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MATHEMATICAL METHODS
Written examination 2

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SHORT SOLUTIONS

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Section A

Q1 $\frac{\pi}{\frac{\pi}{2}} = 2$

$\therefore$ B

Q2 $\ln(x) + \ln(2x) = \ln(2x^2)$

$\therefore$ C

Q3 $n = 48$, 6 successes

$\therefore$ A

Q4 $h(2) = 0$

$\therefore$ B

Q5 $-f(x) = -f(x)$, $f(x)^2 = f(x^2)$

$\therefore$ C

Q6 $X \sim Bi(10, 0.25)$

$\Pr(X = 4) = 0.146$

$\therefore$ A

Q7 y-int of tangent is $a-1$

$a=1$

$\therefore B$

Q8 $\frac{d}{dx}(\ln(x-a)) = \frac{1}{x-a}, \quad x>a$

$\therefore E$

Q9 $h(x): (x+2)^2-4$

s.p at $(-2,-4)$

$h(-5)=5$

$y \in [-4, 5]$

$\therefore E$

Q10 $x \geq -2 \cap x \leq \frac{1}{2}$

$-2 \leq x \leq 1/2$

$\therefore D$

Q11 $3\int_0^a f(x)\,dx + 2a$

$= 3k+2a$

$\therefore A$

Q12 $\sqrt{\frac{3/5 \times 2/5}{n}} < 0.08$

$n > 37.5, \quad n \geq 38$

$\therefore D$

Q13 $\frac{f(12)-f(0)}{12-0} = 10.22$

$\therefore B$

Q14 $\frac{1}{\pi}\int_0^{\pi} \cos\left(x-\frac{\pi}{2}\right)dx = \frac{1}{\pi}\int_0^{\pi} \sin(x)\,dx$

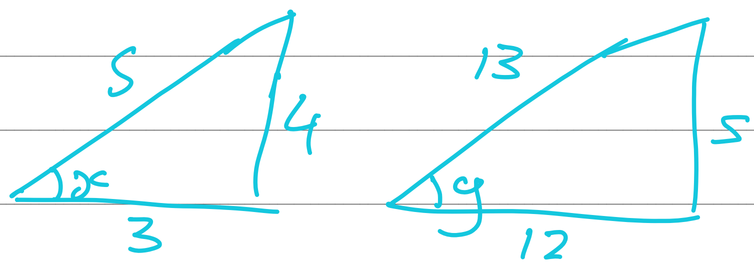
$\therefore E$

Q15 $X \sim Bi(4, 1/2)$

$\Pr(X=2 \mid X \geq 1) = 2/5$

$\therefore D$

Q16



$\sin(x) = \frac{-4}{5}$, $\cos(y) = \frac{12}{13}$

$12/13 - 4/5 = 8/65$

$\therefore$ A

Q17 $X \sim Bi(n, 0.1)$

$Pr(X \geq 2) > 0.5$

$n = 17$ $\therefore$ C

Q18 using graph screen

3 solns

$\therefore$ D

Q19 A and B are not differentiable at $x=0$

C is not continuous at $x=0$

D is not differentiable at $x=0$

$\therefore$ E

Q20 $Pr(A \cap B) = p^3$

$Pr(A' \cup B) = Pr(A') + Pr(B) - Pr(A' \cap B)$

$= 1 - p + p^2 - p^2 + p^3$

$= 1 - p + p^3$

$\therefore$ D

	A	A'	
B	p^3	$p^2 - p^3$	p^2
B'			
	p		1

Section B

Q1a. $V = x(2h-2x)(h-2x)$
$= x(50-2x)(25-2x)$
$= 2x(25-x)(25-2x)$

b. $x \in (0, \frac{25}{2})$

c. $V_{box}'(x) = 12x^2 - 300x + 1250$

d. $V_{box}'(x) = 0$
$x = -\frac{25}{6}(\sqrt{3}-3)$, $x = \frac{25}{6}(\sqrt{3}+3)$
as $x \in (0, \frac{25}{2})$
$x = -\frac{25}{6}(\sqrt{3}-3)$ cm
$V_{box}(-\frac{25}{6}(\sqrt{3}-3)) = \frac{15625\sqrt{3}}{9}$ cm^3

e. wasted part: $4x^2 = 100$ cm^2
total area: $2h^2 = 1250$ cm^2

$\therefore \frac{100}{1250} \times 100\% = 8\%$

f. i. $x \in (0, \frac{h}{2})$

ii. $V_1(x) = x(2h-2x)(h-2x)$
$= 2x(h-x)(h-2x)$
$V_1'(x) = 0$, $x = \frac{h}{6}(\sqrt{3}+3)$, $x = -\frac{h}{6}(\sqrt{3}-3)$
as $x \in (0, \frac{h}{2})$
$V_1(-\frac{h}{6}(\sqrt{3}-3)) = \frac{h^3\sqrt{3}}{9}$ cm^3

g. $V_2(x) = x(h-2x)^2, \ x \in (0, \frac{h}{2})$

$V_2'(x) = (2x-h)(6x-h) = 0$

$\therefore x = \frac{h}{6}, \ x = \frac{h}{2}$

as $x \in (0, \frac{h}{2})$

$x = \frac{h}{6}$ for max

Q2. a. width $= \frac{1}{4}$ units

b. $A_1 = \frac{1}{4}\left((\frac{1}{4})^2 + (\frac{1}{2})^2 + (\frac{3}{4})^2 + 1^2\right) = \frac{15}{32}$ units^2

c. $A_2 = \int_0^1 x^2 \, dx$

$= \frac{1}{3}$ units^2

d. $A_3 = f(-1) + f(0) + f(1) + f(2)$

$= 6 + 2 - 4 - 6 = -2$

e. $A_4 = \int_0^1 \sqrt{x} - x^2 \, dx = \frac{1}{3}$ units^2

f. $\begin{cases} y = \sqrt{x} \\ y = ax^2 \end{cases}, \ x = a^{-\frac{2}{3}}, \ x = 0$

$\therefore \int_0^a \sqrt{x} - ax^2 \, dx = \frac{1}{3}$

$a = 0.77, \ a = 1$

OR

$\int_0^{a^{-2/3}} \sqrt{x} - ax^2 \, dx + \int_{a^{-2/3}}^a ax^2 - \sqrt{x} \, dx = \frac{1}{3}$

$a = 1.13$

$\therefore a = 0.77, \ 1.00, \ 1.13$

Q3 $g(x) = \ln(x^2-1) - \ln(1-x)$

a. $x \in (-\infty, -1)$

$y \in \mathbb{R}$

b. i. $y = -x - 2$

ii. $f(-2) = 0$

$\therefore y = x + 2$

c. There are two distinct x-values which give the same y-value.

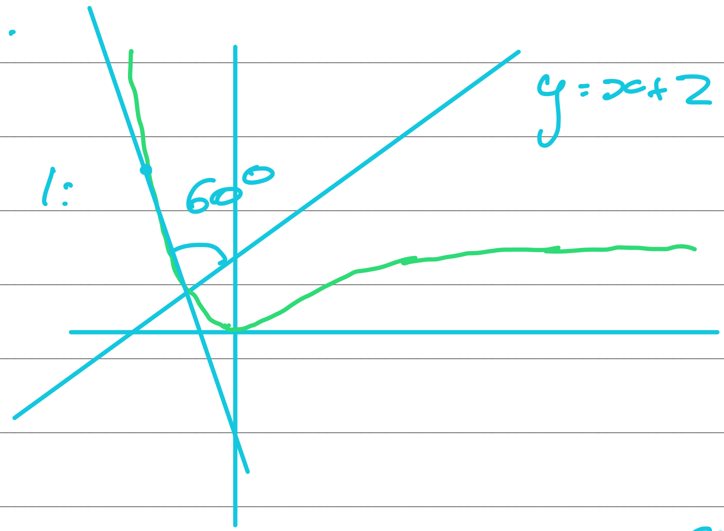
$p(x) = \frac{1}{4}$

$\Rightarrow x = \ln\left(\frac{2}{3}\right), \ln(2)$

d. $p'(a) = 2e^{-a} - 2e^{-2a}$

e.

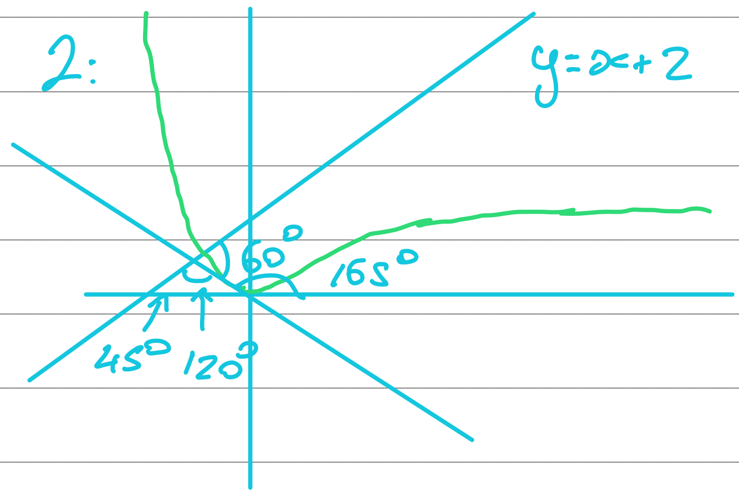
case 1:



$$\therefore \tan(60^\circ + 45^\circ) = 2e^{-a} - 2e^{-2a}$$

$$\boxed{a = -0.67}$$

case 2:



$$\therefore \tan(165^\circ) = 2e^{-a} - 2e^{-2a}$$

$$\boxed{a = -0.11}$$

f. $p(x) = x+2$

$x = -0.7504\ldots$

$$\therefore A = \int_{-2}^{-0.7504\ldots} x+2 \, dx + \int_{-0.7504\ldots}^{0} p(x) \, dx$$

$= 1.038$ units^2

Q4

a. $W \sim N(10, 0.8^2)$

$\Pr(W \geq 11) = 0.106$

b. $\Pr(W < k) = 0.8$

$\Rightarrow k = 10.7 \text{ m s}^{-1}$

c. $E(\hat{P}) = \frac{2}{25}$

$sd(\hat{P}) = \sqrt{\frac{2/25 \times (1 - 2/25)}{25}} = \frac{\sqrt{46}}{125}$

d. let $Y \sim Bi(25, \frac{2}{25})$

$\Pr(\hat{P} > 0.1) = \Pr(Y > 2.5)$

$= \Pr(Y \geq 3)$

$= 0.323$

e. 50 s^{-1}

f. $\int_0^m f(x)\,dx = 1/2$

$\Rightarrow m = 22.6 \text{ s}^{-1}$

g. $sd(X) = \sqrt{\int_0^{50} x^2 f(x)\,dx + \left(\int_0^{50} x f(x)\,dx\right)^2}$

$= 10.3 \text{ s}^{-1}$

h. $a = \frac{1}{6}$ to preserve area.

$\therefore \int_0^{30} \frac{1}{6} f(\frac{x}{6})\,dx = 1/2$

$b = 1.33, \quad a = 0.75$

Q5

a. 4π

b. -1.722

c. $h_{min} = 2\pi$

d. $a = 2$

e i. $\int \sin\left(\frac{x}{a}\right) + \cos(ax)\,dx$

$= -a\cos\left(\frac{x}{a}\right) + \frac{1}{a}\sin(ax)$

ii. $\int_0^{2a\pi} g_a(x)\,dx$

$= \left[\frac{1}{a}\sin(ax) - a\cos\left(\frac{x}{a}\right)\right]_0^{2a\pi}$

$= \frac{1}{a}\sin(2a^2\pi) - a\cos(2\pi) - \left(\frac{1}{a}\sin(0) - a\cos(0)\right)$

$= -a - (-a) = 0$

$\therefore$ area under = area over

for $x \in [0, 2\pi a]$

f. $-1 \le \sin\left(\frac{x}{a}\right) \le 1$, $-1 \le \cos(ax) \le 1$

$\Rightarrow -1-1 \le \sin\left(\frac{x}{a}\right) + \cos(ax) \le 1+1$

$\Rightarrow -2 \le \sin\left(\frac{x}{a}\right) + \cos(ax) \le 2$

g. min value $= -\sqrt{2}$ when $a = 1$