The Mathematical Association of Victoria

Trial Examination 2023

MATHEMATICAL METHODS

Trial Written Examination 2 - SOLUTIONS

SECTION A: Multiple Choice

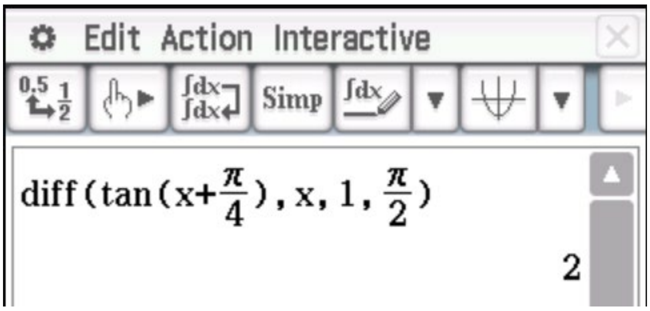
Question	Answer	Question	Answer
1	A	**11**	C
2	D	**12**	E
3	B	**13**	D
4	C	**14**	B
5	C	**15**	A
6	E	**16**	A
7	E	**17**	D
8	D	**18**	C
9	A	**19**	E
10	B	**20**	B

Question 1 **Answer A**

Gradient of $y = \tan\left(x + \frac{\pi}{4}\right)$ at $x = \frac{\pi}{2}$ is 2 using technology

OR

$\frac{dy}{dx} = \sec^2\left(x + \frac{\pi}{4}\right)$ at $x = \frac{\pi}{2}$, $\frac{dy}{dx} = 2$



Question 2 **Answer D**

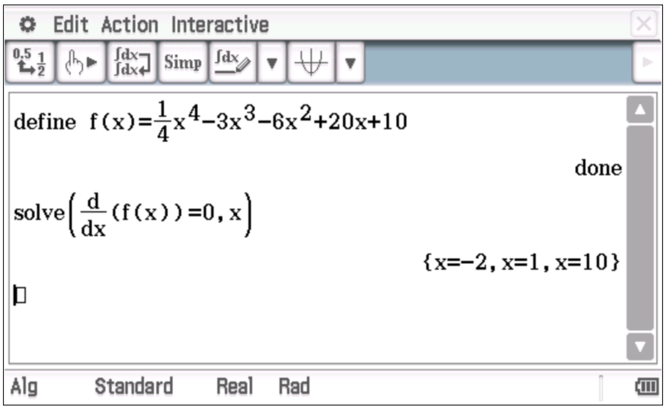
Period of $g(x) = -3\sin\left(\frac{\pi}{8}x + 1\right)$

Period = $\dfrac{2\pi}{\frac{\pi}{8}} = 16$

Question 3 **Answer B**

$f:(-\infty,a]\to R, f(x)=\frac{1}{4}x^4-3x^3-6x^2+20x+10$

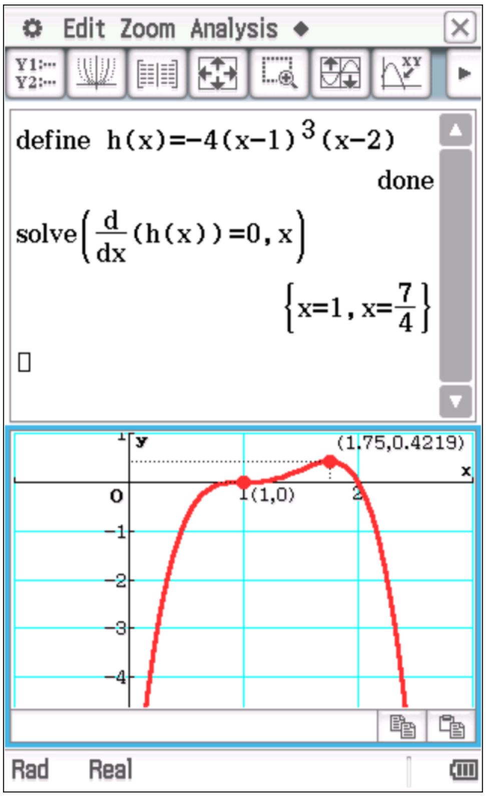
f^{-1} exists when original function f is one-to-one.
Stationary points are at $x=-2$, $x=1$ and $x=10$
Closest stationary point from $-\infty$ is at $x=-2$
So $a=-2$



Question 4 **Answer C**

$h(x)=-4(x-1)^3(x-2)$ has stationary points at $x=1$ and $x=\frac{7}{4}$

Strictly increasing for $\left(-\infty,\frac{7}{4}\right]$



Question 5 **Answer C**

$2x+ky=3$

$-3x-y=m$

Using ratios, gradients will be the same when

$-\frac{2}{3}=-k$

$k=\frac{2}{3}$

y-intercepts will be the same when

$-\frac{2}{3}=\frac{3}{m}$

$-2m=9$

$m=-\frac{9}{2}$

OR

Using $y=mx+c$

$y=\frac{-2}{k}x+\frac{3}{k}$

$y=-3x-m$

Gradients will be the same when

$-\frac{2}{k}=-3$

$k=\frac{2}{3}$

y-intercepts will be the same when

$\frac{3}{k}=-m$

$\frac{3}{\frac{2}{3}}=-m$

$m=-\frac{9}{2}$

Question 6 **Answer E**

$f:[0,\infty)\to R, f(x)=\sqrt{x}$

Translate 2 units to the right and 3 units down

$f_1:[2,\infty)\to R, f_1(x)=\sqrt{x-2}-3$

Dilate by a factor of 4 from the x-axis

$g:[2,\infty)\to R, g(x)=4\sqrt{x-2}-12$

Question 7 **Answer E**

Given $\int_a^5 f(x)dx = 3$ and $\int_5^b f(x)dx = -4$ where $a < 5 < b$

$\int_a^b \left(2f(x)+1\right)dx = \int_a^5 \left(2f(x)+1\right)dx + \int_5^b \left(2f(x)+1\right)dx$

$= \int_a^5 \left(2f(x)\right)dx + \int_a^5 (1)\,dx + \int_5^b \left(2f(x)\right)dx + \int_5^b (1)\,dx$

$= 2\int_a^5 \left(f(x)\right)dx + 2\int_5^b \left(f(x)\right)dx + \int_a^b (1)\,dx$

$= 2\times 3 + 2\times -4 + b - a$

$= -2 + b - a$

$= -2 - a + b$

Question 8 **Answer D**

The points are $(0,5)$ and $\left(x,(x-2)^2\right)$.

$d = \sqrt{x^2 + \left((x-2)^2 - 5\right)^2}$

Solve $\dfrac{d}{dx}\sqrt{x^2 + \left((x-2)^2 - 5\right)^2} = 0$ for x.

$x = \dfrac{\pm\sqrt{6}+2}{2}$, $x = 4$

Minimum occurs when $x = \dfrac{-\sqrt{6}+2}{2}$.

OR

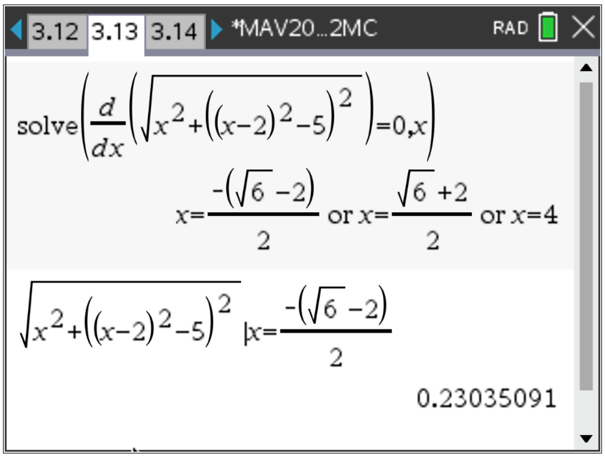
$f'(x) = 2(x-2)$

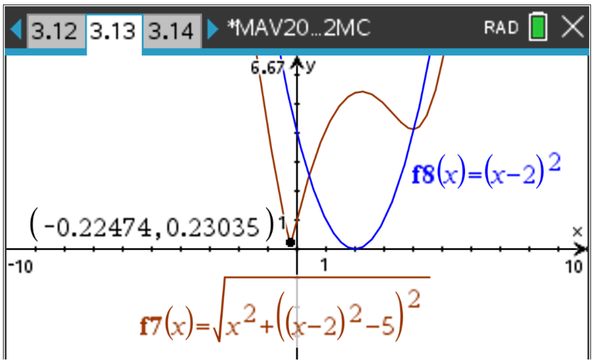
$y = -\dfrac{x}{2(x-2)} + 5$.

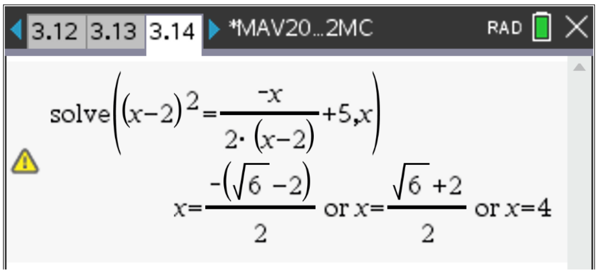
Solve $(x-2)^2 = -\dfrac{x}{2(x-2)} + 5$

$x = \dfrac{\pm\sqrt{6}+2}{2}$, $x = 4$

Minimum occurs when $x = \dfrac{-\sqrt{6}+2}{2}$.







Question 9 **Answer A**

$g: R\backslash\{1\} \rightarrow R, g(x) = \dfrac{1}{(x-1)^2}$

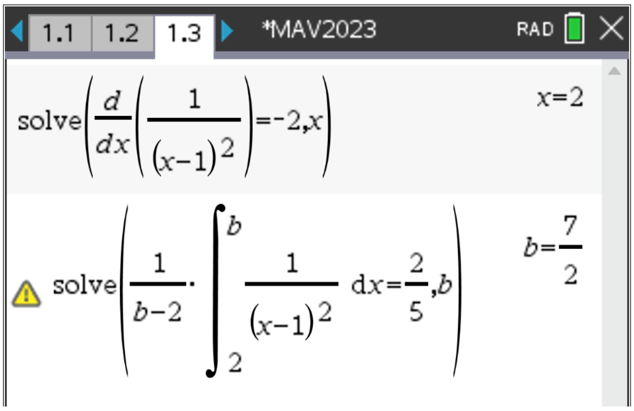
Average value $= \dfrac{1}{b-a}\displaystyle\int_a^b \left(\dfrac{1}{(x-1)^2}\right)dx = \dfrac{2}{5}$

Solve $\dfrac{d}{dx}\dfrac{1}{(x-1)^2} = -2$ for x

$a = 2$

Solve $\dfrac{1}{b-2}\displaystyle\int_2^b \left(\dfrac{1}{(x-1)^2}\right)dx = \dfrac{2}{5}$ for b

$b = \dfrac{7}{2}$



Question 10 **Answer B**

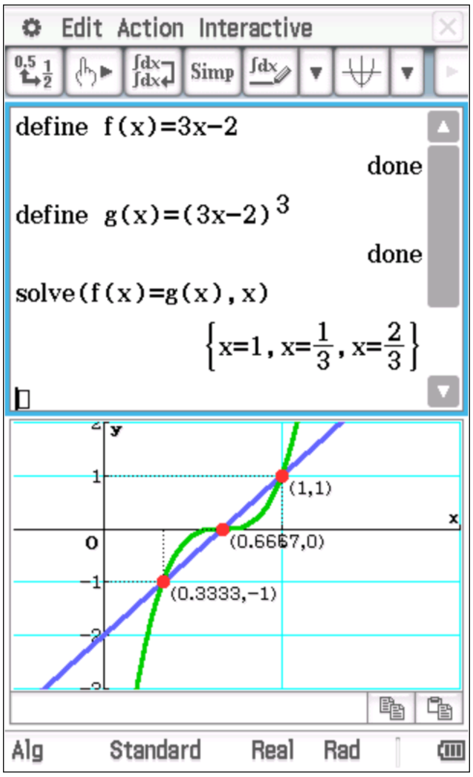
Intersection points between graphs of $f(x) = 3x - 2$ and $g(x) = (3x-2)^3$ are at $x = \frac{1}{3}, x = \frac{2}{3}$ and $x = 1$

Area found by $\int_{\frac{1}{3}}^{\frac{2}{3}} (\text{cubic} - \text{linear})\,dx + \int_{\frac{2}{3}}^{1} (\text{linear} - \text{cubic})\,dx$

Area = $\int_{\frac{1}{3}}^{\frac{2}{3}} (g(x) - f(x))\,dx + \int_{\frac{2}{3}}^{1} (f(x) - g(x))\,dx$

As $\int_{\frac{1}{3}}^{\frac{2}{3}} (g(x) - f(x))\,dx = \int_{\frac{2}{3}}^{1} (f(x) - g(x))\,dx$

Area = $2\int_{\frac{2}{3}}^{1} (f(x) - g(x))\,dx = 2\int_{1}^{\frac{2}{3}} (g(x) - f(x))\,dx$



Question 11 **Answer C**

$y+z=2$

$-2x-3y=8$

Let $y=\lambda$

$\lambda+z=2$

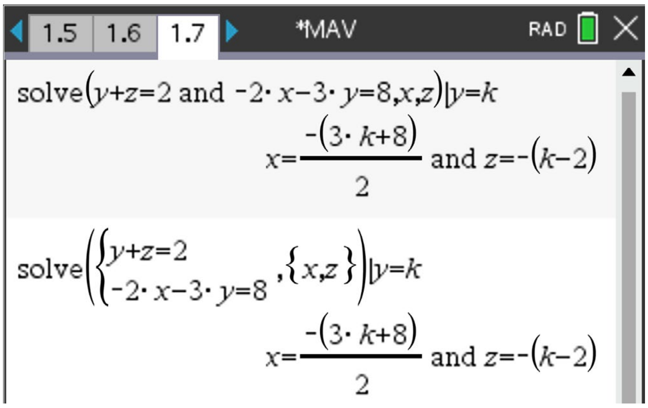
$z=2-\lambda$

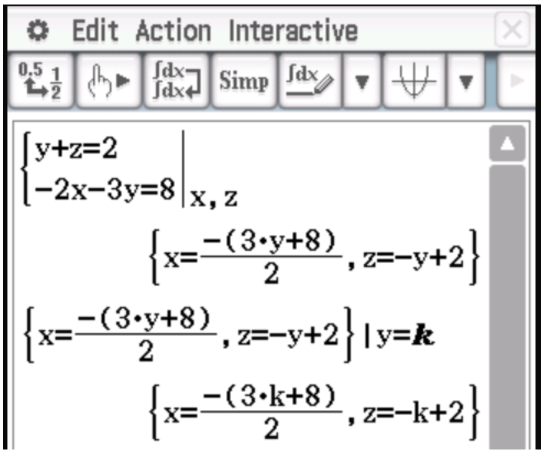
$-2x-3\lambda=8$

$-2x=8+3\lambda$

$x=-\dfrac{8+3\lambda}{2}$

$y=\lambda$, $x=-\dfrac{3\lambda+8}{2}$, $z=-\lambda+2$ where $\lambda\in R$





Question 12 **Answer E**

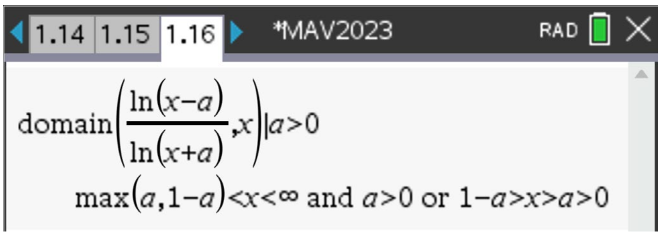
$h(x)=\dfrac{\log_e(x-a)}{\log_e(x+a)}$, $a>0$

Maximal domain for the intersection of:

Numerator: $x-a>0 \ \therefore\ x>a$

Denominator: $x+a>0 \ \therefore\ x>-a$ and $\log_e(x+a)\neq 0$

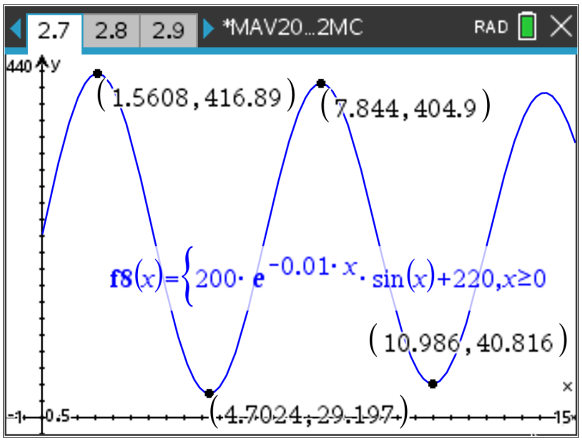
Giving (a,∞)

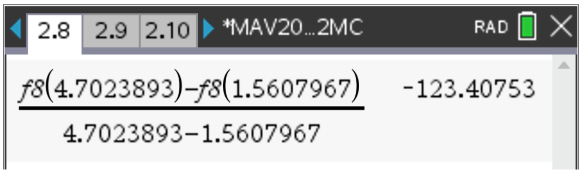


Question 13 **Answer D**

Average rate of change $=\dfrac{s(4.70...)-s(1.56...)}{4.70...-1.56...}$

$=-123.408$ correct to three decimal places

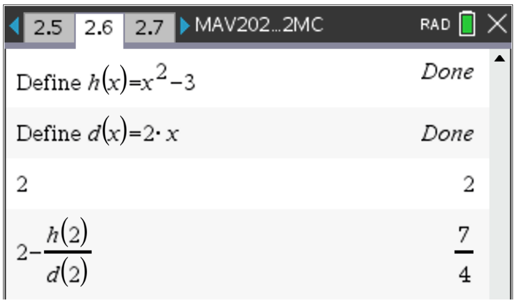


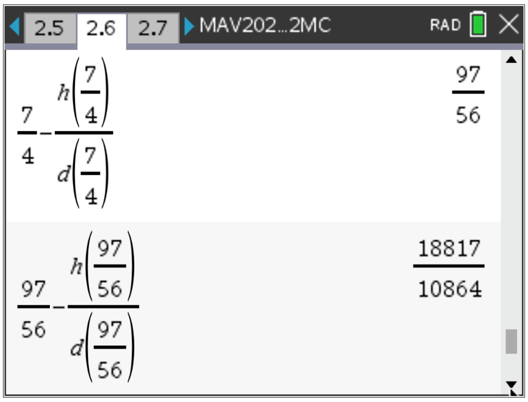


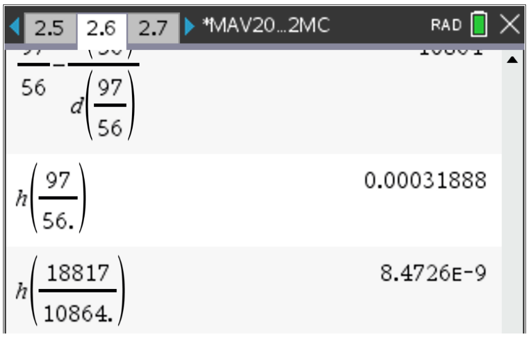
Question 14 **Answer B**

The program will stop when $h(x) < -0.0001$ or $h(x) > 0.0001$.

Iteration	x	$h(x)$
	2	1
1	1.75	0.0625
2	$\frac{97}{56} = 1.7321...$	0.0003…
3	$\frac{18817}{10864} = 1.7320...$	$8.4... \times 10^{-9}$







Question 15 **Answer A**

Area $= \frac{b-a}{2n}\left(f(1)+2f(2)+2f(3)+f(4)\right)$

$= \frac{4-1}{6}\left(0+2f(2)+2f(3)+f(4)\right)$

$= \frac{1}{2}\left(2f(2)+2f(3)+f(4)\right)$

$= f(2)+f(3)+\frac{1}{2}f(4)$

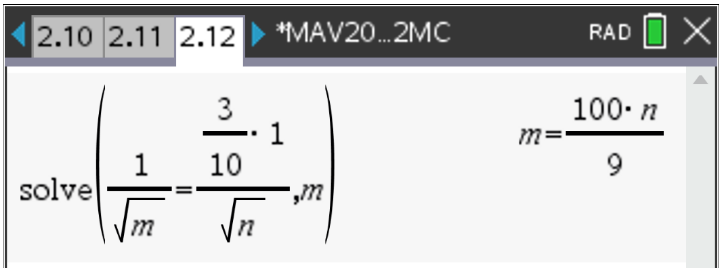
Question 16 **Answer A**

$W_1 = 2z\frac{\sigma}{\sqrt{n_1}}$, $W_2 = 2z\frac{\sigma}{\sqrt{n_2}}$

$W_2 = 2z\frac{\sigma}{\sqrt{n_2}} = 0.3\times 2z\frac{\sigma}{\sqrt{n_1}}$

$\frac{1}{\sqrt{n_2}} = 0.3\frac{1}{\sqrt{n_1}}$

$n_2 = \frac{100}{9}n_1$

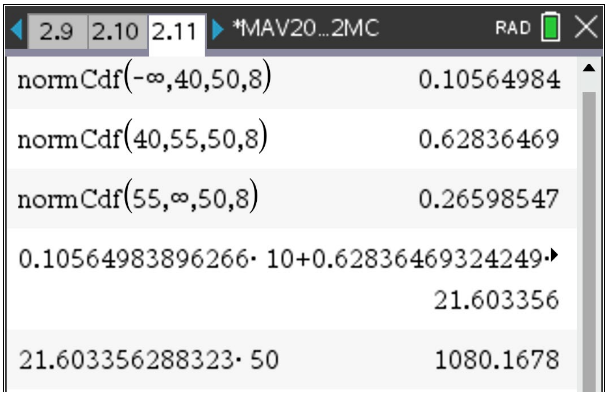


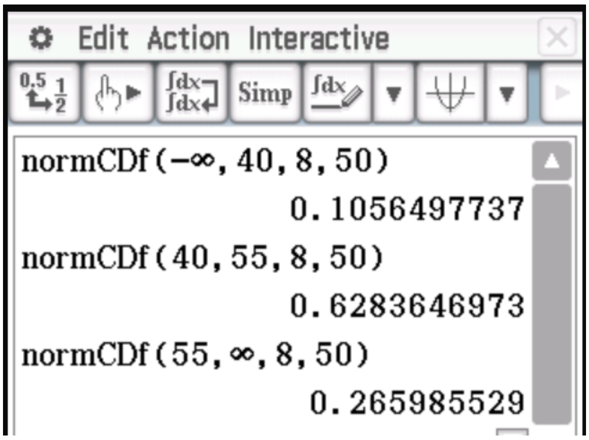
Question 17 **Answer D**

Height (cm)	less than 40 cm	from 40 cm to 55 cm	greater than 55 cm
Proportion	0.105…	0.628…	0.265…
Cost ($)	10	20	30

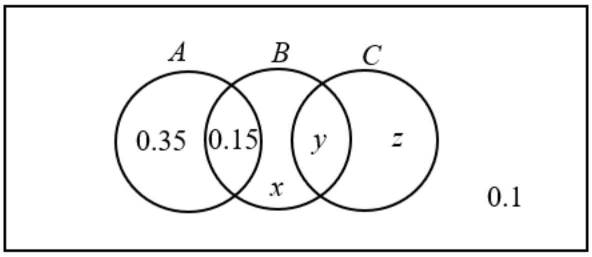
Cost $=\left(0.105...\times 10+0.628...\times 20+0.265...\times 30\right)\times 50$

= $1080.17





Question 18 **Answer C**



$\Pr(A) = 0.5$, $\Pr(B) = 0.3$ and $\Pr(C) = 0.35$

$\Pr(A \cap B) = \Pr(A) \times \Pr(B) = 0.5 \times 0.3 = 0.15$ independent events

$x + y = 0.15$

$0.5 + 0.15 + z + 0.1 = 1$

$z = 0.25$

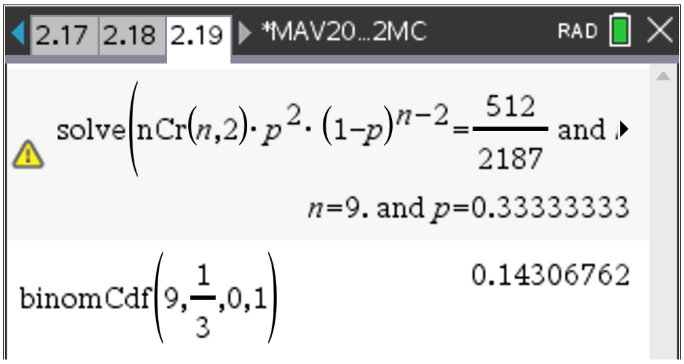
$y = \Pr(B \cap C) = 0.1$

Question 19 **Answer E**

Solve $np(1-p) = 2$ and $\binom{n}{2} p^2 (1-p)^{n-2} = \dfrac{512}{2187}$

$n = 9$ and $p = \dfrac{1}{3}$

$\Pr(X < 2) = \Pr(X \le 1) = 0.1431$ correct to four decimal places



Question 20 **Answer B**

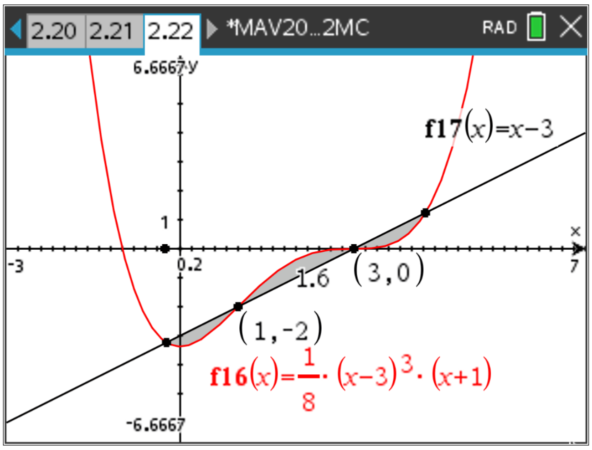
$$h(x)=\frac{1}{8}x^4-x^3+\frac{9}{4}x^2-\frac{27}{8}=\frac{1}{8}(x-3)^3(x+1)$$

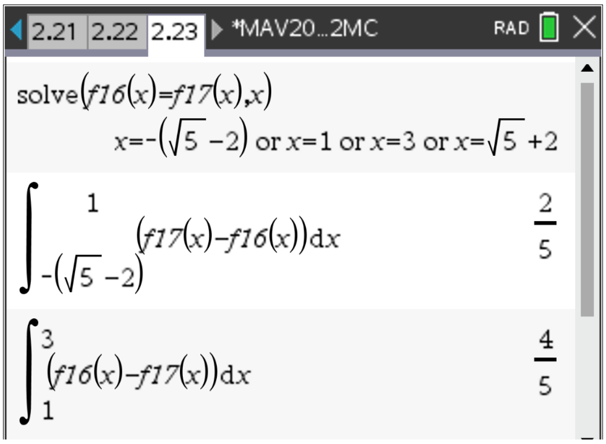
Points of inflection are at $(1,-2)$ and $(3,0)$.

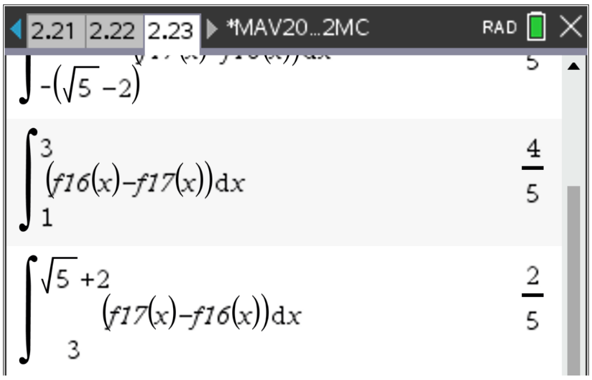
The equation of the line passing through these points is $y=x-3$.

$$\text{Area } = \int_{-\sqrt{5}+2}^{1}\left(x-3-h(x)\right)dx+\int_{1}^{3}\left(h(x)-(x-3)\right)dx+\int_{3}^{\sqrt{5}+2}\left(x-3-h(x)\right)dx$$

= 1.6







END OF SECTION A SOLUTIONS

SECTION B

Question 1

$$f(x)=\begin{cases} k & 0 \le x \le 100 \\ 20\cos\left(\dfrac{\pi(x-100)}{200}\right)+10 & 100 < x \le 400 \end{cases}$$

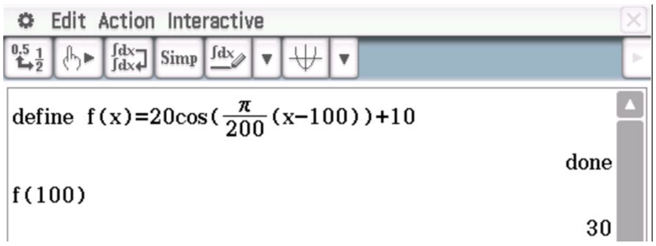
a. $f(x)$ is **continuous** at $x = 100$.

At the joining point

$f(100) = 20\cos\left(\dfrac{\pi(100-100)}{200}\right)+10$

$= 20\cos(0)+10$

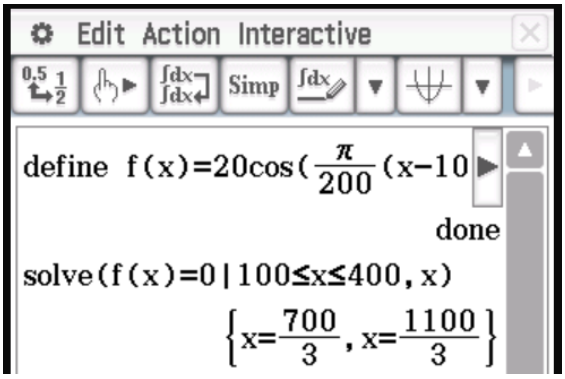
$k = 30$ **1A**



b. Solve $f(x) = 0$ to find the section of graph that runs under ground level.

Gives $x = \dfrac{700}{3}, x = \dfrac{1100}{3}$

Answer $x \in \left(\dfrac{700}{3}, \dfrac{1100}{3}\right)$ **1A**



c. Smooth at $x = 100$ requires same gradient as $x \to 100^-$ and $x \to 100^+$

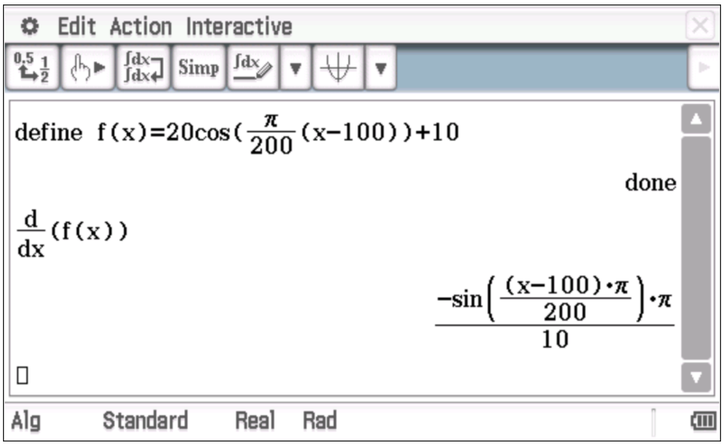
As $x \to 100^-$, $y = 30$ giving gradient equal to zero.

As $x \to 100^+$,

$$f'(x) = -\frac{\pi}{10}\sin\left(\frac{\pi(x-100)}{200}\right)$$

$$f'(100) = -\frac{\pi}{10}\sin\left(\frac{\pi(100-100)}{200}\right) = 0$$ giving gradient equal to zero.

Smooth at $x = 100$ **1M**



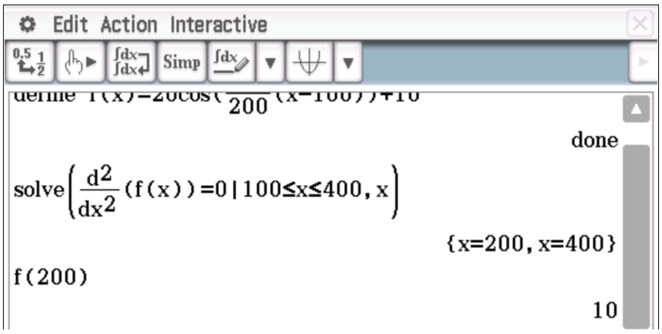
d.i. Points of inflection at $f''(x) = 0$

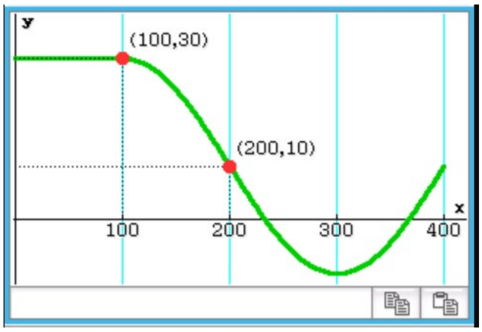
gives $x = 200$

(Note $f'(x)$ and $f''(x)$ do not exist at $x = 400$)

Point of inflection at $x = 200$ with concavity of graph changing either side of $x = 200$

Coordinates of point of inflection $(200,10)$ **1A**

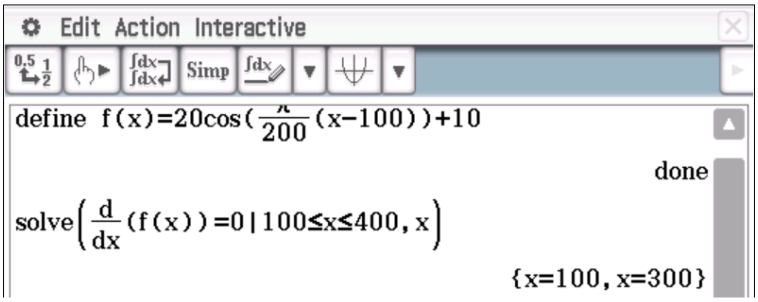




d.ii. Stationary points at $f'(x)=0$

gives $x=100,\ x=300$

Strictly decreasing for $x\in[100,300]$ **1A**



e. Three supports of equal widths begin at $x=100$ and end at $x=\frac{700}{3}$

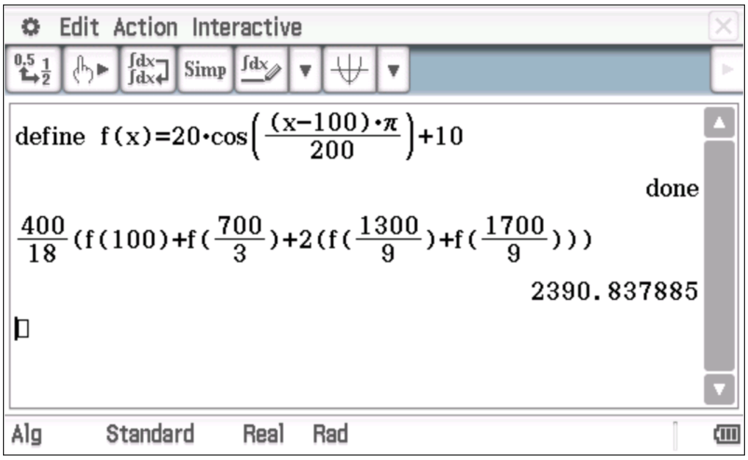
Width of each section $=\dfrac{\frac{700}{3}-100}{3}=\dfrac{400}{9}=44\tfrac{4}{9}$ **1A**

Edges of sections: $x=100,\ x=\frac{1300}{9},\ x=\frac{1700}{9},\ x=\frac{700}{3}$

Area of the cross-sections $=\dfrac{\frac{400}{9}}{2}\left(f(100)+f\left(\frac{700}{3}\right)+2\left(f\left(\frac{1300}{9}\right)+f\left(\frac{1700}{9}\right)\right)\right)$ **1M**

$=\dfrac{\frac{400}{9}}{2}\left(f(100)+2\left(f\left(\frac{1300}{9}\right)+f\left(\frac{1700}{9}\right)\right)\right)$ as $f\left(\frac{700}{3}\right)=0$

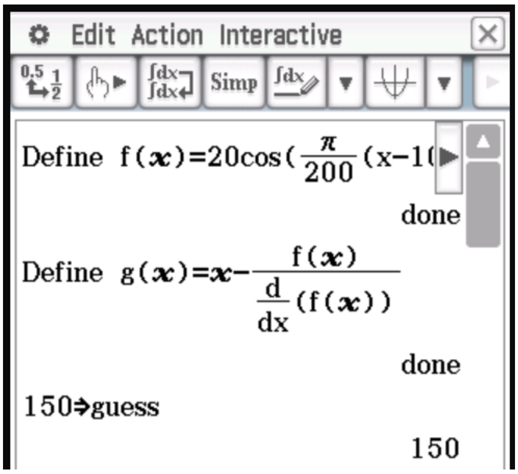
$=2390.8$ square metres correct to one decimal place **1A**

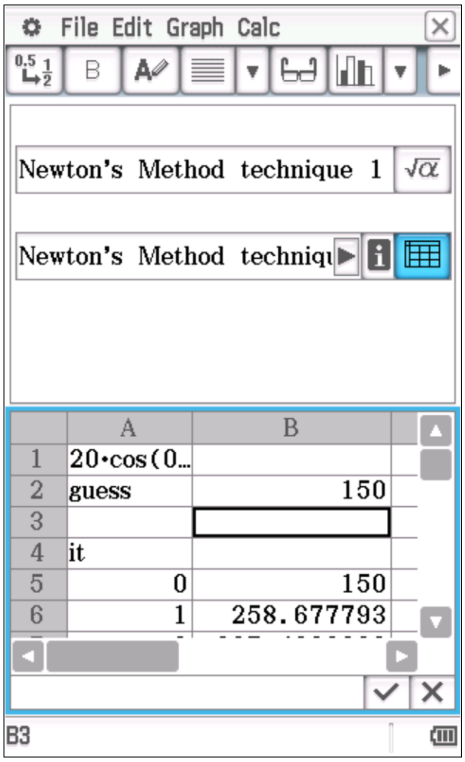


f.i. Newtons method, $x_0=150$

$x_1=150-\dfrac{f(150)}{f'(150)}$ **1M**

$x_1=258.678$ **1A**





f.ii. Newtons method, $x_0 = 150$

$x_3 = 233.1994015...$

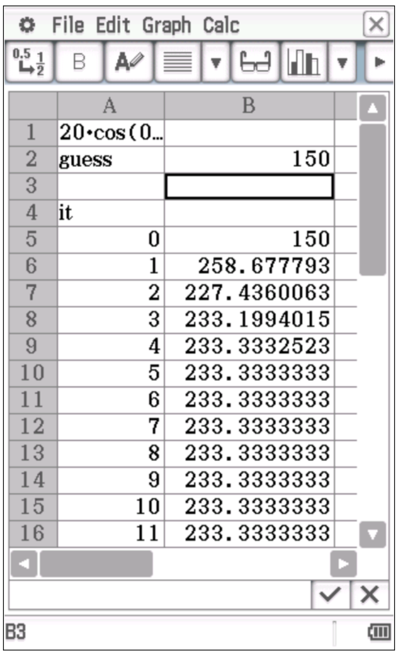
Solving $f(x) = 0$ gives x-intercepts of $x = \dfrac{700}{3}$ and $x = \dfrac{1100}{3}$

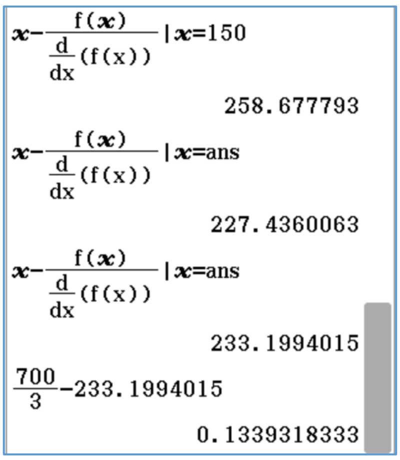
Near x-intercept $x = \dfrac{700}{3} = 233.333333.......$

Horizontal distance $233.333333... - 233.1994015...$ **1M**

Distance $= 0.133932...$

$= 0.1339$ metres correct to four decimal places **1A**





Question 2

$f(x)=\dfrac{p}{x-20}-40$ and $g(x)=\dfrac{q}{x+30}+10$

a. Graph of f has x-intercept at $x=25$.

$f(25)=0$

Gives $0=\dfrac{p}{25-20}-40 \Rightarrow \dfrac{p}{5}=40$

Shown $p=200$. **1M**

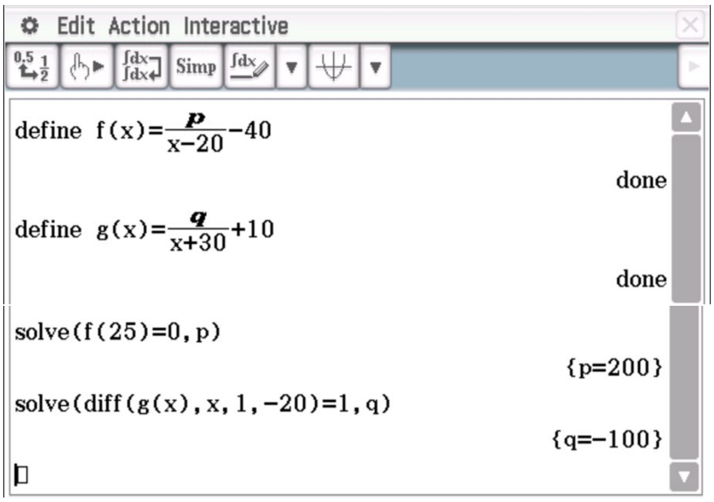
b.i. Graph of g has a gradient of 1 at $x=-20$.

$g'(x)=-\dfrac{q}{(x+30)^2}$

$g'(-20)=1$

Gives $1=-\dfrac{q}{(-20+30)^2} \Rightarrow 1=-\dfrac{q}{100}$

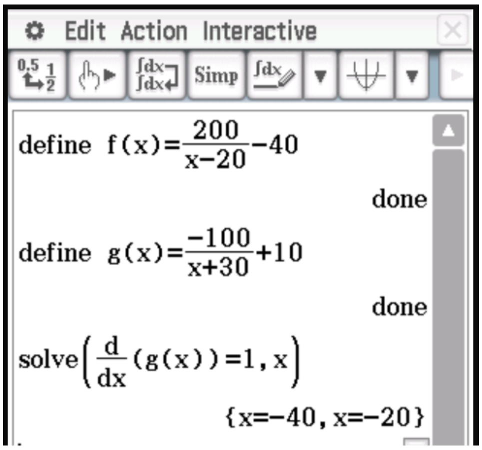
Shown $q=-100$. **1M**



b.ii. Solve $g'(x)=1$

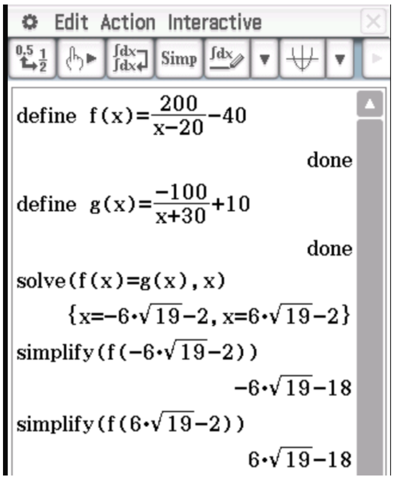
$x=-40,\ x=-20$

Other value $x=-40$ **1A**

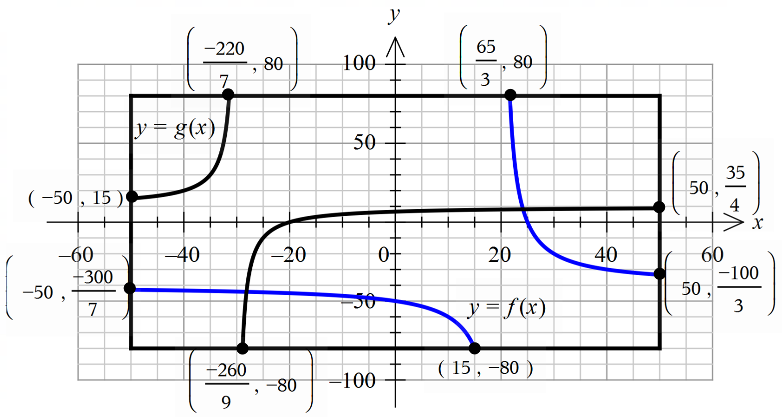


c.i. Points of intersection between f and g at $x=-2\pm 6\sqrt{19}$

$\left(-6\sqrt{19}-2,-6\sqrt{19}-18\right)$ and $\left(6\sqrt{19}-2,6\sqrt{19}-18\right)$ **1A**



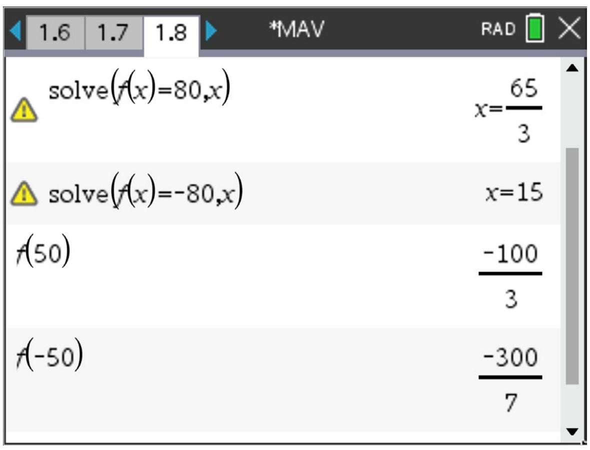
c.ii. Shape **1A**
Endpoints **1A**



Solve $f(x)=-80$ gives $x=15$

Solve $f(x)=80$ gives $x=\dfrac{65}{3}$

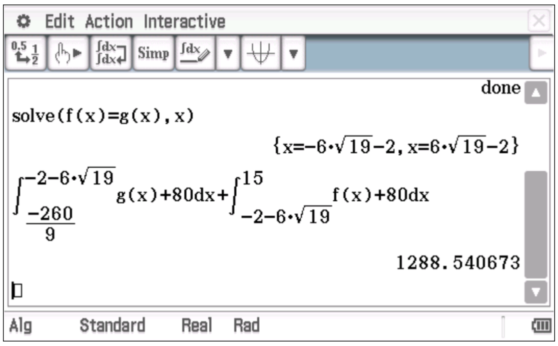
$f(-50)=-\dfrac{300}{7}$, $f(50)=-\dfrac{100}{3}$

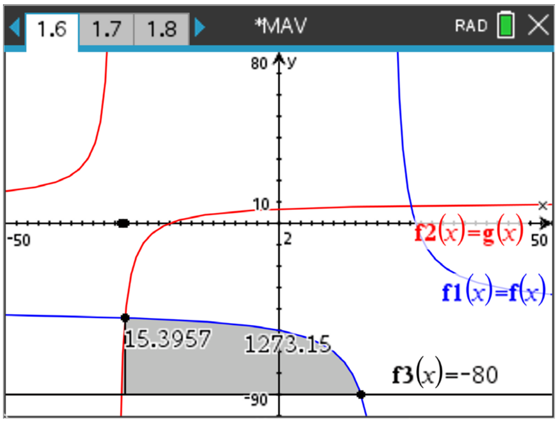


d.i. Area $= \displaystyle\int_{\frac{-260}{9}}^{-2-6\sqrt{19}} \big(g(x)-(-80)\big)\,dx + \int_{-2-6\sqrt{19}}^{15} \big(f(x)-(-80)\big)\,dx$ **1A** correct terminals

$= \displaystyle\int_{\frac{-260}{9}}^{-2-6\sqrt{19}} \big(g(x)+80\big)\,dx + \int_{-2-6\sqrt{19}}^{15} \big(f(x)+80\big)\,dx$ **1A** correct functions

d.ii. Area =1289 sq km **1A**





d.iii. Area of the crop = 1288.54… sq km

Area of farmland defined by the lines $x=\pm 50$ and $y=\pm 80$

Area of the farm = $100\times 160 = 16\ 000$ sq km

$\dfrac{1288.54...}{16\ 000}\times 100\% = 8.05...\%$

= 8% to the nearest percentage **1H**

e. Tangent to $f(x)$ at $x=0$

$y=-\dfrac{x}{2}-50$

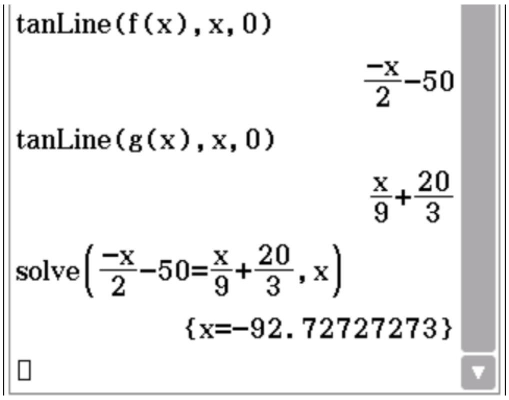
Tangent to $g(x)$ at $x=0$

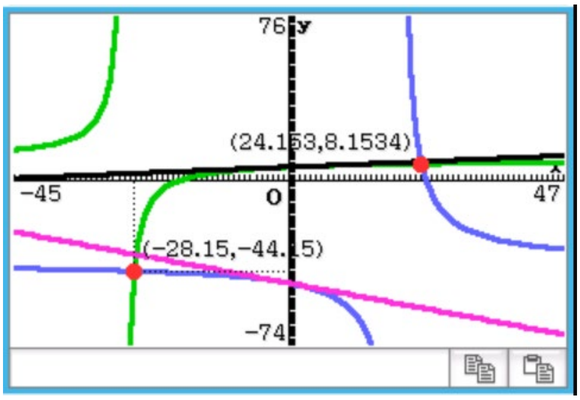
$y=\dfrac{x}{9}+\dfrac{20}{3}$

Equate tangents give point of intersection at $x=-92.7272...$ **1M**

Outside the of domain of $x\in[-50,50]$

Water pipes do not meet on his property **1A**



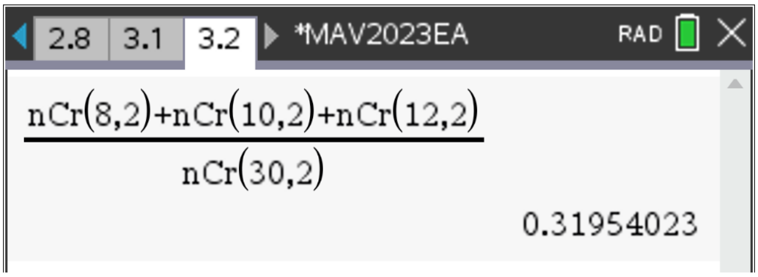


Question 3

a. Pr(2 black socks) + Pr(2 green socks) + Pr(2 blue socks)

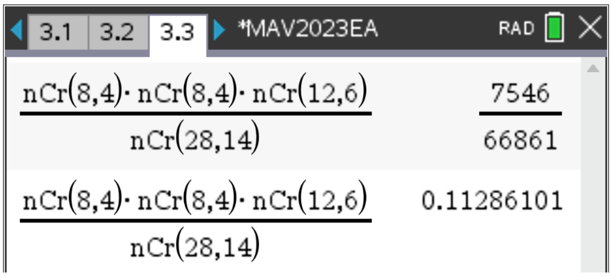
$$= \frac{\binom{8}{2}+\binom{10}{2}+\binom{12}{2}}{\binom{30}{2}}$$ **1M**

= 0.3195 **1A**



b. $\frac{\binom{8}{4}\binom{8}{4}\binom{12}{6}}{\binom{28}{14}}$

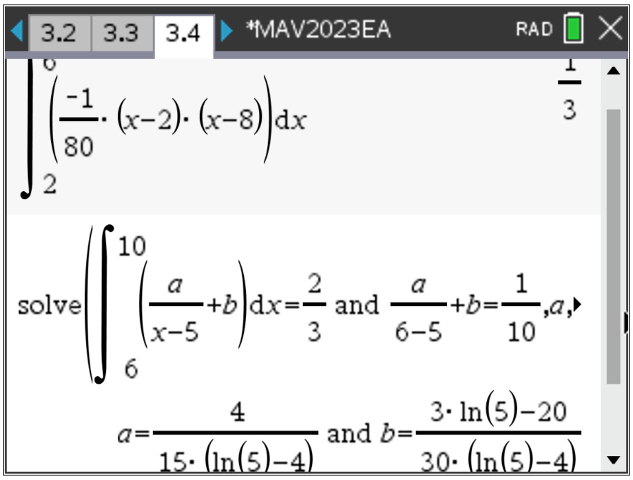
= 0.1129 **1A**

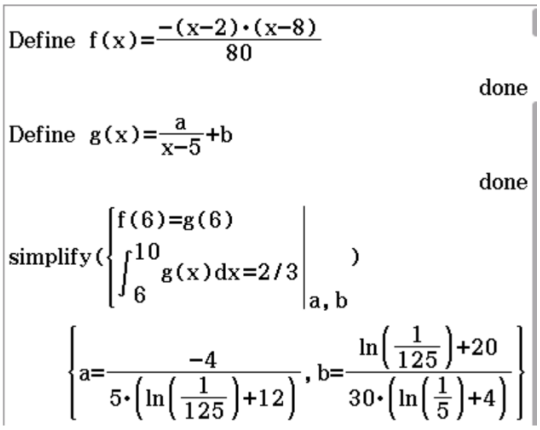


c. $\int_2^6 \left(-\frac{1}{80}(x-2)(x-8)\right)dx = \frac{1}{3}$, $-\frac{1}{80}(6-2)(6-8) = \frac{1}{10}$

Solve $\int_6^{10} \left(\frac{a}{x-5}+b\right)dx = \frac{2}{3}$ and $\frac{a}{6-5}+b = \frac{1}{10}$ for a and b **1M**

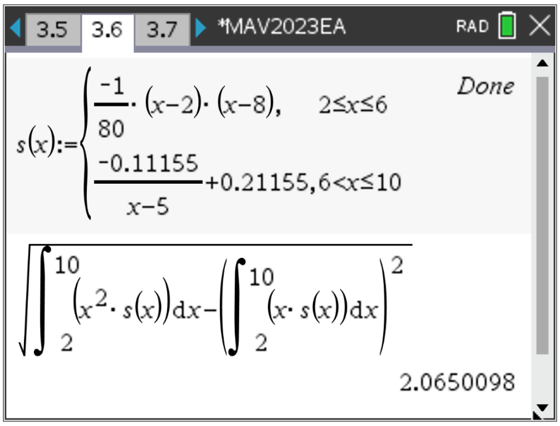
$a = \frac{4}{15(\log_e(5)-4)} = \frac{-4}{5\left(\log_e\left(\frac{1}{125}\right)+12\right)}$ and $b = \frac{3\log_e(5)-20}{30(\log_e(5)-4)} = \frac{\log_e\left(\frac{1}{125}\right)+20}{30\left(\log_e\left(\frac{1}{5}\right)+4\right)}$ **1A**





d.i. $\text{sd}(X) = \sqrt{\int_2^{10}\left(x^2 \times s(x)\right)dx - \left(\int_2^{10}\left(x \times s(x)\right)dx\right)^2}$ **1M**

$= 2.065$ **1A**

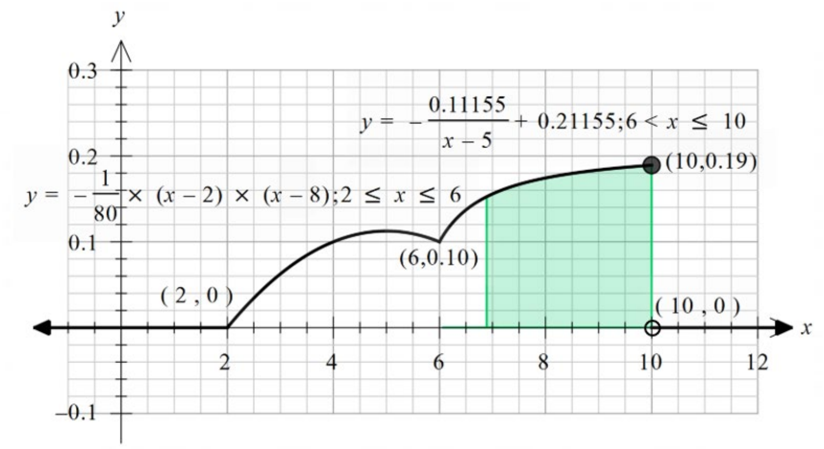


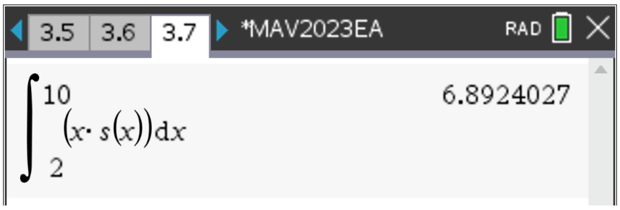
d.ii. Coordinates with open and closed circles **1A**

Shape (must draw along axis) **1A**

Shading $\text{E}(X) \approx 6.892$ **1H**

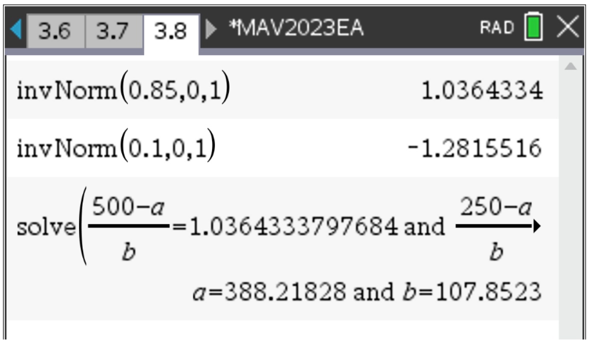
There is no need to show the equations.

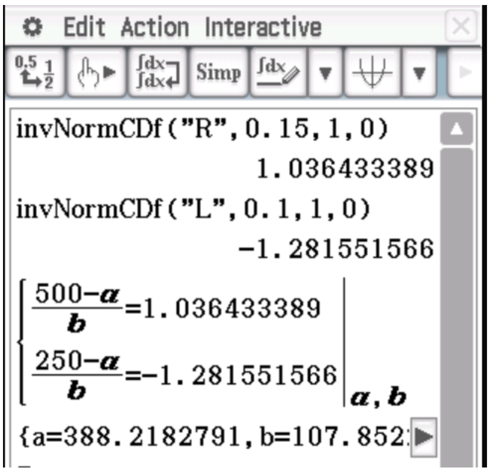




e. Solve $\frac{500-\mu}{\sigma} = 1.036...$ and $\frac{250-\mu}{\sigma} = -1.281...$ **1M**

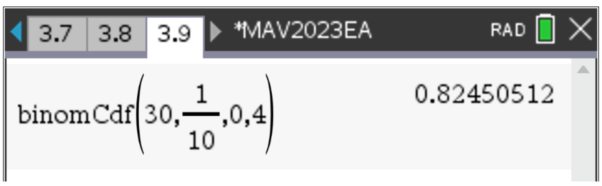
$\mu = 388$ mm and $\sigma = 108$ mm **1A**

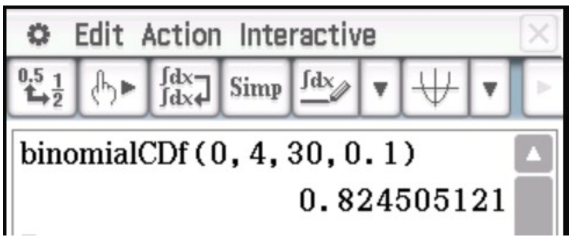




f. $L \sim \text{Bi}\left(30, \frac{1}{10}\right)$ **1M**

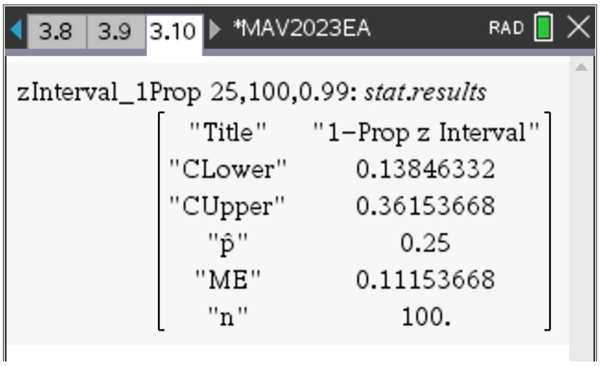
$\Pr(L < 5) = 0.8245$ **1A**





g. (0.1385, 0.3615) **1A**

0.04 is outside the confidence interval. If Maya did 100 such samples, she would expect 99 of the confidence intervals to contain p. It is highly likely the farmer is incorrect but further statistical testing needs to be carried out. **1M**

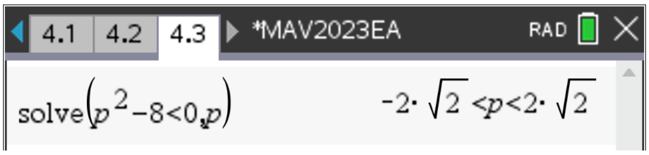


Question 4

a. $g:(-\infty,\infty)\to R, g(x)=\log_e(x^2+px+2)$

$\Delta = p^2 - 8 < 0$ **1M**

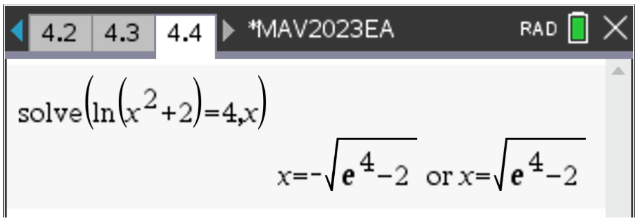
$-2\sqrt{2} < p < 2\sqrt{2}$ **1A**



b. g^{-1} will not exist, as g will be a 'many to one function' or g 'fails the horizontal line test' or 'there exist two x-values for some y-values'. **1A**

c. Solve $\log_e(x^2+2)=4$

$a=-\sqrt{e^4-2}$ and $b=\sqrt{e^4-2}$ **1A**



d. h is symmetrical about the y-axis.

Average value $=\dfrac{1}{\sqrt{e^4-2}}\displaystyle\int_0^{\sqrt{e^4-2}}\left(\log_e\left(x^2+2\right)\right)dx = 2.537...$ **1H**

Sketch the graphs of $y=2.537...$ and h and find the bounded area.

Area = 6.480… **1M**

Volume = $2\times 6.480...$

$=12.96\ \text{dm}^3$ **1A**

OR

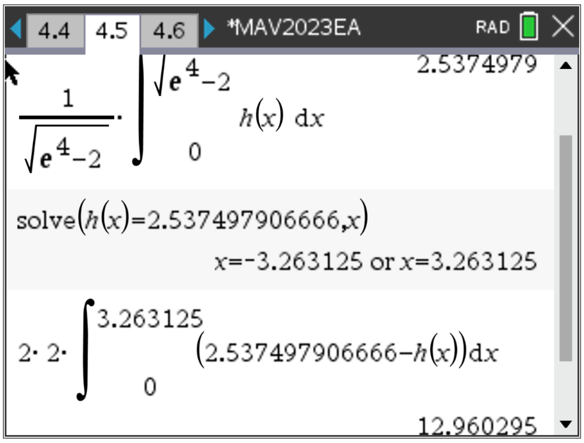
Solve $h(x) = 2.537...$ for x.

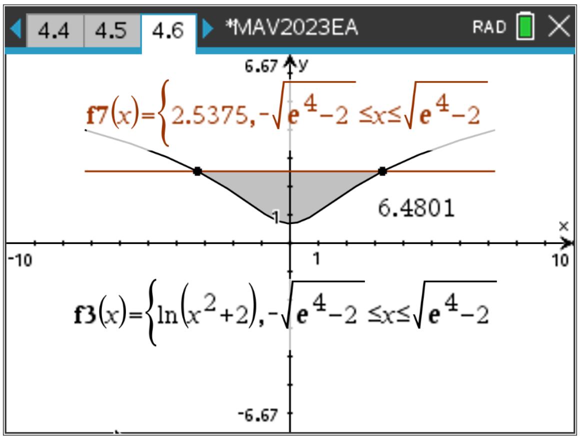
$x = -3.263...$ or $x = 3.263...$

Volume = length $\times$ area bounded by $y = 2.537...$ and $h(x)$

Volume $= 2 \times 2 \times \int_{0}^{3.263...} \left(2.537... - h(x)\right)dx$ **1M**

$= 12.96\ \text{dm}^3$ **1A**



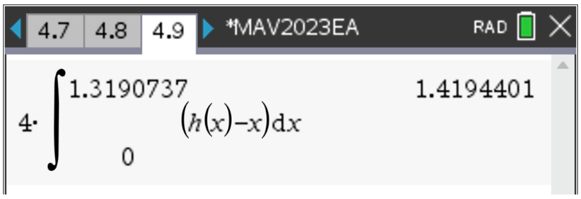


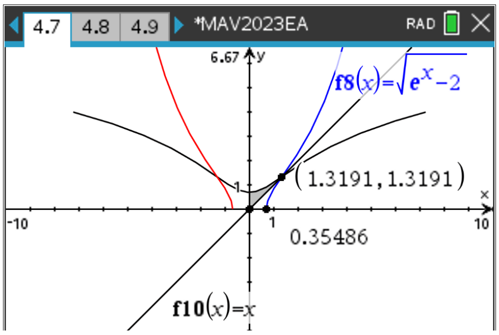
e. Sketch h and $y = x$ and find the bounded area between the y-axis, h and $y = x$. **1M**

$\int_{0}^{1.319...} \left(h(x) - x\right)dx = 0.3548...$ **1A**

$4\int_{0}^{1.319...} \left(h(x) - x\right)dx$

$= 1.42\ \text{dm}^2$ **1H**





OR

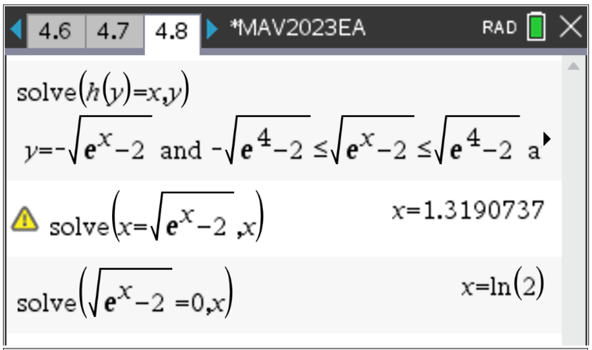
The equation of RHS branch is $y=\sqrt{e^x-2}$ (part of the inverse relation of h) **1A**

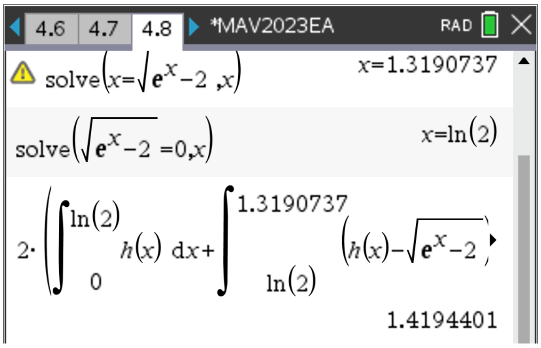
Solve $\sqrt{e^x-2}=0$, $x=\log_e(2)$

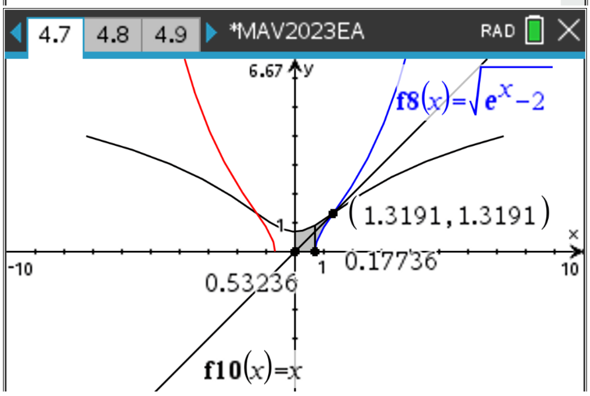
Solve $\sqrt{e^x-2}=x=h(x)$, $x=1.319...$

$$2\left(\int_0^{\log_e(2)}\left(h(x)\right)dx+\int_{\log_e(2)}^{1.319...}\left(h(x)-\sqrt{e^x-2}\right)dx\right)$$ **1M**

$=1.42\ \text{dm}^2$ **1A**



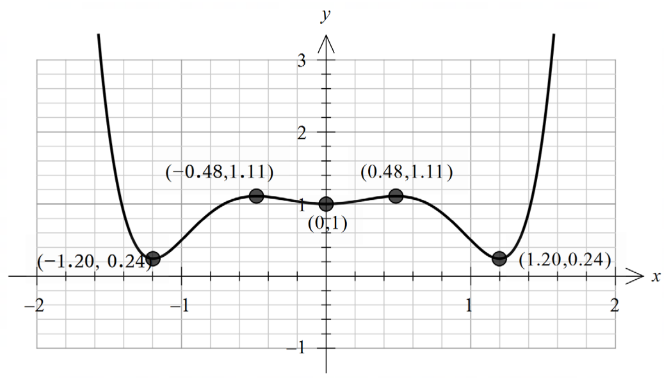




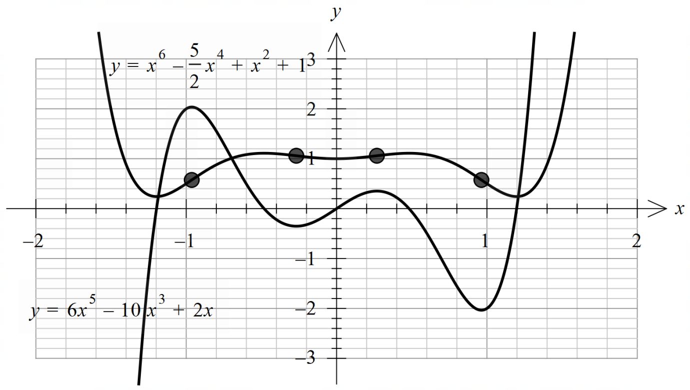
Question 5

$f:R\to R, f(x)=x^6-\frac{5}{2}x^4+x^2+1$

a. Points labelled correctly. **1A**

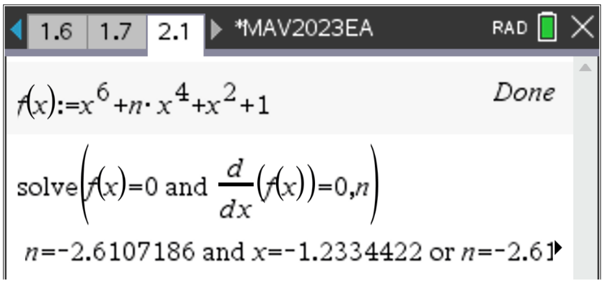


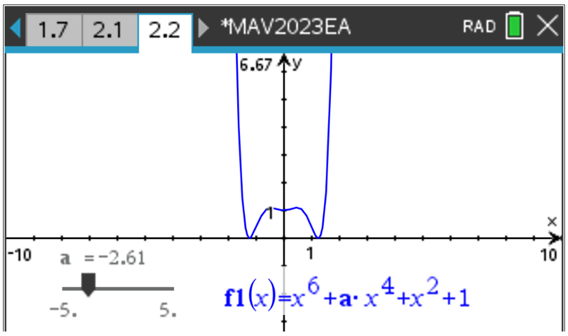
b. 4 **1A**



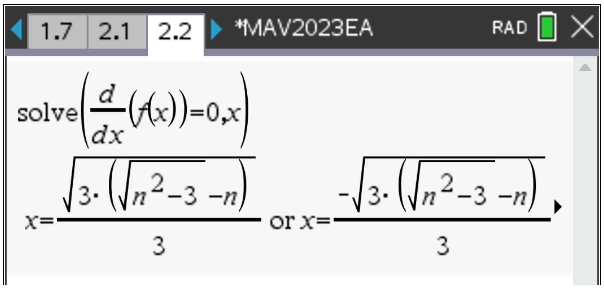
c. Solve $f(x)=0$ and $f'(x)=0$ **1M**

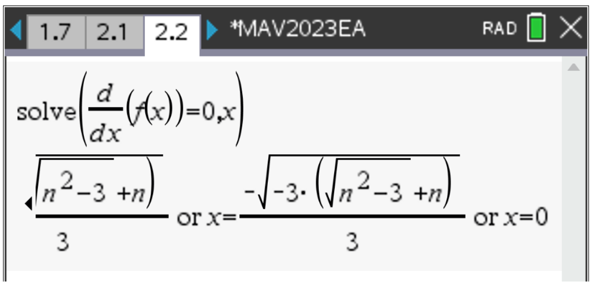
$a \geq -2.611$ **1A**





d. $x = \frac{\pm\sqrt{3\left(\sqrt{a^2-3}-a\right)}}{3}$, $x = \frac{\pm\sqrt{-3\left(\sqrt{a^2-3}+a\right)}}{3}$, $x = 0$ **1A**





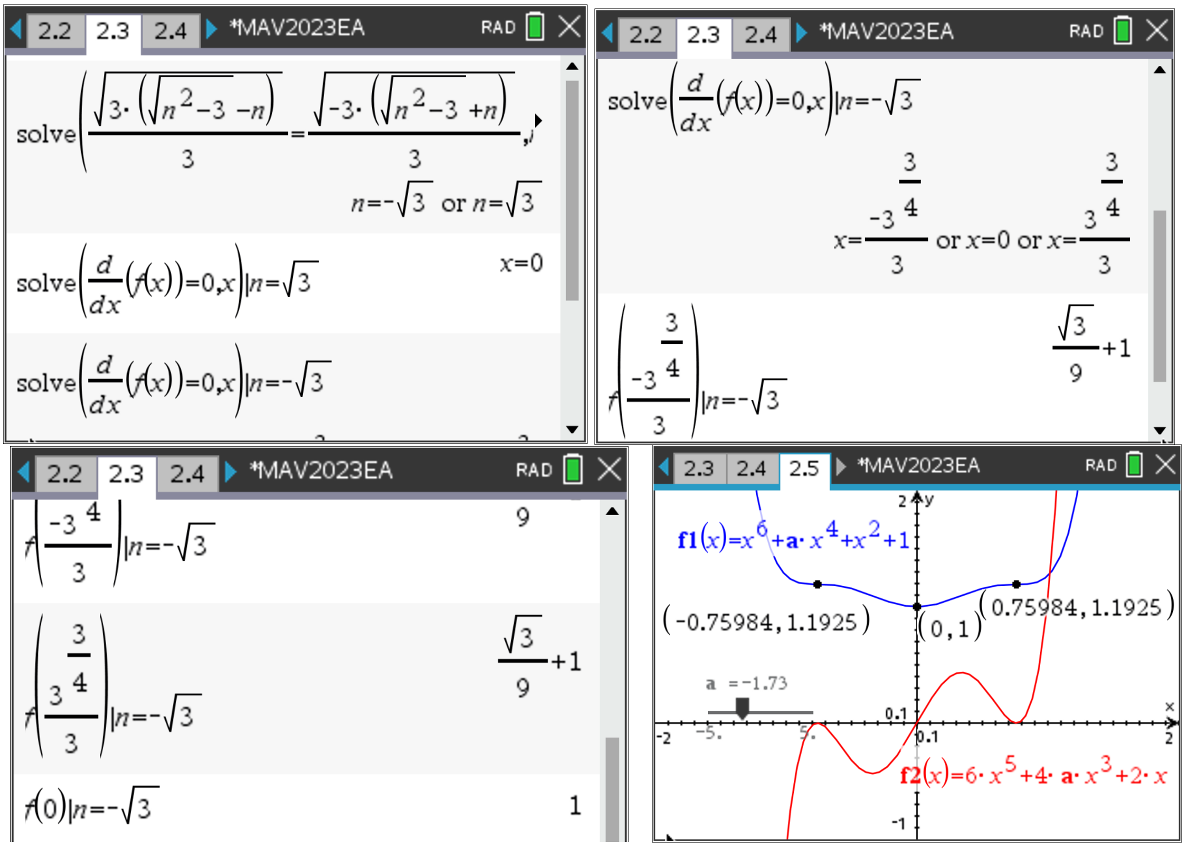
e. Solve $\dfrac{\sqrt{3\left(\sqrt{a^2-3}-a\right)}}{3} = \dfrac{\sqrt{-3\left(\sqrt{a^2-3}+a\right)}}{3}$ for a

$a = -\sqrt{3}$ **1A**

$\left(-3^{-\frac{1}{4}}, \dfrac{\sqrt{3}}{9}+1\right)$ stationary point of inflection

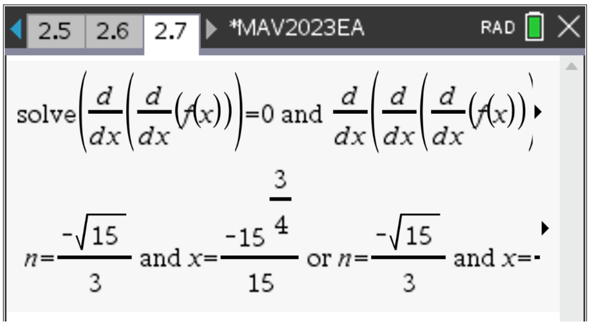
$\left(3^{-\frac{1}{4}}, \dfrac{\sqrt{3}}{9}+1\right)$ stationary point of inflection **1A** both

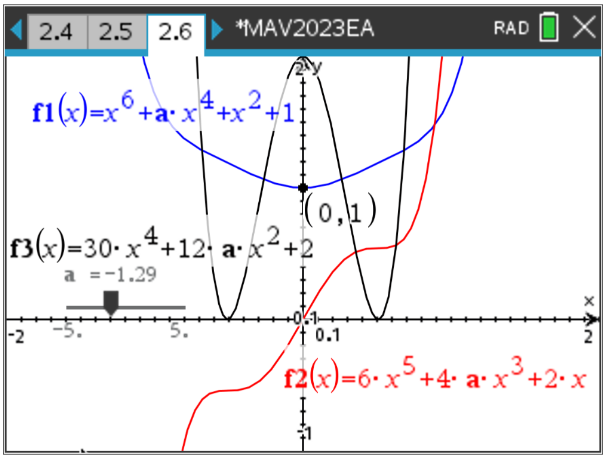
$(0,1)$ local minimum **1A**



f. Solve $f''(x)=0$ and $f'''(x)=0$ for a. **1M**

$a \geq \frac{-\sqrt{15}}{3}$ **1A**





END OF SOLUTIONS