FULL NAME: _____TEACHER: _____CLASS NUMBER: ____

MELBOURNE HIGH SCHOOL

MATHEMATICAL METHODS

2021 Unit 4
SAC 2
Online Component

Wednesday 8 September
Time: 120 minutes

Instructions

Answer all questions in the spaces provided.

In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

CAS syntax will not be accepted as correct working out. When working out is required (e.g. for questions worth more than 1 mark), show your method clearly, including defining any random variables and/or how they are distributed.

Johan, the intrepid adventurer, buys all his strawberries in packets from the *Koles* supermarket. *Koles* supermarket sells large packets and small packets of strawberries. The number X of strawberries in a **large** packet is a discrete random variable with probability distribution as shown in the following table:

x	24	25	26	27
$\Pr(X = x)$	$\frac{1}{12}$	$\frac{1}{2-k}$	$\frac{5}{21}$	$\frac{1}{4}+k$

The number Y of strawberries in a **small** packet is a discrete random variable with probability distribution as shown in the following table:

y	14	15	16	17
$\Pr(Y = y)$	0.21	0.38	0.23	0.18

Question 1 (3 marks)

Find all possible values of k.

Question 2 (6 marks)

Johan buys two **small** packets of strawberries. Find the probability, correct to four decimal places, that:

a. each packet has 14 strawberries. 1 mark

b. the total number of strawberries in both packets is equal to 32. 2 marks

c. if the total number of strawberries in both packets is equal to 32 then one of the packets has 15 strawberries. 3 marks

Question 3 (5 marks)
On another occasion Johan buys seven **small** packets of strawberries. Find the probability, correct to four decimal places, that:

a. at least three of the packets have 16 strawberries. 2 marks

b. if at least two of the packets have 15 strawberries then exactly four of the packets have 15 strawberries. 3 marks

Max, another adventurer, also buys all his strawberries from the *Koles* supermarket. He buys a number of **small** packets of strawberries in readiness for a new adventure.

Question 4 (3 marks)

Find the **least** number of packets that Max could have bought if the probability that at least half of his packets have less than 16 strawberries is greater than 0.86.

A **regular** strawberry has a width of at least 2 cm. One of the **small** packets that Max buys has 15 strawberries in it and 12 of those strawberries are **regular**.

Question 5 (3 marks)

Max selects a random sample of 11 strawberries from this packet. Find the probability that more than 80% of the strawberries in the sample are **regular**.

Koles supermarket claims (truthfully) that 90% of all the strawberries it sells are **regular**.
Johan collects a random sample of 51 strawberries after buying two **large** packets.

Question 6 (10 marks)

a. Find, correct to three decimal places, the expected value and standard deviation of the sampling distribution of the proportion of **regular** strawberries in his sample. 2 marks

b. Find, correct to four decimal places, the probability that at least 85% of the strawberries in his sample are **regular**. 3 marks

c. Use the normal approximation to find the approximate probability that at least 80% of the strawberries in his sample are **regular**. Give your answer to three decimal places. 2 marks

Johan counts 47 **regular** strawberries in his sample.

d. Construct a 98% confidence interval for the proportion of **regular** strawberries sold by *Koles*. 3 marks

Johan has millions of followers on Twitter. Twitter is an online social networking and micro-blogging service that enables its users to send and read short text-based messages known as *tweets*. On any given day, the number T of tweets that Johan makes is a random variable with probability distribution given by

t	0	1	2	3
$\Pr(T = t)$	0.1	0.3	0.4	0.2

Question 7 (3 marks)

Johan tweets on both Thursday and Friday.
What is the probability that he makes a total of four tweets over these two days?

Johan and Max know that in their next adventure they will need strawberries. They also know that they will need to correctly answer problems set by the feared *Stochastica* tribe, a tribe of natives that worships probability. They have heard rumours that Tasmania Jones, a rival adventurer who recently disappeared, met this tribe, tried to answer their problems and failed (and in failing, died a horrible, horrible death). It has been said that many problems set by the *Stochastica* tribe are similar to the following two questions.

Question 8 (5 marks)

For events *A* and *B* from a sample space, $\Pr(A) = \frac{1}{5}$ and $\Pr(B) = \frac{2}{3}$.

If A' denotes the complement of *A*, calculate $\Pr(A' \mid B)$ when

a. *A* and *B* are independent. 1 mark

b. *A* and *B* are mutually exclusive. 2 marks

c. $\Pr(A \cup B) = \frac{3}{4}$. 2 marks

Question 9 (10 marks)

Let the random variable *X* be normally distributed with mean 1.5 and standard deviation 0.4.

Let *Z* be the standard normal random variable, such that $Z \sim N(0,\ 1)$.

a. Find *a* such that $\Pr(X < a) = \Pr\left(Z > \frac{a}{3}\right)$. 3 marks

b. Using the fact that, correct to three decimal places, $\Pr\left(Z > \frac{5}{3}\right) = 0.048$ and $\Pr\left(Z > \frac{5}{4}\right) = 0.106$, find *k* such that $\Pr(k < X < 2) = 0.846$. 4 marks

c. Find all possible values of *b* so that the function

$$f(x) = \begin{cases} 4bx^3 + 3x^2 - b^2 & \text{if} \quad 0 < x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

is a probability density function. 3 marks

Johan and Max come face-to-face with the *Stochastica* tribe and it turns out that the rumours about the types of questions asked are incorrect. Instead the problems are actually much harder and have a context that makes sense (rather than being abstract, context-free questions). Furthermore, the tribe also worships statistical inference, as well as probability. This is why Tasmania Jones, who was part of the old study design, was not able to answer their questions and was subsequently killed horribly.

Johan and Max are asked the following question by the *Stochastica* tribe about coconuts:
(Fortunately, they have already spent so long answering questions about another fruit)

Question 10 (13 marks)

On our island we have many coconut trees. We have found that the weights of our coconuts are normally distributed with a mean of 800g.

a. Approximately 10% of coconuts weigh more than 1kg. Use this fact to show that the standard deviation of the weights of coconuts from this tree is 156g. *Set out your reasoning clearly.* 2 marks

b. We gather all the coconuts over 1kg and give them to the tribal elders. If one of these coconuts is selected at random, then what is the probability it weighs more than 1.1kg? Answer to 4 decimal places. 2 marks

c. One particular tree produces 100 coconuts in one year. How many coconuts from this tree are expected to weigh less than 700g? Answer to the nearest coconut. 1 mark

There are many coconut trees on our fabulous island and we like to study them. A tree which produces more than 100 coconuts per year is called a "good tree," a tree which produces less than 20 coconuts per year is called a "bad tree," and all other trees are labeled "okay trees." Every year we select some trees at random and create three confidence intervals for the proportion of each type of tree, each with the same level of confidence.

Each year we choose a different number of trees and a different level of confidence to the year before.

d. Last year, in 2020, we selected 60 trees at random to be studied. 12 of them were good trees, 43 were okay trees, and 5 were bad trees. Giving your limits rounded to 4 decimal places, find a 95% confidence interval for

i. the proportion of good trees 1 mark

ii. the proportion of okay trees 1 mark

iii. the proportion of bad trees 1 mark

Unfortunately we have lost part of our data from previous years.

e. In 2019, we studied 80 trees: 19 were good, 53 were okay, and 8 were bad. However, we can't remember what level of confidence we used that year. The confidence interval for the proportion of good trees is shown below but we lost the other two confidence intervals. Write them in the table below, giving your limits to 4 decimal places, and show all your working out in the space provided. If you do not show us how you worked it out clearly, then we will assume you are just making it up and kill you like we killed Tasmania Jones. 2 marks

Good Trees	$[0.1268, 0.3482]$
Okay Trees	
Bad Trees	

f. For 2018, we can't remember the number of trees we randomly selected, but we know that we used a 92% confidence level and that there were five times as many okay trees as bad trees that year. Write in the missing confidence intervals in the table below, giving your limits to 4 decimal places. Again, though, show all your working out. If you do not show us how you worked it out clearly, then we will assume you are just making it up and kill you like we killed Tasmania Jones. 3 marks

Good Trees	[0.1200, 0.2917]
Okay Trees	
Bad Trees	

Question 11 (0 marks)
If you do not get full marks on question 10 then the tribe will kill you like they killed Tasmania Jones (i.e. horribly). How likely are you to be killed horribly? Answer as a probability from 0 to 1.

END OF QUESTION AND ANSWER BOOKLET
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