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YEAR 12 *Trial Exam Paper*

2022

MATHEMATICAL METHODS

Written examination 2

Worked solutions

This book presents:

- worked solutions
- mark allocations
- tips.

SECTION A – Multiple-choice questions

Question	Answer
1	*E*
2	*D*
3	*B*
4	*E*
5	*C*
6	*C*
7	*D*
8	*A*
9	*B*
10	*B*
11	*D*
12	*E*
13	*A*
14	*B*
15	*B*
16	*D*
17	*B*
18	*E*
19	*A*
20	*D*

Question 1

Answer: E

Explanatory notes

The period of the function is $\frac{11\pi}{12}-\frac{5\pi}{12}=\frac{\pi}{2}$.

Therefore

$$\tan\left(2\left(x-\frac{\pi}{6}\right)\right)=\tan\left(2x-\frac{\pi}{3}\right)$$

Tips

- *The period of the function can be found quickly.*
- *Remember that the period of* $y=\tan(nx)$ *is* $\frac{\pi}{n}$.
- *If an analytic solution is too difficult or time consuming, you can examine options C, D or E by sketching them using CAS.*

Question 2

Answer: D

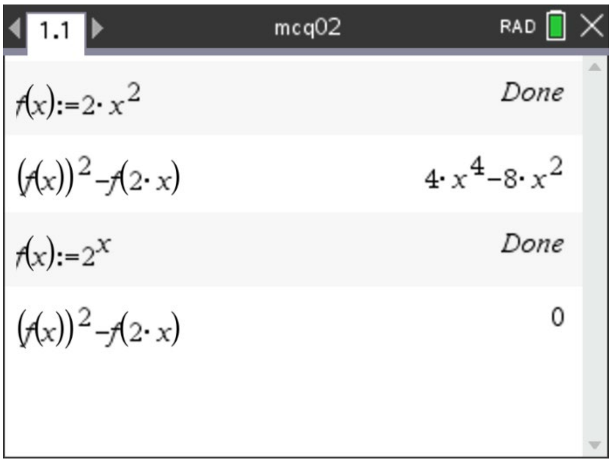
Explanatory notes

Note that $(f(x))^2=(2^x)^2=2^{2x}=f(2x)$.

This may be found by examining each option.

Tip

- *Defining the function* $f(x)$ *in CAS and then evaluating the difference* $(f(x))^2-f(2x)$ *is an efficient way to find the correct function. If the result is zero, then the equation is satisfied. In the screenshot below, the difference is evaluated for both* $f(x)=2x^2$ *and* $f(x)=2^x$*. The latter is correct.*



Question 3

Answer: B

Explanatory notes

Use log laws to see that

$$2+3\log_2(x)-\frac{1}{2}\log_2(y)=\log_2(4)+\log_2\left(x^3\right)-\log_2\left(\sqrt{y}\right)$$
$$=\log_2\left(\frac{4x^3}{\sqrt{y}}\right)$$

Tips

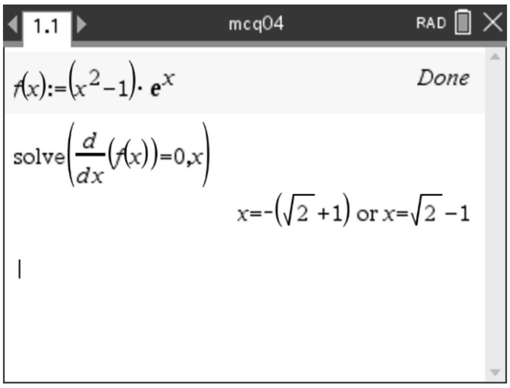
- *Knowing log laws is essential here.*
- *Don't forget that* $2=\log_2\left(2^2\right)=\log_2(4)$.

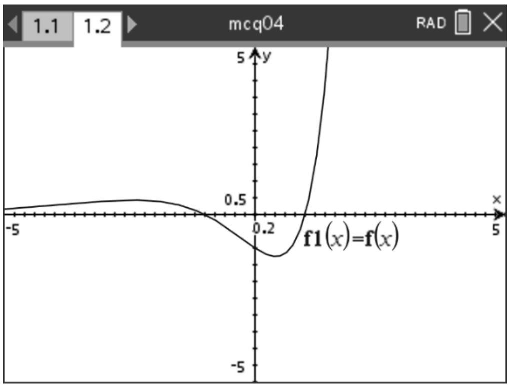
Question 4

Answer: E

Explanatory notes

The graph of $f(x)=\left(x^2-1\right)e^x$ has turning points when $x=-1-\sqrt{2}$ and $x=-1+\sqrt{2}$.





If $x\in\left[-1-\sqrt{2},-1+\sqrt{2}\right]$, then $f(x)$ will have an inverse function.

In all of the other cases, the function fails to be one-to-one.

Tip

- *Use CAS to find the turning points. A quick graph is very useful.*

Question 5

Answer: C

Explanatory notes

This is a conditional probability problem.

The probability of selecting three red balls (*RRR*) is

$$\frac{5}{9}\times\frac{4}{8}\times\frac{3}{7}=\frac{5}{42}$$

The probability of selecting two red balls (*RRY*, *RYR*, *YRR*) is

$$\left(\frac{5}{9}\times\frac{4}{8}\times\frac{4}{7}\right)+\left(\frac{5}{9}\times\frac{4}{8}\times\frac{4}{7}\right)+\left(\frac{4}{9}\times\frac{5}{8}\times\frac{4}{7}\right)=3\left(\frac{5}{9}\times\frac{4}{8}\times\frac{4}{7}\right)=\frac{10}{21}$$

Thus the probability of selecting *at least* two red balls is

$$\frac{5}{42}+\frac{10}{21}=\frac{25}{42}$$

Therefore the probability of selecting three red balls given that at least two red balls were selected is

$$\frac{\frac{5}{42}}{\frac{25}{42}}=\frac{1}{5}$$

Tip

- *Considering the various options (RRY, RYR, YRR) is essential.*

Question 6

Answer: C

Explanatory notes

The average value of $f(2x+1)$ on the interval $[-1,1]$ is $\frac{3}{2}$.

Consequently, the average value of $f(x)$ on the interval $[-1,3]$ is $\frac{3}{2}$.

Then $a\int_{-1}^{3}\frac{3}{2}dx = 4a\cdot\frac{3}{2} = 8$

Therefore $a = \frac{4}{3}$

Tip

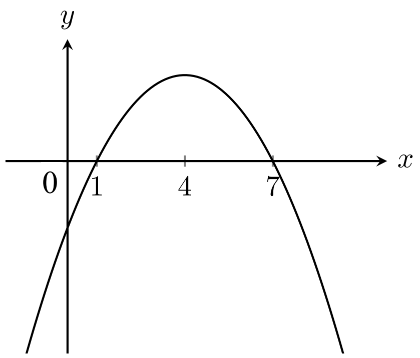
- *Since the average value of $f(2x+1)$ on the interval $[-1,1]$ is $\frac{3}{2}$, it is safe to set $f(2x+1)=\frac{3}{2}$. Thus we can set $f(x)=\frac{3}{2}$ on the interval $[-1,3]$.*

Question 7

Answer: D

Explanatory notes

The derivative $f'(x)$ is a quadratic with x-intercepts at $x=1$ and $x=7$. The turning point occurs when $x=4$ (midway between the x-intercepts).



The graph of $y=f'(x)$ is strictly increasing for $x\in(-\infty,4]$.

Tips

- *Read the question carefully. We are looking for the interval on which the derivative of 'f' is strictly increasing.*
- *The interval on which a function is strictly increasing (or decreasing) may also include endpoints, stationary points (turning points) and stationary points of inflection.*

Question 8

Answer: A

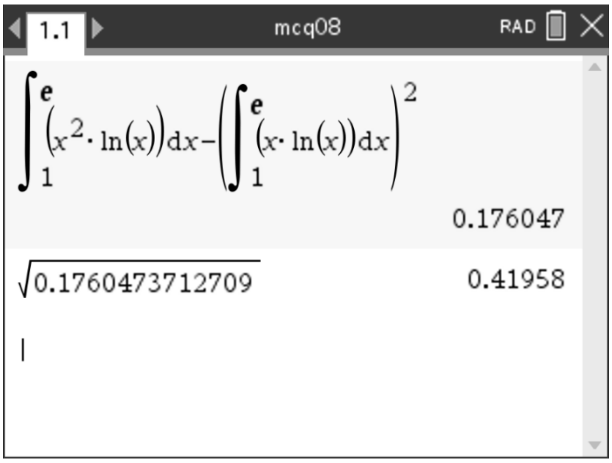
Explanatory notes

The variance is

$$\begin{aligned}\text{Var}(X) &= E\left(X^2\right) - \left(E(X)\right)^2 \\ &= \int_1^e x^2 \log_e(x)dx - \left(\int_1^e x\log_e(x)dx\right)^2 \\ &\approx 0.1760\end{aligned}$$

Therefore the standard deviation is

$$\sigma = \sqrt{\text{Var}(X)} \approx 0.4196$$



Tip

- *The formula for the variance of a probability distribution is given on the formula sheet.*

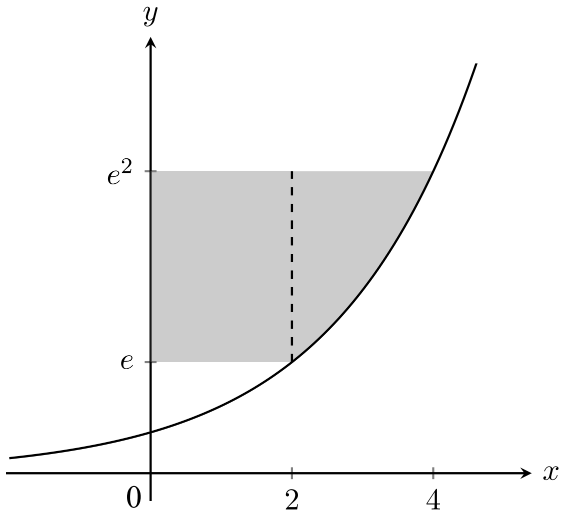
Question 9

Answer: B

Explanatory notes

If $y = f(x) = 2\log_e(x)$, then the inverse function is $f^{-1}(x) = e^{\frac{x}{2}}$.

The area required is equal to the shaded area shown below.



The shaded area can be split into two, with a rectangle on the left. The area of the shaded region is thus

$$A = 2\left(e^2 - e\right) + \int_2^4 \left(e^2 - e^{\frac{x}{2}}\right) dx$$

$$= 2e^2 - 2e + \int_2^4 \left(e^2 - e^{\frac{x}{2}}\right) dx$$

Tip

- *An area bounded by a graph can also be found on a graph of the inverse function. This makes calculation easier and avoids trying to bind to the y-axis.*

Question 10

Answer: B

Explanatory notes

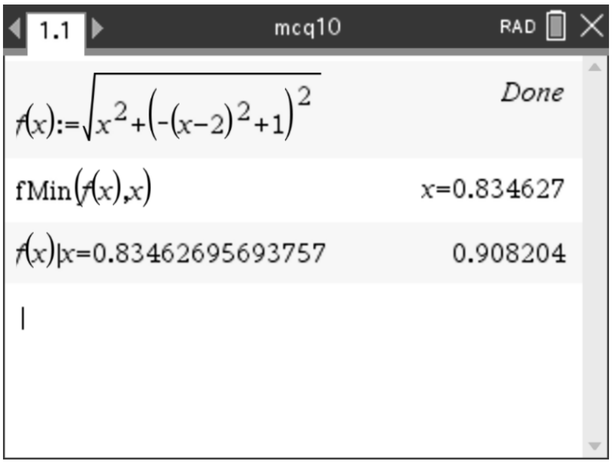
A point on the parabola has coordinates $\left(x, -(x-2)^2+1\right)$.

The distance from the origin $O(0,0)$ to a point on the parabola is

$\sqrt{x^2+\left(-(x-2)^2+1\right)^2}$

Using CAS, we find that this distance is least when $x = 0.8346$

The shortest distance is 0.9082



Tip

- *Use the* fMin *command on CAS.*

Question 11

Answer: D

Explanatory notes

The domain of f is $(-\infty, 5)$ and the domain of g is $\left[-\frac{3}{4}, \infty\right)$.

The maximal domain of $h = \frac{f}{g}$ is $\left(-\frac{3}{4}, 5\right)$.

Tip

- *Note that the domain above is the intersection of the two original domains excluding the point $x = -\frac{3}{4}$ where the function $g(x)$ equals zero.*

Question 12

Answer: E

Explanatory notes

The distribution is described as follows:

$$f(x)=\begin{cases}\dfrac{1}{2} & 0\le x\le 1\\ -\dfrac{1}{4}(x-3) & 1<x\le 3\\ 0 & \text{otherwise}\end{cases}$$

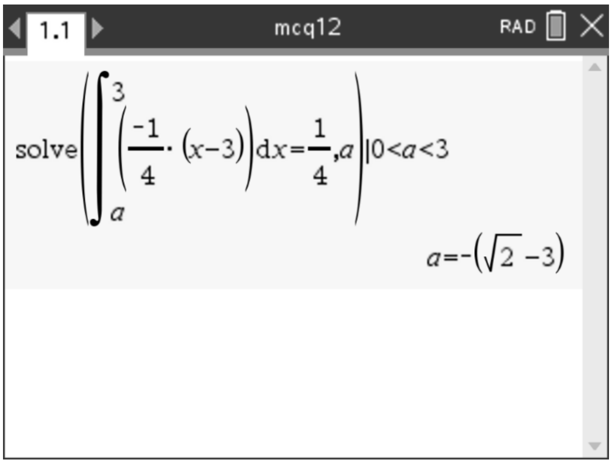
If $\Pr(X<a)=0.75$, then $\Pr(X>a)=0.25$

That is

$$\int_a^3 -\frac{1}{4}(x-3)dx=\frac{1}{4}$$

giving $a=3-\sqrt{2}$

This equation can be solved quickly using CAS:



Tip

- *Use CAS to solve equations like this.*

Question 13

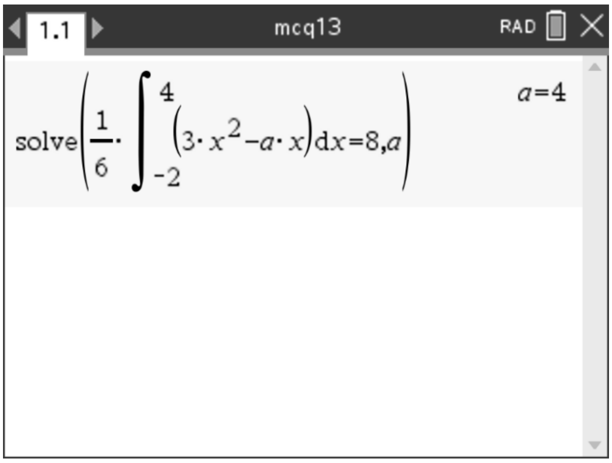
Answer: A

Explanatory notes

Since the average value of $f(x)$ over the interval $[-2,4]$ is 8, we know that

$$\frac{1}{6}\int_{-2}^{4}\left(3x^2-ax\right)dx=8$$

Solving using CAS gives $a=4$.



Tip

- *Use the formula for the average value of a function over a given domain.*

Question 14

Answer: B

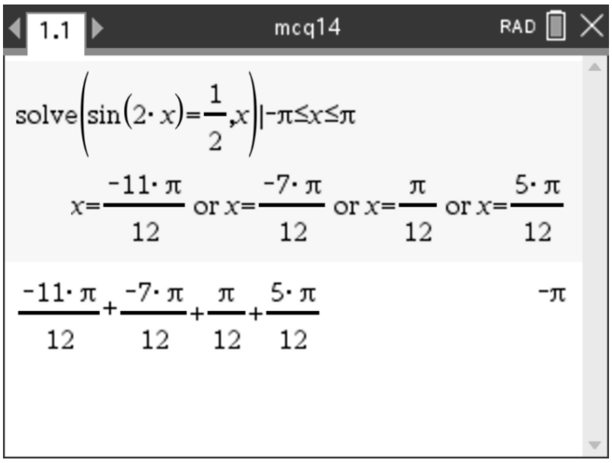
Explanatory notes

Solving $\sin(2x)=\frac{1}{2}$ for $-\pi \le x \le \pi$ gives

$$x=-\frac{11\pi}{12},-\frac{7\pi}{12},\frac{\pi}{12},\frac{5\pi}{12}$$

The sum of the solutions is

$$-\frac{11\pi}{12}-\frac{7\pi}{12}+\frac{\pi}{12}+\frac{5\pi}{12}=-\pi.$$



Tip

- *It is easiest to simply find the solutions and sum them using CAS, as shown above.*

Question 15

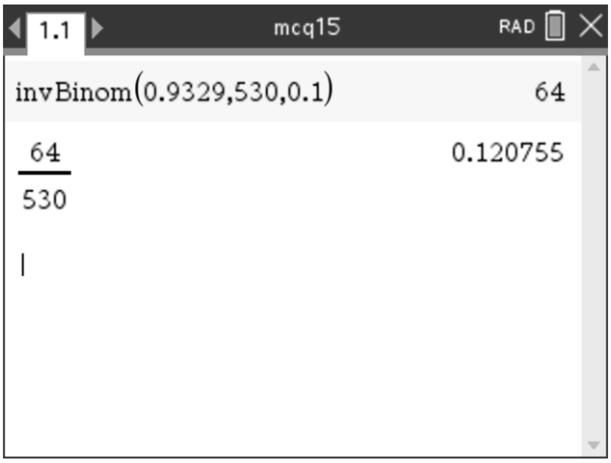
Answer: B

Explanatory notes

Let $X \sim \text{Bi}(0.1, 530)$. If $\Pr\left(\hat{P} > a\right) = 0.0671$, then

$$\Pr(X > 530 \times a) = 0.0671$$
$$\Pr(X \le 530 \times a) = 1 - 0.0671 = 0.9329$$

Use the inverse binomial function on CAS to find the value of $530 \times a$:



$$a \times 530 = 64$$
$$a = \frac{64}{530} = 0.1208$$

Therefore a is closest to 0.12

Question 16

Answer: D

Explanatory notes

Let $y = f\left(\log_e(2x)\right)$ and $u = \log_e(2x)$.

Hence $\frac{du}{dx} = \frac{2}{2x} = \frac{1}{x}$ and $\frac{dy}{du} = f'(u) = f'\left(\log_e(2x)\right)$.

Therefore

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{f'\left(\log_e(2x)\right)}{x}$$

Tip

- *This is a straightforward application of the chain rule.*

Question 17

Answer: B

Explanatory notes

Let $X \sim N\left(85, 5^2\right)$ and $Z \sim N(0,1)$. Then

$$\Pr(75 < Z < 100) = \Pr\left(\frac{75-85}{5} < Z < \frac{100-85}{5}\right) = \Pr(-2 < Z < 3).$$

Consider each option in turn:

A. $\Pr(-3 < Z < 2) = \Pr(-2 < Z < 3)$ ✓

B. $2\Pr(0 < Z < 3) + \Pr(2 < Z < 3) \neq \Pr(-2 < Z < 3)$ ✗

C. $2\Pr(0 < Z < 2) + \Pr(2 < Z < 3) = \Pr(-2 < Z < 3)$ ✓

D. $2\Pr(-3 < Z < 0) - \Pr(-3 < Z < -2) = \Pr(-2 < Z < 3)$ ✓

E. $\Pr(-2 < Z < 2) + \Pr(2 < Z < 3) = \Pr(-2 < Z < 3)$ ✓

Question 18

Answer: E

Explanatory notes

Solving $a + \frac{15a+2}{25} + \frac{25a^2+9}{25} + \frac{125a^3+4}{25} = 1$ gives $a = \frac{1}{5}$.

Substituting this value into the equations for *x* gives

x	1	2	3	4
$\Pr(X = x)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{5}$

Thus the expected value, $E(X)$, $= \frac{1}{5} + \frac{2}{5} + \frac{6}{5} + \frac{4}{5} = \frac{13}{5}$.

Tips

- *The sum of probabilities equals 1. Use CAS to solve the cubic equation.*
- *The formula for the expected value is given in the formula sheet.*

Question 19

Answer: A

Explanatory notes

Stationary points occur when $\frac{dy}{dx}=0$. Differentiating gives

$$\frac{dy}{dx}=6x^2+2(p-1)x+\frac{1}{3}(p+3)$$

For there to be two stationary points, the discriminant of this quadratic must be positive. The discriminant is

$$\begin{aligned}\Delta &= 4(p-1)^2+4\cdot 6\cdot\frac{1}{3}(p+3)\\ &= 4p^2-16p-20\end{aligned}$$

Solving $4p^2-16p-20=0$ gives $p=-1$ or $p=5$. Thus $\Delta>0$ if $p\in(-\infty,-1)\cup(5,\infty)$ or $p\in R\setminus[-1,5]$.

Tip

- *Consider drawing a quick sketch of the quadratic for the discriminant.*

Question 20

Answer: D

Explanatory notes

Note that

$$\begin{aligned}3\cos(2x)+1 &= 3\sin\left(2x+\frac{\pi}{2}\right)+1\\ &= 3\sin\left(2x-\frac{3\pi}{2}\right)+1\\ &= 3\sin\left(2\left(x-\frac{3\pi}{4}\right)\right)+1\end{aligned}$$

That is:

- dilation by a factor of 3 from the *x*-axis
- dilation by a factor of 2 from the *y*-axis and
- translation by $\frac{3\pi}{4}$ units to the right and 1 unit up.

Tip

- *Use the symmetry properties of trigonometric functions to answer this question.*

SECTION B

Question 1a.

Worked solution

The function is a cosine function with amplitude 2, shifted up by 5 units.

Maximum: 7

Minimum: 3

Mark allocation: 1 mark

- 1 mark for both answers

Tip

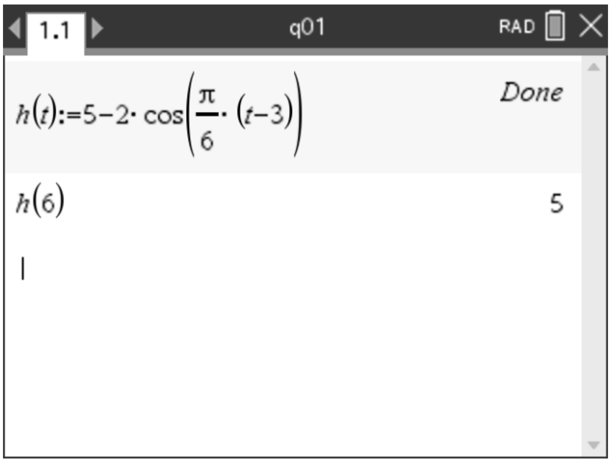
- *Note that this particular cosine graph moves 2 units above and 5 units below.*

Question 1b.

Worked solution

$$h(5) = 5 - 2\cos\left(\frac{\pi}{6}(6-3)\right)$$
$$= 5 - 2\cos\left(\frac{\pi}{2}\right)$$
$$= 5$$

The depth of water is 5 metres.



Mark allocation: 1 mark

- 1 mark for the correct answer

Tip

- *While this question can be done by hand, using CAS helps avoid careless errors.*

Question 1c.

Worked solution

$$5-2\cos\left(\frac{\pi}{6}(t-3)\right)=4$$

$$\cos\left(\frac{\pi}{6}(t-3)\right)=\frac{1}{2}$$

$$\frac{\pi}{6}(t-3)=-\frac{\pi}{3},\ \frac{\pi}{3},\ \frac{5\pi}{3},\ \frac{7\pi}{3}$$

$$t-3=-2,\ 2,\ 10,\ 14$$

$$t=1,\ 5,\ 13,\ 17$$

The depth of water is four metres at 1.00 am, 5.00 am, 1.00 pm and 5.00 pm.

Mark allocation: 2 marks

- 1 mark for 1.00 am and 5.00 am
- 1 mark for 1.00 pm and 5.00 pm (or the equivalent 24-hour versions)

Tip

- *Use CAS and restrict the domain to ensure that all solutions are obtained.*

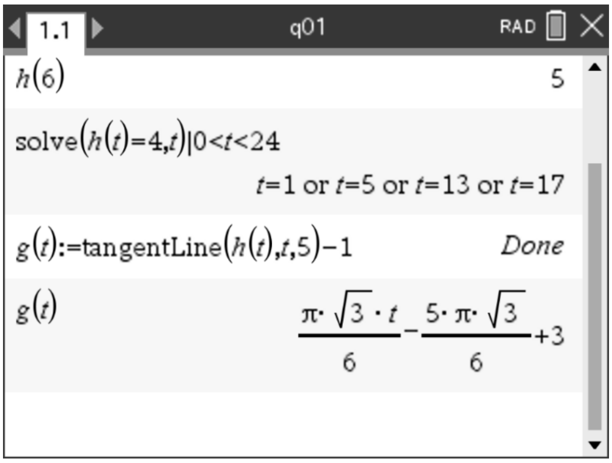
Question 1d.

Worked solution

$$g(t)=\frac{\pi\sqrt{3}t}{6}-\frac{5\pi\sqrt{3}}{6}+3$$

$$=\frac{\pi\sqrt{3}t}{6}-\frac{5\pi\sqrt{3}-18}{6}$$

This is most easily found by using the tangentLine command on CAS:



Mark allocation: 2 marks

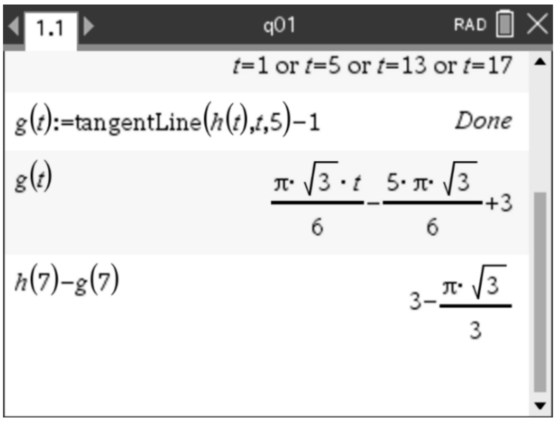
- 1 mark for the correct gradient
- 1 mark for the fully correct answer

Question 1e.

Worked solution

$$h(7)-g(7)=3-\frac{\pi\sqrt{3}}{3}$$

This should be done using CAS:



Mark allocation: 2 marks

- 1 mark for stating the difference between the two functions at $t=7$
- 1 mark for the correct answer

Tip

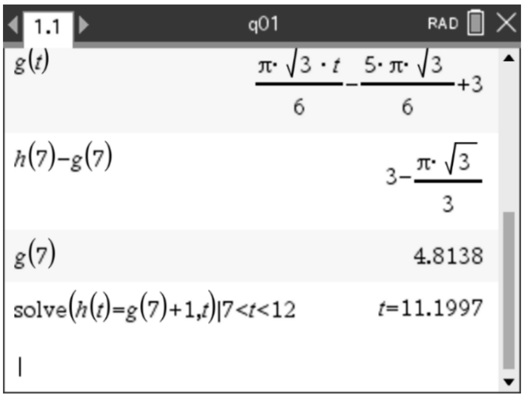
- *Remember to give the answer in the form specified in the question.*

Question 1f.

Worked solution

$C(11.20, 4.81)$

Use CAS to find these coordinates:



Mark allocation: 1 mark

- 1 mark for the correct coordinates

Tip

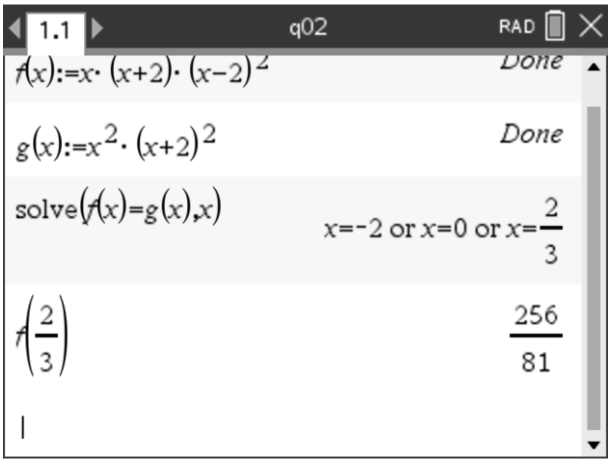
- *Remember to give coordinates and not just values.*

Question 2a.

Worked solution

Solve $f(x)=g(x)$ to find the values of a and b.

$a=\frac{2}{3}$, $b=\frac{256}{81}$



Mark allocation: 2 marks

- 1 mark for the correct answer for a
- 1 mark for the correct answer for b

Tip

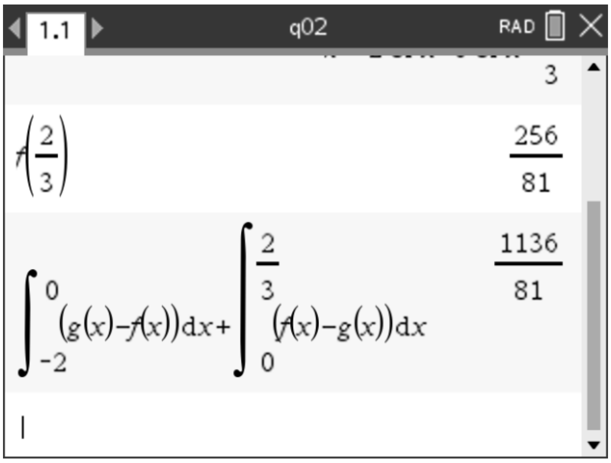
- *This question requires an exact answer to be given. Decimal approximations are not acceptable.*

Question 2b.

Worked solution

The area is found by evaluating two integrals (one for each bounded area):

$$\int_{-2}^{0}\left(g(x)-f(x)\right)dx+\int_{0}^{\frac{2}{3}}\left(f(x)-g(x)\right)dx=\frac{1136}{81}$$



Mark allocation: 2 marks

- 1 mark for using two integrals
- 1 mark for the correct answer

Tip

- *Be careful to correctly identify the upper and lower functions.*

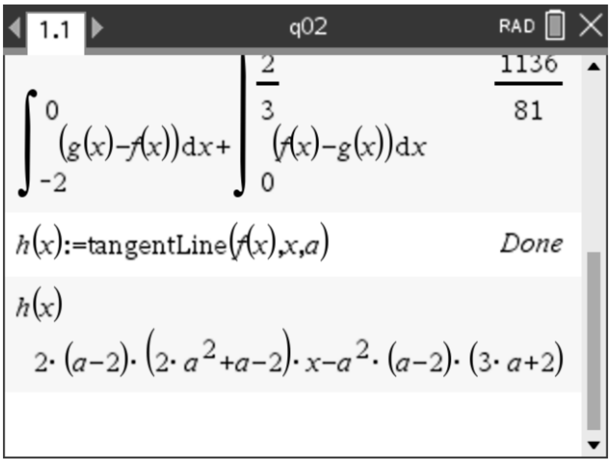
Question 2c.

Worked solution

The equation of the tangent in terms of a is

$$h(x) = -a^2(a-2)(3a+2) + 2(a-2)\left(2a^2 + a - 2\right)x$$

This is found using CAS:



Mark allocation: 1 mark

- 1 mark for the correct answer

Tip

- *The* tangentLine *command allows the use of a variable rather than a specified point.*

Question 2d.

Worked solution

The graphs meet when $x = 1 - a \pm \sqrt{-2a^2 + 2a + 5}$ and when $x = a$ (a general point).

If $1 - a + \sqrt{-2a^2 + 2a + 5} = 1 - a - \sqrt{-2a^2 + 2a + 5}$ then

$$a = \frac{1}{2}\left(1 \pm \sqrt{11}\right)$$

If $1 - a + \sqrt{-2a^2 + 2a + 5} = a$ then

$$a = \frac{3 - \sqrt{33}}{6}$$

If $1 - a - \sqrt{-2a^2 + 2a + 5} = a$ then

$$a = \frac{3 + \sqrt{33}}{6}$$

$$\text{solve}(f(x)=h(x),x)$$

$$x=-\left(\sqrt{-2\cdot a^2+2\cdot a+5}+a-1\right) \text{ or } x=\sqrt{-2\cdot a^2+2\cdot a+5}-a+1 \text{ or } x=a$$

$$\text{solve}\left(-\left(\sqrt{-2\cdot a^2+2\cdot a+5}+a-1\right)=\sqrt{-2\cdot a^2+2\cdot a+5}-a+1,a\right)$$

$$a=\frac{-\left(\sqrt{11}-1\right)}{2} \text{ or } a=\frac{\sqrt{11}+1}{2}$$

$$\text{solve}\left(-\left(\sqrt{-2\cdot a^2+2\cdot a+5}+a-1\right)=a,a\right) \quad a=\frac{-\left(\sqrt{33}-3\right)}{6}$$

$$\text{solve}\left(\sqrt{-2\cdot a^2+2\cdot a+5}-a+1=a,a\right) \quad a=\frac{\sqrt{33}+3}{6}$$

Mark allocation: 4 marks

- 1 mark for equating $1 - a + \sqrt{-2a^2 + 2a + 5}$ and $1 - a - \sqrt{-2a^2 + 2a + 5}$
- 1 mark for the solution $a = \frac{1}{2}\left(1 \pm \sqrt{11}\right)$
- 1 mark for the equations $1 - a + \sqrt{-2a^2 + 2a + 5} = a$ and $1 - a - \sqrt{-2a^2 + 2a + 5} = a$
- 1 mark for the solution $a = \frac{1}{6}\left(3 \pm \sqrt{33}\right)$

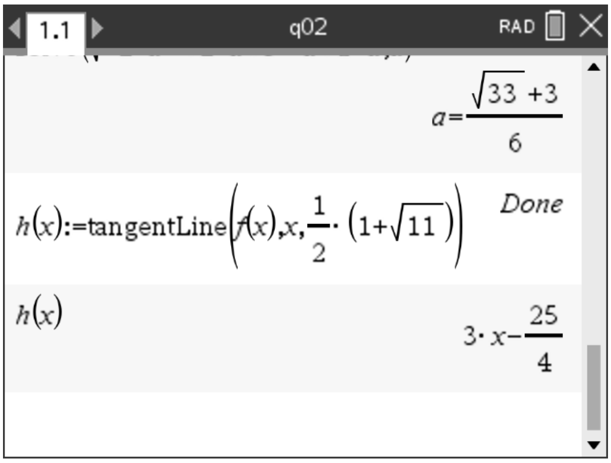
Tips

- *Use CAS to find where, in terms of 'a', the tangent and the curve meet.*
- *Consider all the options to find the points where the tangent meets the curve twice.*

Question 2e.

Worked solution

$h(x) = 3x - \frac{25}{4}$



Mark allocation: 1 mark

- 1 mark for the correct answer

Tip

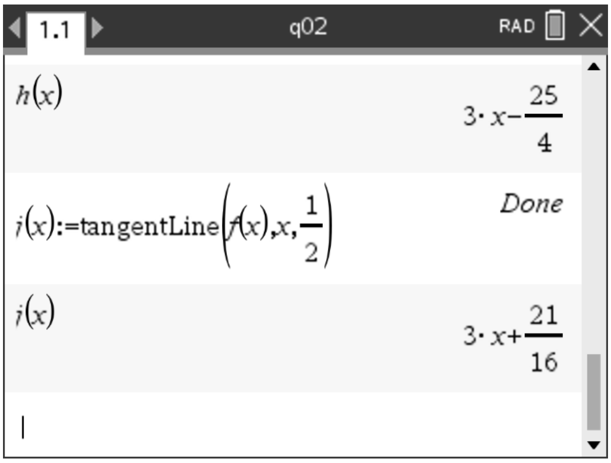
- *Having the function $f(x)$ defined makes entry into CAS much easier.*

Question 2f.

Worked solution

The gradient is 3 and $f'(x) = 3$ when $x = \frac{1}{2}$. Therefore

$j(x) = 3x + \frac{21}{16}$



Mark allocation: 1 mark

- 1 mark for the correct answer

Question 2g.

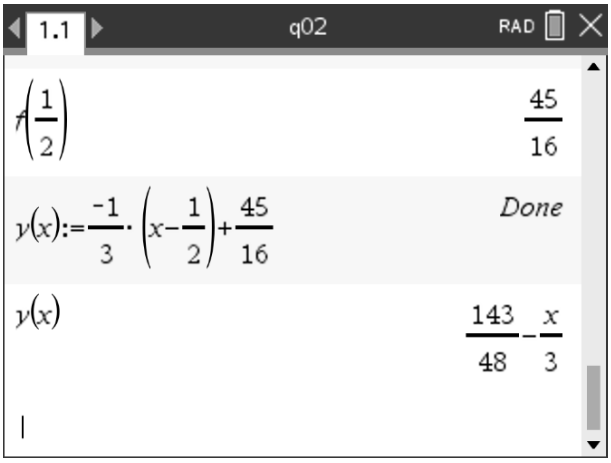
Worked solution

The point of intersection of $j(x)$ and $f(x)$ is $\left(\frac{1}{2}, \frac{45}{16}\right)$.

The equation of the line perpendicular to the graph of $y = j(x)$ and which passes through $\left(\frac{1}{2}, \frac{45}{16}\right)$ is

$$y = -\frac{1}{3}\left(x - \frac{1}{2}\right) + \frac{45}{16}$$

$$= -\frac{x}{3} + \frac{143}{48}$$



Mark allocation: 1 mark

- 1 mark for the equation of the line

Tip

- *Use the fact that the product of the gradients of two perpendicular lines is -1.*

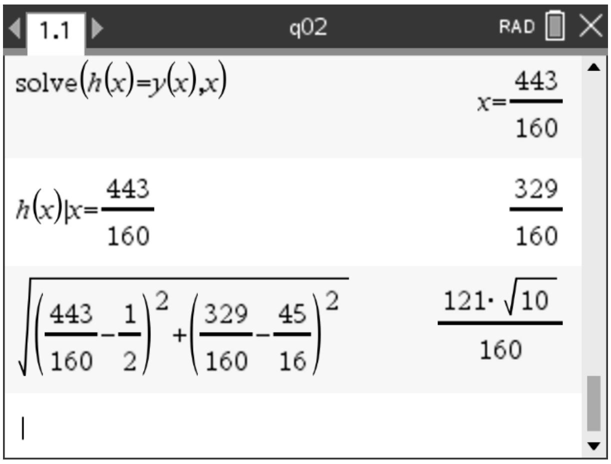
Question 2h.

Worked solution

The line in **part 2g.** meets $h(x)$ at $\left(\frac{443}{160},\frac{329}{160}\right)$.

The distance between $\left(\frac{1}{2},\frac{45}{16}\right)$ and $\left(\frac{443}{160},\frac{329}{160}\right)$ is

$$\sqrt{\left(\frac{443}{160}-\frac{1}{2}\right)^2+\left(\frac{329}{160}-\frac{45}{16}\right)^2}=\frac{121\sqrt{10}}{160}$$



Mark allocation: 2 marks

- 1 mark for finding the point $\left(\frac{443}{160},\frac{329}{160}\right)$
- 1 mark for using the distance formula to derive the correct answer

Tips

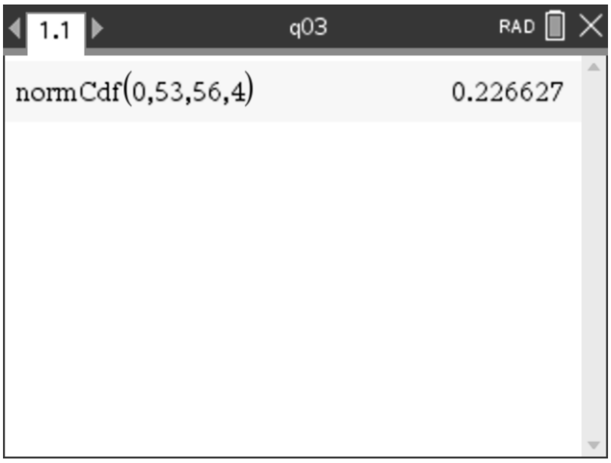
- *The distance is the shortest perpendicular distance.*
- *Use the results from earlier to find the point of intersection of the lines and then use the distance formula.*

Question 3a.

Worked solution

Suppose $X \sim N\left(56, 4^2\right)$

$\Pr(X < 53) = 0.2266$



Mark allocation: 1 mark

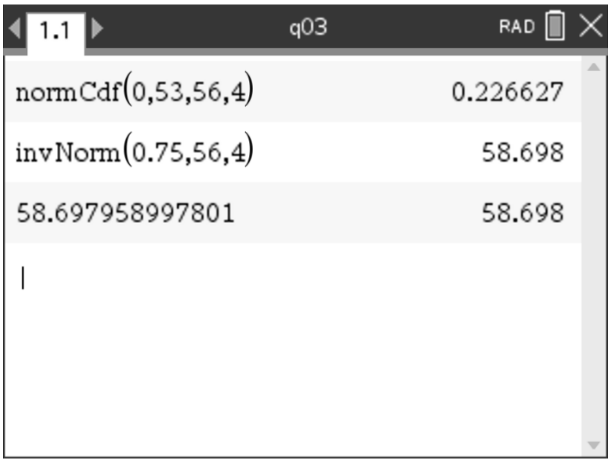
- 1 mark for the correct answer

Question 3b.

Worked solution

$\Pr(X < m) = 0.75$

$m = 58.6980$



Mark allocation: 1 mark

- 1 mark for the correct answer

Tip

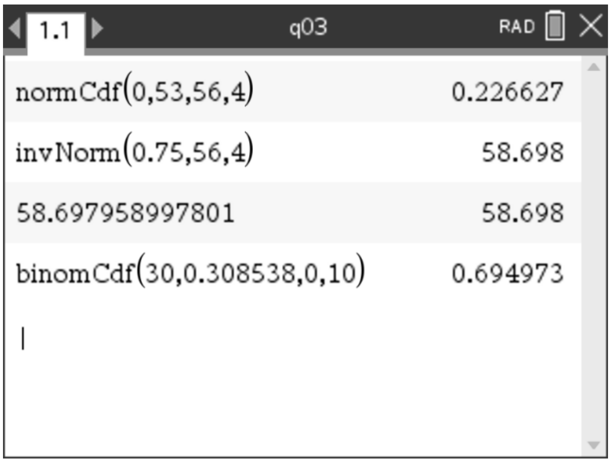
- *The inverse normal function is used here.*

Question 3c.

Worked solution

Let $W \sim \text{Bi}(30, 0.308538)$

$\Pr(W \le 10) = 0.6950$



Mark allocation: 2 marks

- 1 mark for using the binomial distribution
- 1 mark for the correct answer

Tip

- *Note that the distribution is now binomial. The parameters are n and p.*

Question 3d.

Worked solution

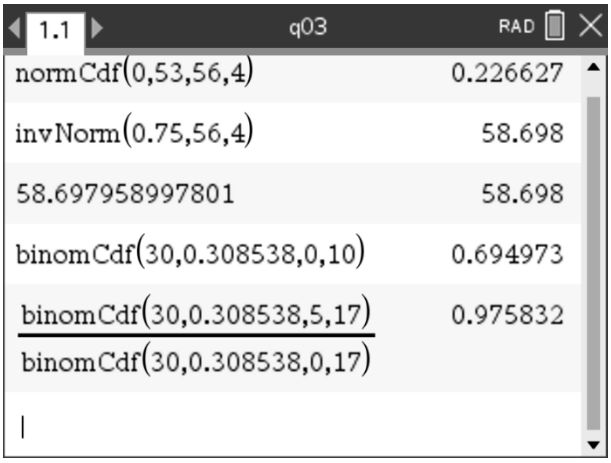
$$\Pr\left(\hat{P} > 0.15 | \hat{P} < 0.6\right) = \Pr\left(\frac{W}{30} > 0.15 | \frac{W}{30} < 0.6\right)$$

$$= \Pr\left(W > 4.5 | W < 18\right)$$

$$= \Pr(W \geq 5 | W \leq 17)$$

$$= \frac{\Pr\left(5 \leq W \leq 17\right)}{\Pr\left(W \leq 17\right)}$$

$$= 0.9758$$



Mark allocation: 2 marks

- 1 mark for using the conditional probability formula
- 1 mark for the correct answer

Tip

- *Note that conditional probability is required here. Be familiar with the conditional probability formula.*

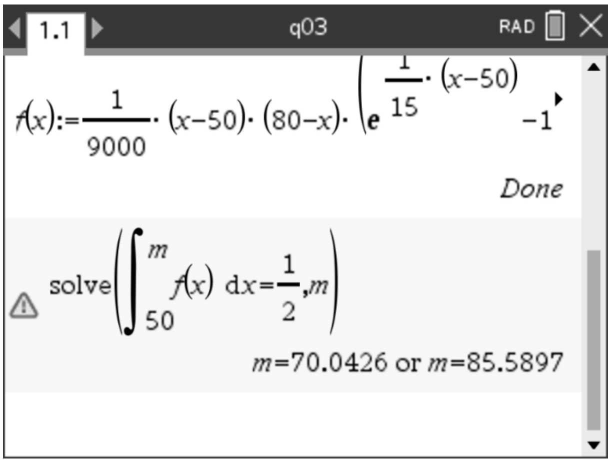
Question 3e.

Worked solution

The median time is m where

$$\int_{50}^{m} f(x)dx = \frac{1}{2}$$

$$m = 70.04$$



Mark allocation: 2 marks

- 1 mark for the appropriate integral (with the right-hand side equal to $\frac{1}{2}$)
- 1 mark for the correct answer

Tip

- *Define the function in CAS, as it will be used several times.*

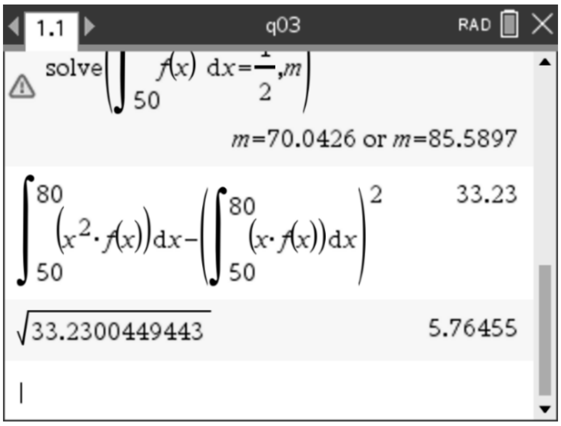
Question 3f.

Worked solution

Denote the probability distribution by X. Then

$$\begin{aligned}\mathrm{Var}(X) &= E\left(X^2\right) - \left(E(X)\right)^2 \\ &= 33.23\end{aligned}$$

Therefore $\sigma_X = 5.7646$



Mark allocation: 2 marks

- 1 mark for the appropriate use of the variance formula
- 1 mark for the correct answer

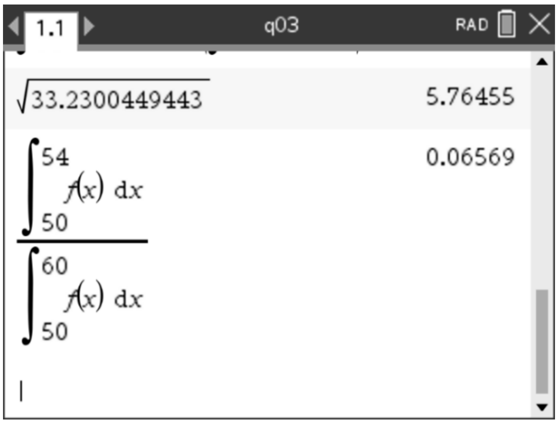
Tip

- *Remember that the standard deviation is the square root of the variance.*

Question 3g.

Worked solution

$$\begin{aligned}\Pr\left(X < 54 | X < 60\right) &= \frac{\Pr\left(X < 54\right)}{\Pr\left(X < 60\right)} \\ &= 0.0657\end{aligned}$$



Mark allocation: 2 marks

- 1 mark for using the conditional probability formula
- 1 mark for the correct answer

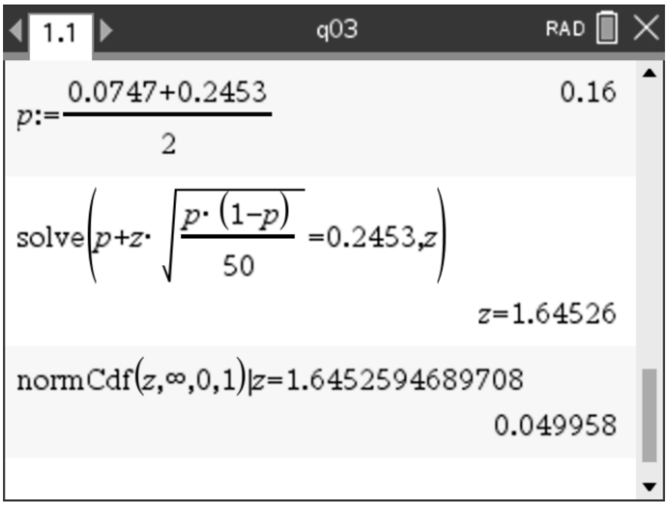
Question 3h.

Worked solution

The value of $\hat{p}$ is $\dfrac{0.0747+0.2453}{2}\approx 0.16$

Solving $\hat{p}+z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{50}}=0.2453$ gives $z\approx 1.6453$

Then $\Pr(Z>1.6453)\approx 0.05$ is the area in each tail of the standard normal distribution, and so we have a 90% confidence interval.



Mark allocation: 2 marks

- 1 mark for finding $\hat{p}$ and z
- 1 mark for the correct answer

Tips

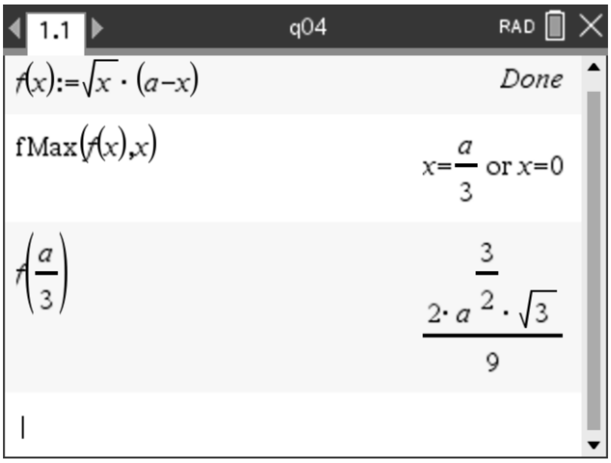
The value of $\hat{p}$ is the midpoint of the confidence interval.
The formula for calculating a confidence interval is on the formula sheet.

Question 4a.

Worked solution

The maximum occurs when $x = \frac{a}{3}$.

The coordinates of A are $A\left(0, \frac{2\sqrt{3}}{9}a^{\frac{3}{2}}\right)$.



Mark allocation: 2 marks

- 1 mark for finding $x = \frac{a}{3}$
- 1 mark for the correct answer

Question 4b.

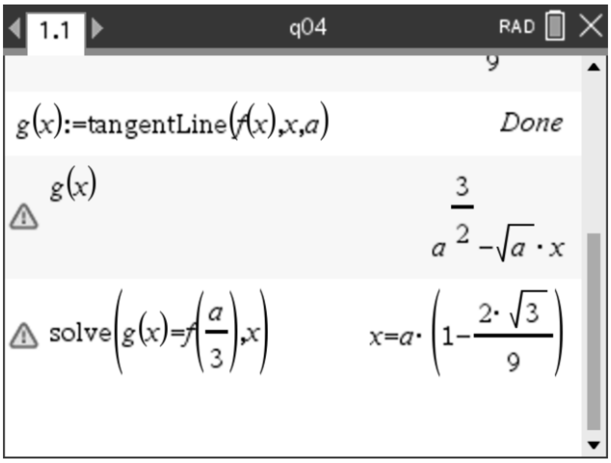
Worked solution

The equation of the tangent to f at $x = a$ is

$$g(x) = -\sqrt{a}x + a^{\frac{3}{2}}$$

The coordinates of B are

$$B\left(\frac{\left(9-2\sqrt{3}\right)a}{9}, \frac{2\sqrt{3}}{9}a^{\frac{3}{2}}\right)$$



Mark allocation: 2 marks

- 1 mark for finding the equation of the tangent to f at $x = a$
- 1 mark for the correct coordinates

Tip

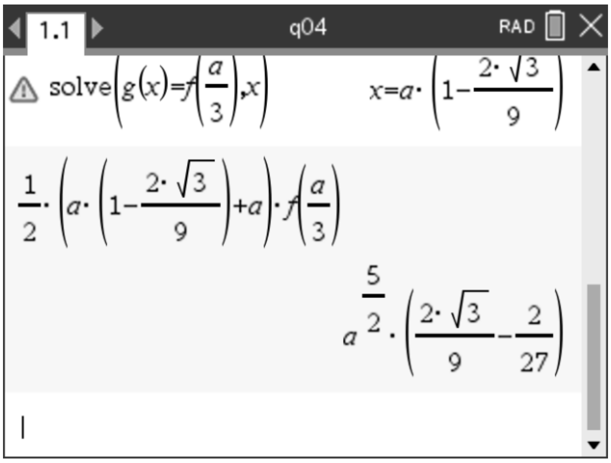
- *To find the coordinates of B, the tangent must be found first. You can find this quickly using CAS.*

Question 4c.

Worked solution

The area A_1 of the trapezium $OABC$ is

$$\frac{1}{2}\left(\frac{9-2\sqrt{3}}{9}a+a\right)\frac{2\sqrt{3}}{9}a^{\frac{3}{2}}=\frac{6\sqrt{3}-2}{27}a^{\frac{5}{2}}$$



Mark allocation: 2 marks

- 1 mark for correctly using the trapezium formula
- 1 mark for the correct answer

Tips

- *Remember to give your answer in the form specified in the question.*
- *The formula for the area of a trapezium is found on the formula sheet.*

Question 4d.

Worked solution

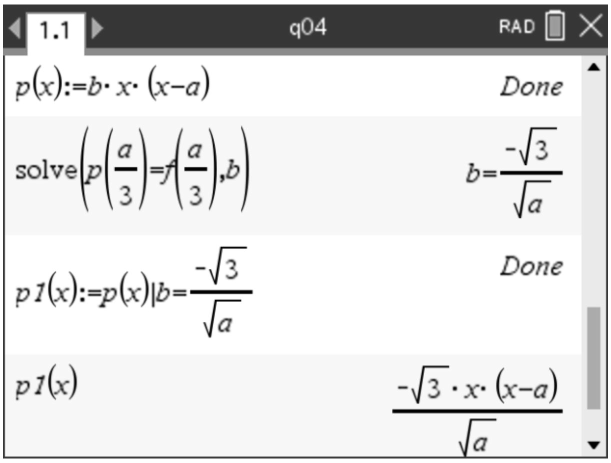
There are various ways to approach this problem. One way is to consider that

$$p(x) = bx(x - a)$$

$$p\left(\frac{a}{3}\right) = \frac{2\sqrt{3}}{9}a^{\frac{3}{2}}$$

$$b = -\frac{\sqrt{3}}{\sqrt{a}}$$

$$p(x) = -\frac{\sqrt{3}}{\sqrt{a}}x(x - a)$$



Mark allocation: 2 marks

- 1 mark for providing appropriate working (for example, giving the parabola in intercept form)
- 1 mark for the correct answer

Tip

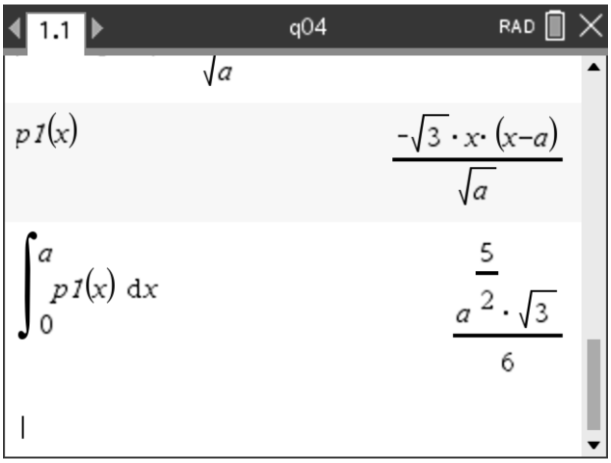
- *The intercept form has been used here. Alternatively, a system of equations could be solved to find the equation of the parabola.*

Question 4e.

Worked solution

The area of the shaded region is

$$A_2 = \int_0^a p(x)dx = \frac{\sqrt{3}}{6}a^{\frac{5}{2}}$$



Mark allocation: 1 mark

- 1 mark for the correct answer

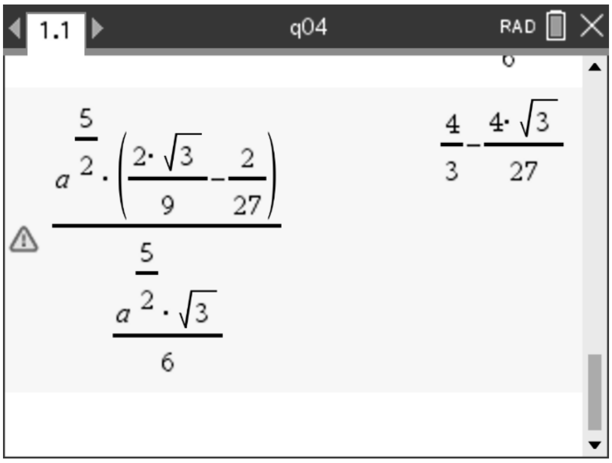
Question 4f.

Worked solution

The ratio is

$$\frac{A_1}{A_2} = \frac{4}{3} - \frac{4\sqrt{3}}{27}$$

$$= \frac{4}{27}\left(9 - \sqrt{3}\right)$$



Mark allocation: 1 mark

- 1 mark for the correct answer

Question 5a.

Worked solution

$y = e^{\frac{x}{2}} - 3$. Swap x and y:

$x = e^{\frac{y}{2}} - 3$

$y = 2\log_e(x+3)$

Therefore $f^{-1}(x) = 2\log_e(x+3)$ or $f^{-1}(x) = 2\ln(x+3)$.

Mark allocation: 1 mark

- 1 mark for the correct answer

Tips

- *Remember to write* $\log_e$ *or* ln.
- *For a single-mark question where no particular working is required, solving an equation of the form* $f(y) = x$ *for* x *on CAS allows the inverse function to be quickly found.*

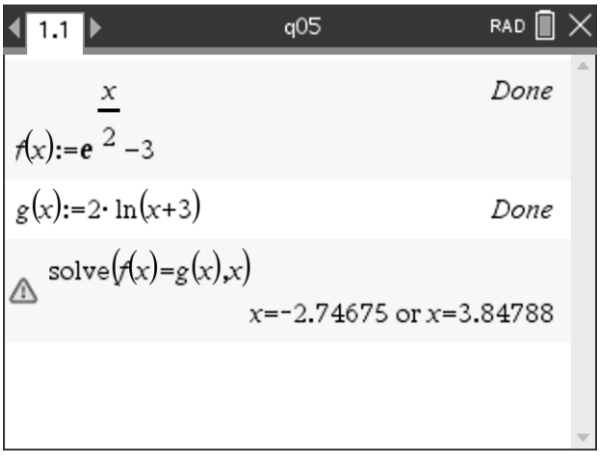
Question 5b.

Worked solution

The points of intersection are

$(-2.747, -2.747)$ and $(3.848, 3.848)$

Use CAS to find these points:



Mark allocation: 2 marks

- 1 mark for each intersection, correct to three decimal places

Tips

- *Ensure that coordinates are given. Simply writing the x-values will not satisfy the requirements of the question.*
- *The x-coordinate and y-coordinate are equal, as the intersection points lie on the line* $y = x$.

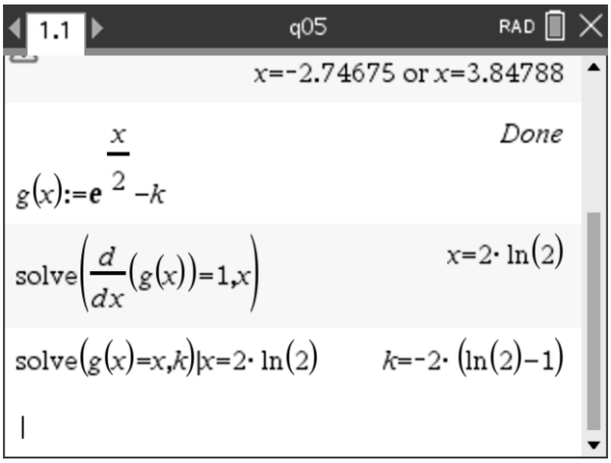
Question 5c.i.

Worked solution

If the graphs of $y = g(x)$ and $y = g^{-1}(x)$ meet only once, they must meet on the line $y = x$ (which is the tangent to $g(x)$ and $g^{-1}(x)$ at this point). The gradient of the line $y = x$ is 1 and so we solve $g'(x) = 1$.

If $g'(x) = 1$ then $x = 2\log_e(2)$ and

$$g\left(2\log_e(2)\right) = 2\log_e(2)$$
$$k = 2 - \log_e(4)$$



Mark allocation: 1 mark

- 1 mark for working that leads to the desired result

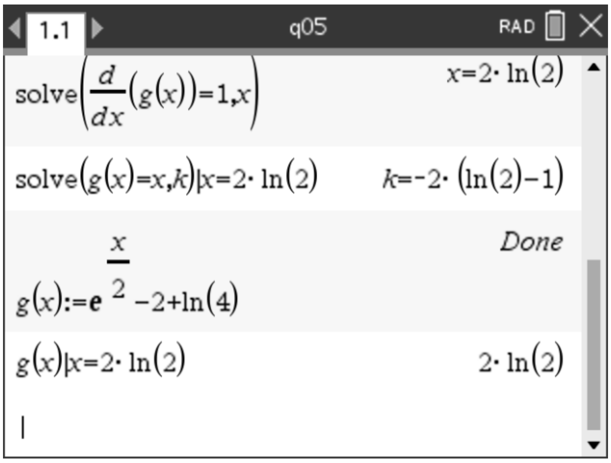
Tip

- *If the two curves meet only once, then they must meet on the line $y = x$.*

Question 5c.ii.

Worked solution

$\left(2\log_e(2),\, 2\log_e(2)\right)$ or $\left(\log_e(4),\, \log_e(4)\right)$



Mark allocation: 1 mark

- 1 mark for the correct coordinates

Tip

- *As $g(x)$ and $g^{-1}(x)$ meet on the line $y = x$, the x and y values of the point of intersection are equal.*

Question 5d.

Worked solution

$k \in \left(-\infty, 2-\log_e(4)\right)$ or $-\infty < k < 2-\log_e(4)$

Mark allocation: 1 mark

- 1 mark for the correct interval

Question 5e.

Worked solution

$$\begin{aligned} y &= -\left(x-\log_e(4)\right)+g\left(\log_e(4)\right) \\ &= -x+\log_e(4)+2-k \end{aligned}$$

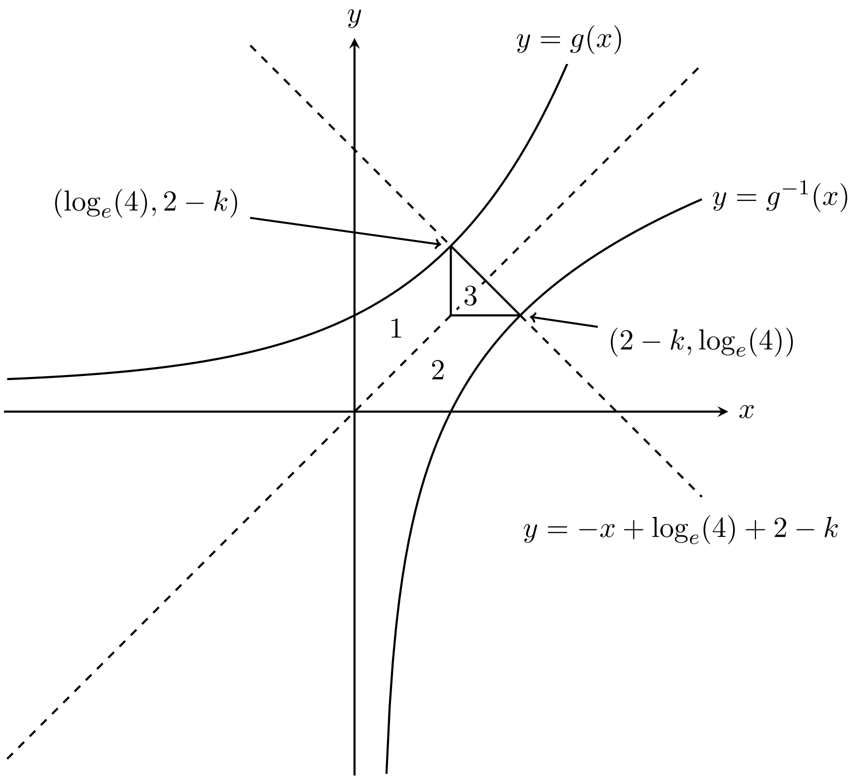
Mark allocation: 1 mark

- 1 mark for working that leads to the desired result

Question 5f.

Worked solution

The area can be found by considering the following diagram.



The area labelled 1 is given by the integral $\int_0^{\log_e(4)} \left(g(x) - x\right) dx$. The area labelled 2 is (by symmetry) equal to area 1.

The area of the triangle labelled 3 is given by $\frac{1}{2}\left(2 - k - \log_e(4)\right)^2$. Therefore

$$
\begin{aligned}
A(k) &= 2\int_0^{\log_e(4)} \left(g(x) - x\right) dx + \frac{1}{2}\left(2 - k - \log_e(4)\right)^2 \\
&= \frac{1}{2}k^2 - \left(2 - \log_e(4)\right) + 6 - k\log_e(4) - \frac{1}{2}\left(\log_e(4)\right)^2
\end{aligned}
$$

$$g(x):=e^{\frac{x}{2}}-k$$

$$\text{solve}\left(\frac{d}{dx}(g(x))=1,x\right) \qquad x=2\cdot\ln(2)$$

$$\text{solve}(g(x)=x,k)|x=2\cdot\ln(2) \qquad k=^{-}2\cdot(\ln(2)-1)$$

$$g(x):=e^{\frac{x}{2}}-2+\ln(4) \qquad \textit{Done}$$

$$g(x)|x=2\cdot\ln(2) \qquad 2\cdot\ln(2)$$

$$g(x):=e^{\frac{x}{2}}-k \qquad \textit{Done}$$

$$2\cdot\int_0^{\ln(4)}(g(x)-x)\,dx+\frac{1}{2}\cdot(2-k-\ln(4))^2 \qquad \frac{k^2}{2}+k\cdot(^{-}2\cdot\ln(2)-2)-2\cdot(\ln(2))^2-4\cdot\ln(2)+6$$

$$a(k):=\frac{k^2}{2}+k\cdot(^{-}2\cdot\ln(2)-2)-2\cdot(\ln(2))^2-4\cdot\ln(2)+6 \qquad \textit{Done}$$

Mark allocation: 2 marks

- 1 mark for the appropriate working
- 1 mark for the correct answer

Tip

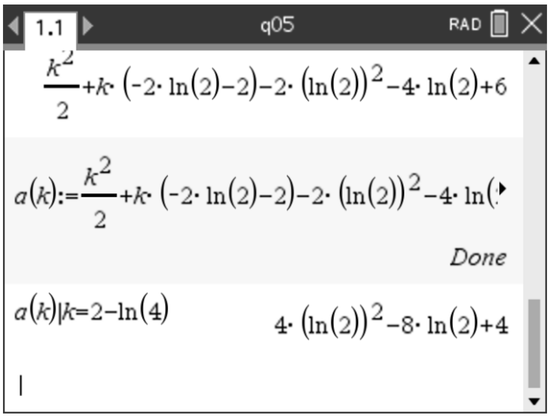
- *This question requires careful consideration of the areas involved. It may be useful to annotate the graph.*

Question 5g.

Worked solution

The minimum value of $A(k)$ is $4\left(\log_e(2)\right)^2 - 8\log_e(2) + 4$.

This occurs when the curves $y = g(x)$ and $y = g^{-1}(x)$ meet; that is, when $k = 2 - \log_e(4)$.



Mark allocation: 2 marks

- 1 mark for the correct minimum value of $A(k)$
- 1 mark for the correct value of k

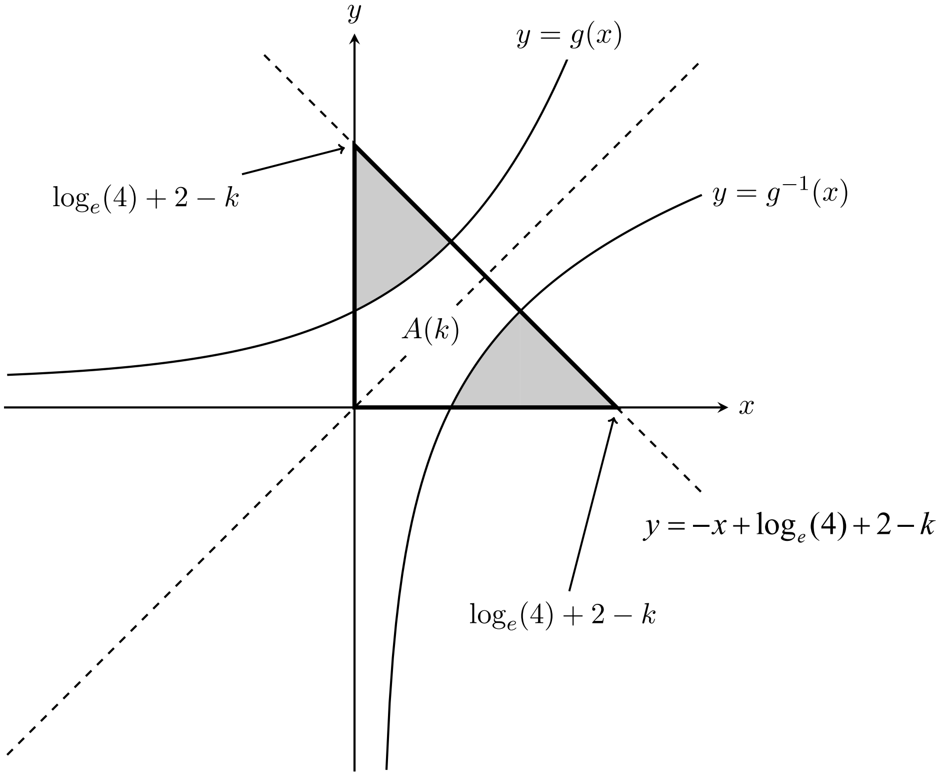
Tip

- *Read the question carefully. Two items of information are required.*

Question 5h.

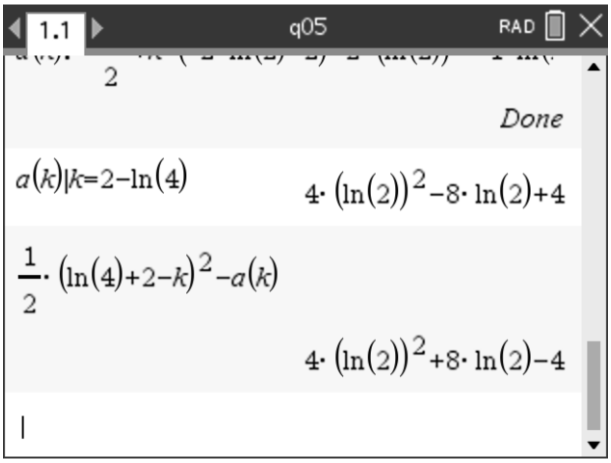
Worked solution

The total shaded area is equal to the difference in the areas of the triangle with vertices at $(0,0)$, $\left(0,\ \log_e(4)+2-k\right)$ and $\left(\log_e(4)+2-k, 0\right)$, and the area $A(k)$ found in **part f.**



The value of B is a constant.

$$B=\frac{1}{2}\left(\log_e(4)+2-k\right)^2-A(k)$$
$$=4\left(\log_e(2)\right)^2+8\log_e(2)-4$$



Mark allocation: 2 marks

- 1 mark for attempting to calculate the area
- 1 mark for the correct answer

END OF WORKED SOLUTIONS