Done' on the right.
   
3.  $f'(x)$  with the result  $6 \cdot x^2 + 2 \cdot p \cdot x - p - 1$  on the right.

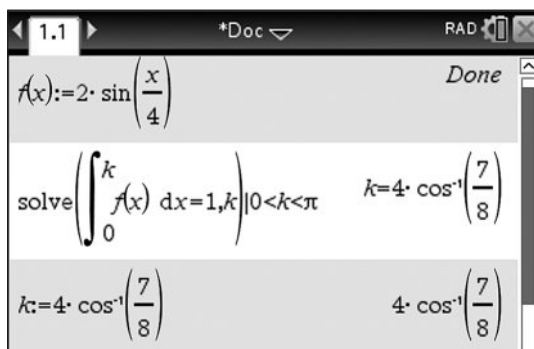
If a function has a turning point, then it is not monotonic; hence, it has no inverse.

So, looking for a positive discriminant gives:

$$\begin{aligned}(2p)^2 - 4 \times 6 \times (-p-1) &> 0 \\ 4p^2 + 24p + 24 &> 0 \\ p^2 + 6p + 6 &> 0\end{aligned}$$

**Question 14 E**

Using a CAS calculator gives:



A screenshot of a CAS calculator interface. The top bar shows '1.1', '\*Doc', and 'RAD'. The main display area shows three lines of input and output:
   
1.  $f(x) := 2 \cdot \sin\left(\frac{x}{4}\right)$  with 'Done' on the right.
   
2.  $\text{solve}\left(\int_0^k f(x) dx = 1, k\right) | 0 < k < \pi$  with the result  $k = 4 \cdot \cos^{-1}\left(\frac{7}{8}\right)$  on the right.
   
3.  $k := 4 \cdot \cos^{-1}\left(\frac{7}{8}\right)$  with the result  $4 \cdot \cos^{-1}\left(\frac{7}{8}\right)$  on the right.



A screenshot of a CAS calculator interface showing a definite integral. The main display area shows  $\int_1^k f(x) dx$  on the left and the result  $0.751299$  on the right.

**Question 15 E**

The three numbers can be expressed as  $x$ ,  $2x$  and  $100 - 3x$ .

Using a CAS calculator gives:

$f(x) := x \cdot 2 \cdot x \cdot (100 - 3 \cdot x)$	Done
$\text{solve}\left(\frac{d}{dx}(f(x))=0, x\right)$	$x=0.$ or $x=22.2222$
$f(22)$	32912
$f(23)$	32798

Hence, the maximum product is 32 912.

**Question 16 A**

The answer can be obtained by trial and error using a CAS calculator.

$f(n) := \text{binomCdf}(n, 0.24, 3, n)$	Done
$f(17)$	0.812347
$f(18)$	0.842994
$f(16)$	0.776767

Hence, the smallest possible value of  $n$  is 17.

**Question 17 C**

Reflection in the  $y$ -axis maps  $y = \sin(2x)$  to  $y = \sin(-2x)$ .

Dilation by a factor of 2 from the  $y$ -axis maps  $y = \sin(-2x)$  to  $y = \sin(-x)$ .

Translation of  $\frac{\pi}{2}$  units in the positive direction of the  $x$ -axis maps  $y = \sin(-x)$  to:

$$\begin{aligned} y &= \sin\left(-\left(x - \frac{\pi}{2}\right)\right) \\ &= \sin\left(\frac{\pi}{2} - x\right) \\ &= \cos(x) \end{aligned}$$

**Question 18 B**

Using a CAS calculator gives:

$f(x) := e^{2 \cdot x} + 1$	Done
$g(x) := \text{tangentLine}(f(x), x=a)$	Done
$\triangle \text{solve}(g(2)=0, a)$	$a=2.50335$
$a := 2.50335$	2.50335
$g(0)$	-597.645

**Question 19 B**

Using a CAS calculator gives:

$\text{domain}\left(5 \cdot \tan\left(\frac{x}{3}\right) - 2, x\right)$	$x \neq \frac{3 \cdot (2 \cdot n1 - 1) \cdot \pi}{2}$
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The answer is not immediately identifiable. Further manipulation gives:

$$\begin{aligned} \frac{3\pi}{2}(2k-1) &= \frac{3\pi}{2}(2k-1) + 2\pi m \quad (\text{where } k \text{ and } m \text{ are any integers}) \\ &= \frac{3\pi}{2}(2k-1) + \frac{3\pi}{2}\left(\frac{2}{3\pi} \times 2\pi m\right) \\ &= \frac{3\pi}{2}\left(2k-1 + \frac{4m}{3}\right) \end{aligned}$$

Choosing  $m = 3$ ,  $\frac{3\pi}{2}(2k+3)$ .**Question 20 C**Stationary points exist when  $\frac{d}{dx}f(g(x)) = 0$ .

$$\frac{d}{dx}f(g(x)) = f'(g(x)) \times g'(x)$$

$$f'(g(x)) \times g'(x) = 0 \Rightarrow \begin{cases} g'(x) = 0 \\ f'(g(x)) = 0 \end{cases}$$

Observing the graphs:

$$\begin{cases} g'(x) = 0 \\ f'(g(x)) = 0 \end{cases} \Rightarrow \begin{cases} 1 \text{ solution: } x = 3 \\ g(x) \approx 0.5 \Rightarrow 2 \text{ solutions} \\ g(x) \approx 3.5 \Rightarrow 1 \text{ solution} \\ g(x) \approx 6.5 \Rightarrow \text{no solution} \end{cases}$$

Note that the question asks for solutions for  $0 \leq x \leq 5$ . Therefore, there are 4 stationary points.

**SECTION B****Question 1** (9 marks)

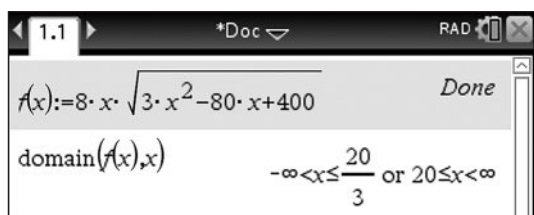
$$\text{a. } KM = \frac{80 - 6x - 2x}{2} = 40 - 4x \quad \text{M1}$$

$$\begin{aligned} \text{b. } KO &= \sqrt{KM^2 - MO^2} \\ &= \sqrt{(40 - 4x)^2 - (2x)^2} \quad \text{M1} \\ &= 2\sqrt{3x^2 - 80x + 400} \quad \text{A1} \end{aligned}$$

$$\begin{aligned} A &= \frac{1}{2} \times KO \times (KL + MN) \\ &= \frac{1}{2} \times (2\sqrt{3x^2 - 80x + 400}) \times 8x \quad \text{M1} \\ &= 8x\sqrt{3x^2 - 80x + 400} \end{aligned}$$

*Note: Consequential on answer to Question 1a.*

c. Using a CAS calculator gives:



The screenshot shows a CAS calculator interface with the following text:

```

1.1 | *Doc | RAD | Done
f(x) := 8 * x * sqrt(3 * x^2 - 80 * x + 400)
domain(f(x), x) | -inf < x <= 20/3 or 20 <= x < inf

```

The valid interval for  $x$  such that the trapezium exists needs to be selected.

$$0 < x < \frac{20}{3} \quad \text{A1}$$

*Note: Responses must use strict inequalities.*



d. Using a CAS calculator gives:

$f(x) := 8 \cdot x \cdot \sqrt{3 \cdot x^2 - 80 \cdot x + 400}$   
 solve $\left(\frac{d}{dx}(f(x))=0, x\right)$   
 $x = \frac{-10 \cdot (\sqrt{3} - 3)}{3}$  or  $x = \frac{10 \cdot (\sqrt{3} + 3)}{3}$

$x = 4.2265$  or  $x = 15.7735$   
 $f\left(\frac{-10 \cdot (\sqrt{3} - 3)}{3}\right) = 363.333$

$$\frac{dA}{dx} = 0 \Rightarrow x = 4.2 \dots \text{ or } x = 15.7 \dots$$

M1

$$\text{Since } 15.7 \dots \notin \left(0, \frac{20}{3}\right), x = 4.2 \dots$$

$$A(4.2 \dots) = 363.3$$

A1

e. Using a CAS calculator gives:

$f(x) := 8 \cdot x \cdot \sqrt{3 \cdot x^2 - 80 \cdot x + 400}$   
 $\frac{1}{\frac{20}{3} - 0} \int_0^{\frac{20}{3}} f(x) dx = 255.631$

$$\frac{1}{\frac{20}{3} - 0} \int_0^{\frac{20}{3}} A(x) dx = 255.6$$

M1 A1

**Question 2** (11 marks)

a. Solving for  $k$  using a CAS calculator gives:

A screenshot of a CAS calculator interface. The top bar shows '1.1', '\*Doc', and 'RAD'. The main display area shows three lines of input and output:

- Line 1:  $g(x) := \frac{-1}{16} \cdot x^2 - \frac{1}{4} \cdot x + \frac{221}{16}$  followed by 'Done'.
- Line 2:  $h(x) := \frac{1}{20} \cdot x^2 - \frac{41}{20} \cdot x + k$  followed by 'Done'.
- Line 3:  $g(8)$  followed by the result  $\frac{125}{16}$ .

A screenshot of a CAS calculator interface showing the definition of  $h(8)$  and solving for  $k$ :

- Line 1:  $h(8)$  followed by the result  $k - \frac{66}{5}$ .
- Line 2:  $\text{solve}(g(8)=h(8),k)$  followed by the result  $k = \frac{1681}{80}$ .

$$g(8) = \frac{125}{16}$$

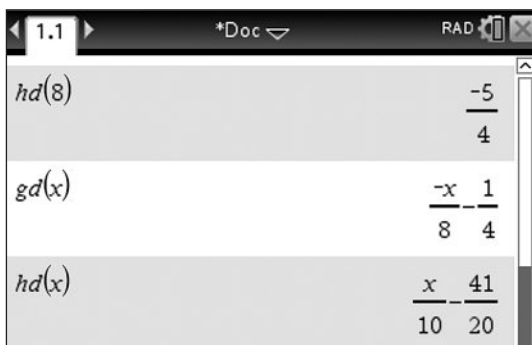
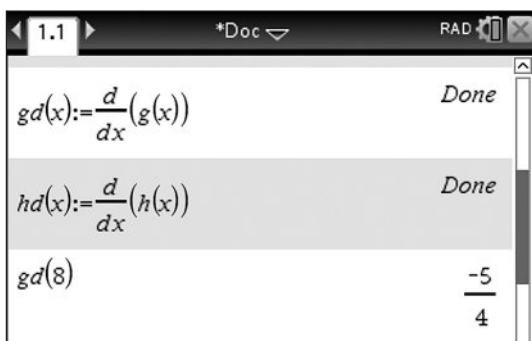
M1

$$h(8) = k - \frac{66}{5}$$

$$g(8) = h(8) \Rightarrow k = \frac{1681}{80}$$

M1

b. Using a CAS calculator gives:



$$g'(x) = -\frac{1}{8}x - \frac{1}{4}$$

$$h'(x) = \frac{1}{10}x - \frac{41}{20}$$

$$g'(8) = -\frac{5}{4}$$

$$h'(8) = -\frac{5}{4}$$

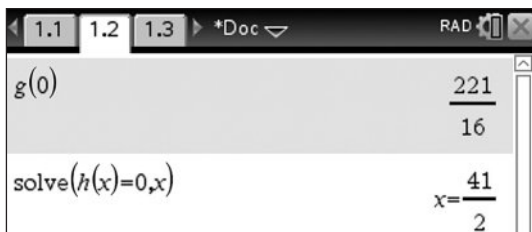
$g'(x)$  and  $h'(x)$  M1

$g'(8)$  and  $h'(8)$  A1

$$\begin{cases} g(8) = h(8) \\ g'(8) = h'(8) \end{cases} \Rightarrow f(x) \text{ is differentiable at } x = 8$$

M1

c. Using a CAS calculator gives:



height:  $g(0) = \frac{221}{16}$  m

A1

length:  $h(x) = 0 \Rightarrow x = \frac{41}{2}$  m

A1

- d. i. The curve is concave down for the given interval, so the trapezium will always be under the curve. Hence, the approximation will be less than the actual area. A1

- ii. Using the CAS calculator gives:

The screenshot shows a CAS calculator interface with the following content:

$$\int_0^8 g(x) dx = \frac{551}{6}$$

$$\frac{1}{2} \cdot (g(0) + 2 \cdot (g(1) + g(2) + g(3) + g(4) + g(5) + g(6)) + g(8)) = \frac{367}{4}$$

The screenshot shows the calculation of the difference between the actual area and the approximate area:

$$\frac{551}{6} - \frac{367}{4} = 0.083333$$

$$\text{actual area} = \int_0^8 g(x) dx = \frac{551}{6}$$

A1

$$\text{approximate area} = \frac{1}{2} \left( g(0) + 2 \left( \begin{array}{l} g(1) + g(2) + g(3) + g(4) \\ + g(5) + g(6) + g(7) \end{array} \right) + g(8) \right) = \frac{367}{4}$$

A1

$$\text{difference} = \frac{551}{6} - \frac{367}{4} = 0.083$$

A1

**Question 3** (14 marks)

$$\text{a. } \int_8^{12} \frac{1}{40}(t-8)dt = \frac{[(t-8)^2]_8^{12}}{80} = \frac{16-0}{80} = \frac{16}{80} \quad \text{A1}$$

$$\int_{12}^{15} \frac{1}{40}(20-t)dt = \frac{[(20-t)^2]_{12}^{15}}{-80} = \frac{25-64}{-80} = \frac{39}{80} \quad \text{A1}$$

$$\begin{aligned} \Pr(T \leq 15) &= \frac{16}{80} + \frac{39}{80} && \text{M1} \\ &= \frac{55}{80} \\ &= \frac{11}{16} \end{aligned}$$

b. Using a CAS calculator gives:

The screenshot shows a CAS calculator window with the following content:

$$f(t) := \begin{cases} \frac{1}{40} \cdot (t-8), & 8 \leq t \leq 12 \\ \frac{1}{40} \cdot (20-t), & 12 \leq t \leq 20 \end{cases}$$

Below the function definition, the integral of  $f(t)$  from 8 to 12 is calculated, resulting in  $\frac{16}{55}$ . The final result shown is  $\frac{11}{16}$ .

$$\frac{\int_8^{12} f(t)dt}{\frac{11}{16}} = \frac{16}{55}$$

M1 A1

c.

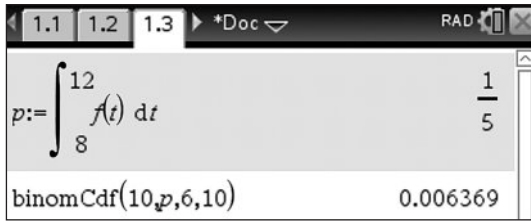
The screenshot shows a CAS calculator window with the following content:

$$\int_8^{20} (t \cdot f(t)) dt = 13.8667$$

$$\int_8^{20} t \times f(t) dt = 13.9 \text{ minutes}$$

M1 A1

d.



$X$  = number of trips completed in less than 12 minutes

$$X \sim \text{Bi}(10, p)$$

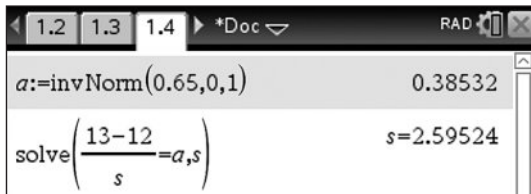
$$p = \int_8^{12} f(t) dt = \frac{1}{5}$$

M1

$$\Pr(X \geq 6) = 0.0064$$

A1

e.



$$\Pr(U < 13) = \Pr(Z < a)$$

$$a = 0.3853 \dots$$

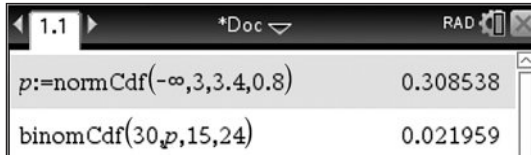
M1

$$\frac{13-12}{\sigma} = a$$

$$\sigma = 2.5952$$

A1

f.



$Y$  = number of times a trip is interrupted by red light

$$Y \sim N(3.4, 0.8^2)$$

M1

$X$  = number of trips with less than three red light interruptions

$$X \sim \text{Bi}(30, p)$$

$$p = \Pr(Y < 3) = 0.3085 \dots$$

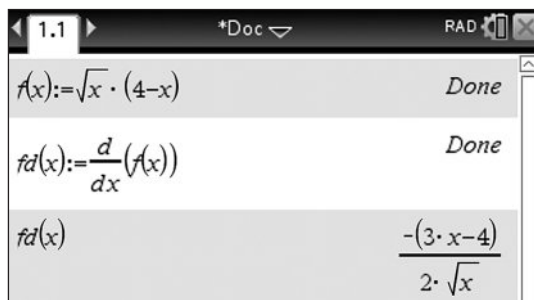
A1

$$\hat{P} = \frac{X}{30} \Rightarrow \Pr(0.5 \leq \hat{P} \leq 0.8) = \Pr(15 \leq X \leq 24) = 0.0220$$

A1

**Question 4** (16 marks)

a. Using a CAS calculator gives:



$$f'(x) = \frac{4-3x}{2\sqrt{x}}$$

M1

$$f'(x) = 0 \Rightarrow x = \frac{4}{3}$$

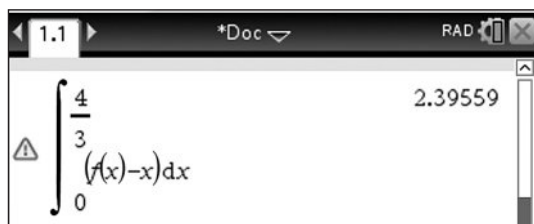
A1

If  $D_f = [0, k]$  does not contain any turning points, then  $f$  has an inverse. Therefore,

$$0 < k \leq \frac{4}{3}.$$

M1

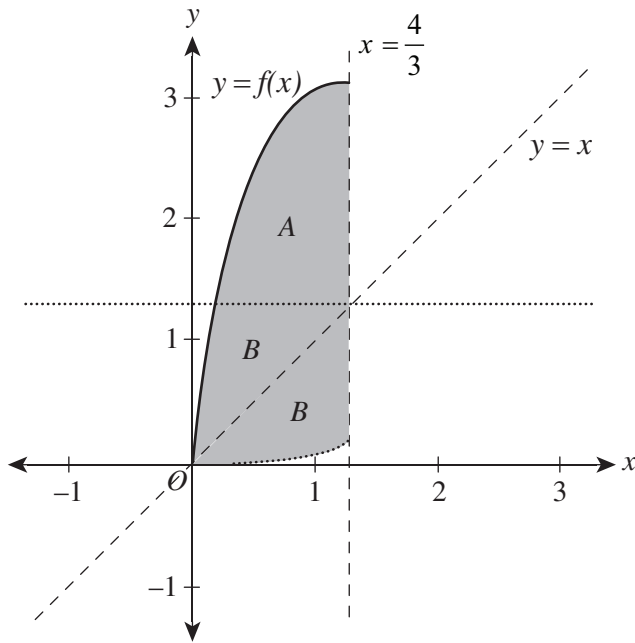
b.



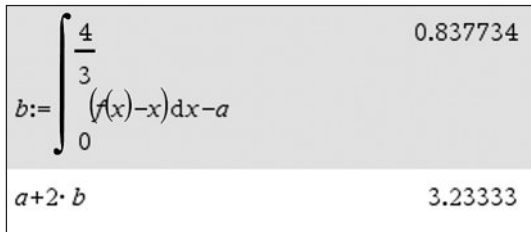
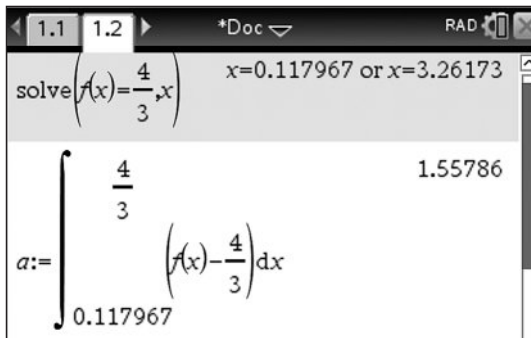
$$\int_0^{\frac{4}{3}} (f(x) - x) dx = 2.40$$

M1 A1

- c. Rather than trying to find a rule for  $f^{-1}$ , the area can be found using symmetrical regions.



Using the graph above, the total area can be represented by  $A + 2B$ .



$$f(x) = \frac{4}{3} \Rightarrow x = 0.1179 \dots \quad \text{M1}$$

$$A = \int_{0.1179 \dots}^{\frac{4}{3}} \left( f(x) - \frac{4}{3} \right) dx = 1.5578 \dots \quad \text{M1}$$

$$B = \int_0^{\frac{4}{3}} (f(x) - x) dx - A = 0.8377$$

$$A + 2B = 3.23 \quad \text{A1}$$



d.

TI-84 Plus calculator screenshot showing the definition of  $g(x) := a \cdot \sqrt{x} \cdot (4 - a \cdot x)$  and the integration of  $(g(x) - x)$  from  $0$  to  $\frac{4}{3 \cdot a}$ . The result is  $s(a) := \frac{256 \cdot \sqrt{\frac{3}{a}}}{135} - \frac{8}{9 \cdot a^2}$ .

$\text{solve}\left(\frac{d}{da}(s(a))=0, a\right)$	$a=1.05429$
$s(1.05429)$	$2.3991$

Let the area be  $S(a)$ :

$$S(a) = \int_0^{\frac{4}{3a}} (g(x) - x) dx$$

$$= \frac{256}{135} \sqrt{\frac{3}{a}} - \frac{8}{9a^2}$$

M1

$$S'(a) = 0 \Rightarrow a = 1.05 \dots$$

A1

$$S(1.05 \dots) = 2.40$$

A1

e.

i.

TI-84 Plus calculator screenshot showing the definition of  $g(x) := -a \cdot \sqrt{x} \cdot (a \cdot x - 4)$  and the solve function  $\text{solve}\left(g\left(\frac{4}{3 \cdot a}\right) < \frac{4}{3 \cdot a}, a\right)$ . The result is  $0 < a < 0.572357$ .

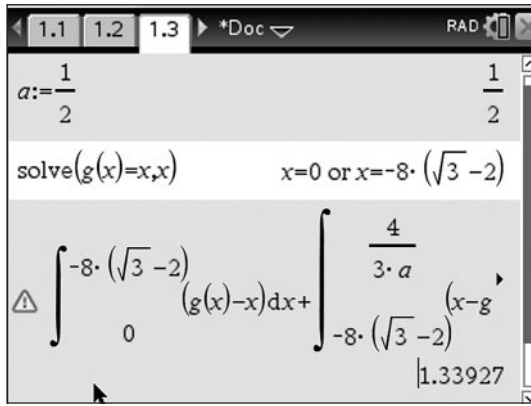
$$g\left(\frac{4}{3a}\right) < \frac{4}{3a}$$

M1

$$0 < a < 0.57$$

A1

ii.



$$g(x) = x \Rightarrow x = 16 - 8\sqrt{3}$$

M1

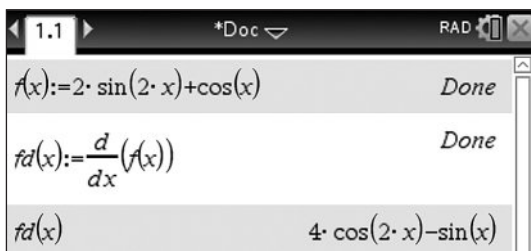
$$\int_0^{16-8\sqrt{3}} (g(x) - x) dx + \int_{16-8\sqrt{3}}^{\frac{8}{3}} (x - g(x)) dx = 1.34$$

M1 A1

Note: Award the second M1 for one correct definite integral.

**Question 5** (10 marks)

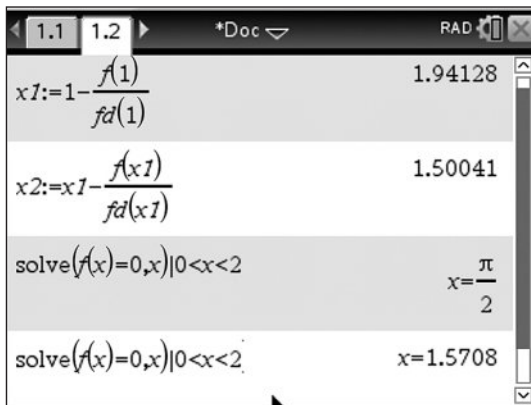
a.



Hence,  $f'(x) = 4\cos(2x) - \sin(x)$ .

A1

b.



$$x_1 = 1 - \frac{f(1)}{f'(1)} = 1.9412 \dots$$

M1

$$x_2 = 1 - \frac{f(x_1)}{f'(x_1)} = 1.5004$$

A1

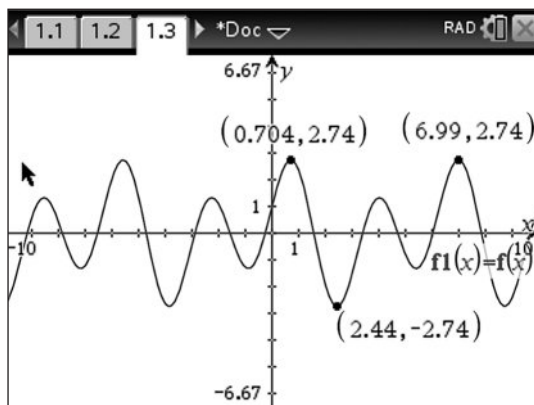
$$f(x) = 0 \Rightarrow \text{closest root is } x = \frac{\pi}{2} \approx 1.5708 > x_2$$

A1

c. i. period =  $LCM(2\pi, \pi) = 2\pi$

A1

ii.



$$-2.74 \leq y \leq 2.74$$

A1

d. i. Observation and trial and error give:

$$a = -1$$

A1

$$b = -\frac{1}{2}$$

A1

ii.  $f\left(\frac{x}{2} - \pi\right) = 2\sin\left(2\left(\frac{x}{2} - \pi\right)\right) + \cos\left(\frac{x}{2} - \pi\right)$

$$= 2\sin(x - 2\pi) + \cos\left(\pi - \frac{x}{2}\right)$$

M1

$$= 2\sin(x) - \cos\left(\frac{x}{2}\right)$$

$$= -\left(-2\sin(x) + \cos\left(\frac{x}{2}\right)\right)$$

$$= -\left(2\sin(-x) + \cos\left(-\frac{x}{2}\right)\right)$$

M1

$$= -f\left(-\frac{x}{2}\right)$$