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2021

Mathematical Methods

Trial Examination 2

(2 hours)

SECTION A Multiple-choice questions

Instructions for Section A

Answer **all** questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Unless otherwise indicated, the diagrams in this examination are **not** drawn to scale.

Question 1 If the graphs of $y = f(x)$ and $y = g(x)$ intersect at (a, b), then the graphs of $y = 2f\left(\frac{x}{\sqrt{2}}\right)$ and $y = 2g\left(\frac{x}{\sqrt{2}}\right)$ intersect at

A. $\left(\sqrt{2}a, \frac{b}{2}\right)$

B. $\left(\frac{a}{2}, \frac{b}{\sqrt{2}}\right)$

C. $\left(\frac{a}{\sqrt{2}}, \frac{b}{2}\right)$

D. $\left(2a, \sqrt{2}b\right)$

E. $\left(\sqrt{2}a, 2b\right)$

Question 2 $f(x)$ is a continuous odd function and $g(x) = (f(x))^2$. $g'(0)$ is

A. 0

B. $(f'(0))^2$

C. 1

D. $2f'(0)$

E. undefined

Question 3 $y = f(x)$ cuts the x-axis at $x = a$ and $x = b$ only, and $y = af\left(\frac{x}{b}\right)$ cuts the x-axis at $x = \alpha$ and $x = \beta$, where $a, b, \alpha, \beta \in R \setminus \{0\}$, $b > a$ and $\beta > \alpha$. The value of $\beta - \alpha$ is

A. $b - a$

B. $ab - a^2$

C. $b^2 - ab$

D. $b^2 - a^2$

E. $b + a$

Question 4 $y = a^{\log_b x}$ where $a, b \in R^+$, **cannot** be written as

A. $\log_a(y - x) = \log_b(b - a)$

B. $x = y^{\log_a b}$

C. $\log_a y = \log_b x$

D. $y = x^{\log_b a}$

E. $\log_y x = \log_a b$

Question 5 The tangent to $f(x)$ at $x = a$ and the tangent to $f^{-1}(x)$ at $x = f(a)$ intersect at (α, β) where $\alpha, \beta \in R \setminus \{a\}$. Which one of the following statements is necessary true?

A. $\alpha > \beta$

B. $\alpha > a$

C. $\alpha - \beta = 0$

D. $\beta > a$

E. $\beta > \alpha$

Question 6 $f(x)$ is a linear function, $f(1) = a$, $f(3) = b$ and $a > b$. The average value of $f(x)$ equals the average rate of change of $f(x)$ with respect to x over the interval $[1, 3]$ when

A. $a = 0$

B. $b = 0$

C. $a > 0$

D. $b > 0$

E. $a + b > 0$

Question 7 Given $\int_a^b f(2x)dx = c$ in $[a, b]$, the value of $\int_{-2a}^{-2b} \frac{1}{2} f(-x)\, dx$ is

A. c

B. $-c$

C. $2c$

D. $-2c$

E. $4c$

Question 8 Given $g(x) = \pi x^2$ for $x \in \left[-\frac{1}{2}, 1\right]$ and $f(x) = \cos(2x)$ for $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$, the domain and range of $f(g(x))$ are respectively

A. $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $(-1, 1)$

B. $\left(-\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$, $(-1, 0)$

C. $\left(-\frac{1}{2}, \frac{1}{\sqrt{2}}\right]$, $(0, 1)$

D. $\left(\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$, $(-1, 0)$

E. $\left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $(-1, 0)$

Question 9 $f(x)$ is a quadratic function. The remainder is 3 when it is divided by $x-1$. The remainder is also 3 when it is divided by $x+1$. When $f(x)$ is divided by $x^2 - 1$, the remainder is

A. 0

B. 1

C. 2

D. 3

E. 5

Question 10 The minimum and maximum values of $\sqrt{(a\sin(nx))^2 + (b\cos(nx))^2}$ for $b > a > 0$ and $n \neq 0$ are respectively

A. $b - a$, $b + a$

B. $\sqrt{b-a}$, $\sqrt{b+a}$

C. a, b

D. $\sqrt{a}$, $\sqrt{b}$

E. a^2, b^2

Question 11 $f(x) = ax^3 + bx^2 + cx + d$ has no stationary points when

A. $3ac > b^2$

B. $c > b$

C. $4c > b^2$

D. $b \leq -2\sqrt{c}$

E. $ad > bc$

Question 12 Given $T_1\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$, $T_2\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} c \\ d \end{bmatrix}$ and $T_3\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$, $T_3T_2T_1\begin{bmatrix} x \\ y \end{bmatrix} =$

A. $\begin{bmatrix} a(x-c) \\ \frac{1}{2}b(d-y) \end{bmatrix}$

B. $\begin{bmatrix} ax-c \\ \frac{1}{2}(d-by) \end{bmatrix}$

C. $\begin{bmatrix} a(x-c) \\ -b(\frac{1}{2}y+d) \end{bmatrix}$

D. $\begin{bmatrix} a(x-c) \\ -\frac{1}{2}by-d \end{bmatrix}$

E. $\begin{bmatrix} ax-c \\ \frac{1}{2}b(d-y) \end{bmatrix}$

Question 13 $y=\log_e ax$ is the inverse of $y=e^{x-b}$. For $a>0$, their graphs intersect at only one point when

E. $ab=1$

B. $a=e^{-1}$ and $b=1$

C. $a=e$ and $b=e^{-1}$

D. $a=e$ and $b=1$

E. $a=e^{-1}$ and $b=e$

Question 14 The area of the largest square which can slide through the region between the graphs of $y=\frac{3}{4}x^2$ and $y=\log_e x$ is

A. $\frac{41}{200}$

B. $\frac{21}{100}$

C. $\frac{43}{200}$

D. $\frac{11}{50}$

E. $\frac{2}{9}$

Question 15 The probability density function of random variable X is given by

$$f(x)=\begin{cases} a\cos(2x)+b & \text{for } 0\le x\le\pi \\ 0 & \text{elsewhere} \end{cases}$$

The possible values of a and b are

A. $-\frac{1}{\pi}\le a\le\frac{1}{\pi}$ and $b<\frac{1}{4}$

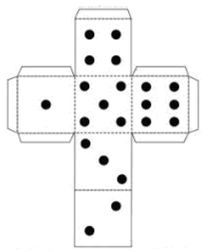
B. $-\frac{1}{\pi}\le a\le\frac{1}{\pi}$ and $b=\frac{1}{\pi}$

C. $-\frac{1}{3}\le a\le\frac{1}{3}$ and $b=\frac{1}{\pi}$

D. $-\frac{1}{3}\le a\le\frac{1}{3}$ and $b<\frac{1}{3}$

E. $-\frac{1}{3}\le a\le\frac{1}{3}$ and $b=\frac{1}{3}$

Question 16 Two dice are made from the net shown. The two dice are randomly stacked vertically on a horizontal table such that 9 square faces can be seen. The probability of getting 33 dots on the 9 faces is



A. $\frac{5}{6}$

B. $\frac{2}{3}$

C. $\frac{1}{2}$

D. $\frac{1}{3}$

E. $\frac{1}{6}$

Question 17 Which one of the following choices is a discrete random variable when 8 coins are tossed?

A. The number of heads equals the number of tails

B. The number of heads is greater than the number of tails

C. Even number of heads and even number of tails

D. The difference between the number of heads and the number of tails

E. The number of heads is two less than the number of tails

Question 18 Random variable X has a binomial distribution with $\mu = a$ and $\sigma = \sqrt{b}$.
Which one of the following statements is **NOT** true?

A. $a > 0$

B. $b > 0$

C. $b > a$

D. $2a - b > 0$

E. $a + b > 0$

Question 19 Random variable X has a probability distribution shown below.

x	1	2	3	4
$\Pr(X = x)$	p	$1.5p^2$	p^3	$0.25p^4$

The value of $\mathrm{E}(X)$ is closest to

A. 1.527

B. 1.528

C. 1.529

D. 1.654

E. 2.007

Question 20 A random sample of size 1000 is taken from a large population.
The proportion of the sample with certain attribute is 0.60.
The approximate 80% confidence interval of the proportion of the population with the certain attribute is closest to

A. $(0.57,\ 0.63)$

B. $(0.58,\ 0.62)$

C. $(0.5595,\ 0.6005)$

D. $(0.5597,\ 0.6003)$

E. $(0.5599,\ 0.6001)$

SECTION B

Instructions for Section B

Answer **all** questions in the spaces provided.
In all questions where a numerical answer is required, an exact value must be given unless otherwise stated.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this examination are **not** drawn to scale.

Question 1 (13 marks)
In Figure 1, AB is a chord of length a units, AC is the diameter of the semi-circle.
The semi-circle has its centre at O and radius of 1 unit.
Circle C_1 with centre P and radius r is the largest possible circle in the segment above AB.
Circle C_2 with centre Q and radius R touches AB, AC and the semi-circle.

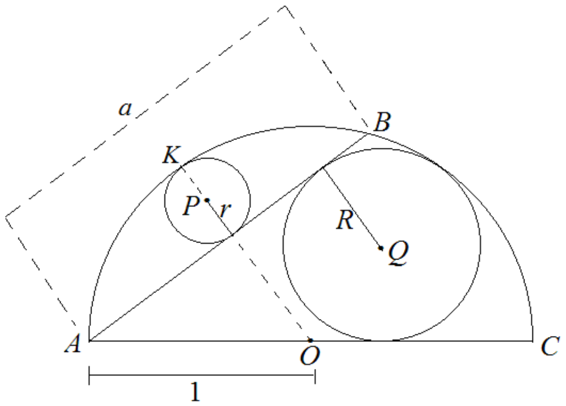


Figure 1

a. Show that $r = \frac{1}{2}\left(1 - \frac{\sqrt{4-a^2}}{2}\right)$. 1 mark

Dotted lines are added to Figure 1.
Lengths x, y and z are as defined in Figure 2. $\angle ABC$ is a right angle.

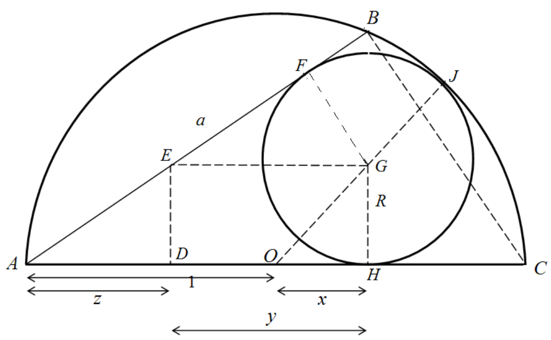


Figure 2

b. Show that $x = \sqrt{1-2R}$. 1 mark

c. Use similar triangles to show that $y = \dfrac{2R}{\sqrt{4-a^2}}$. 1 mark

d. Use similar triangles to show that $z = \dfrac{aR}{\sqrt{4-a^2}}$. 1 mark

e. Study Figure 2 and express x in terms of y and z. 1 mark

f. Solve the equation $\sqrt{1-2R} = \dfrac{aR}{\sqrt{4-a^2}} + \dfrac{2R}{\sqrt{4-a^2}} - 1$ to show that $R = 2\left(\sqrt{\dfrac{2-a}{2+a}} - \dfrac{2-a}{2+a}\right)$.

Show working. 3 marks

g. Find the value of r when R is a maximum. 1 mark

h. Find the total area A of circles C_1 and C_2 in terms of a. 1 mark

i. Find the maximum value of A and the corresponding value of a for $a \geq \frac{1}{2}$.
Correct answer to 3 decimal places.

2 marks

j. Find the minimum value of A for $a \geq \frac{1}{2}$. Correct answer to 3 decimal places. 1 mark

Question 2 (11 marks)

Consider $f(x) = A(x)\cos\left(\frac{\pi x}{2}\right)$.

a. Find the period of $f(x)$ when amplitude $A(x) = 1$, a constant. 1 mark

b. Describe the effects on $f(x)$ when $A(x)$ changes from $A(x) = 1$ to $A(x) = 1 - \frac{x}{2}$. 1 mark

Consider $f(x) = \left(1 - \frac{x}{2}\right)\cos\left(\frac{\pi x}{2}\right)$.

c. Given $g(x) = f(x - b)$, $b \in R$ and $g(x)$ is an odd function, show that $g(x) = \frac{x}{2}\cos\left(\frac{\pi x}{2}\right)$.

2 marks

d. Find the rate of change in the amplitude of $g(x)$ for $x \geq 0$. 1 mark

ei. Find general solution $(x(n), y(n))$ to simultaneous equations $y = g(x)$ and $y = \frac{x}{2}$, where n is an integer.

2 marks

eii. Find $g'(x(n))$ where $x(n)$ is the x-coordinate of the general solution in part ei. 1 mark

f. Show that the area of the regions enclosed by the graph of $g(x)$, $x = 0$, $x = 4n$ above the x-axis and the area of the regions enclosed by the graph of $g(x)$, $x = 0$, $x = 4n$ below the x-axis are equal.

1 mark

gi. Write a definite integral for finding the area of the regions enclosed by the graph of $g(x)$, $x = 0$, $x = 4n$ and $y = \frac{x}{2}$. 1 mark

gii. Use part f only and without integration, evaluate the definite integral in terms of n. 1 mark

Question 3 (13 marks)
A tank in the shape of a truncated cone is shown in Figure 3 below.
The tank is filled with water to a depth of h metres.

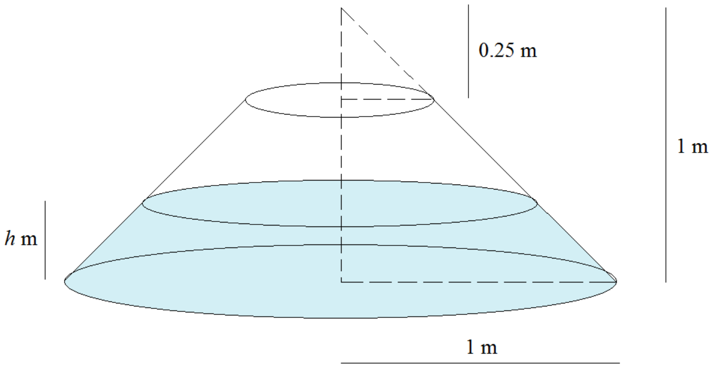


Figure 3

a. Show that the volume (m^3) of water in the tank $V = \pi h\left(1 - h + \dfrac{h^2}{3}\right)$. 2 marks

b. Find the average value of V over $[0, h_{\text{full}}]$ where h_{full} is the depth when the tank is full. 1 mark

c. Find the average value of $\dfrac{dV}{dh}$ over $[0, h_{\text{full}}]$ where h_{full} is the depth when the tank is full. 1 mark

The rate of change in V is related to the rate of change in h with respect to time t by $\dfrac{dV}{dt} = \dfrac{dV}{dh} \times \dfrac{dh}{dt}$.

Water runs into the tank at $\dfrac{\pi}{3600}$ m^3/s. Initially ($t = 0$) the tank is empty.

d. Find $\dfrac{dh}{dt}$ in terms of h. 1 mark

e. Find the time (s) required to fill the tank to a half of its depth.
Do not use calculus. Correct answer to the nearest whole number. 2 marks

f. Find the average value of $\frac{dh}{dt}$ over $[0, t_{\text{full}}]$ where t_{full} is the time when the tank is full. 1 mark

Now water runs into the tank at a controlled rate such that the water level rises at $\frac{1}{1000\pi}$ m/s.

g. Find $\frac{dV}{dt}$ in terms of h. 1 mark

h. Find the time (s) required to fill up the tank. 1 mark

i. Find the time (s) required to fill the tank to a half of its capacity.
Do not use calculus. Correct answer to the nearest whole number. 2 marks

j. Find the average value of $\frac{dV}{dt}$ over $[0, t_{\text{full}}]$ where t_{full} is the time when the tank is full. 1 mark

Question 4 (11 marks)

The graphs of $f(x)=\dfrac{x^3}{27}$ and $g(x)=a\sqrt{x-b}+c$ for $a\in R^+$ intersect at $(3,1)$ only such that $g(x)\le f(x)$.

a. Express a in terms of c. 2 marks

b. Express b in terms of c. 1 mark

c. Show that $g(x)=2\sqrt{x-2}-1$ when $c=-1$. 1 mark

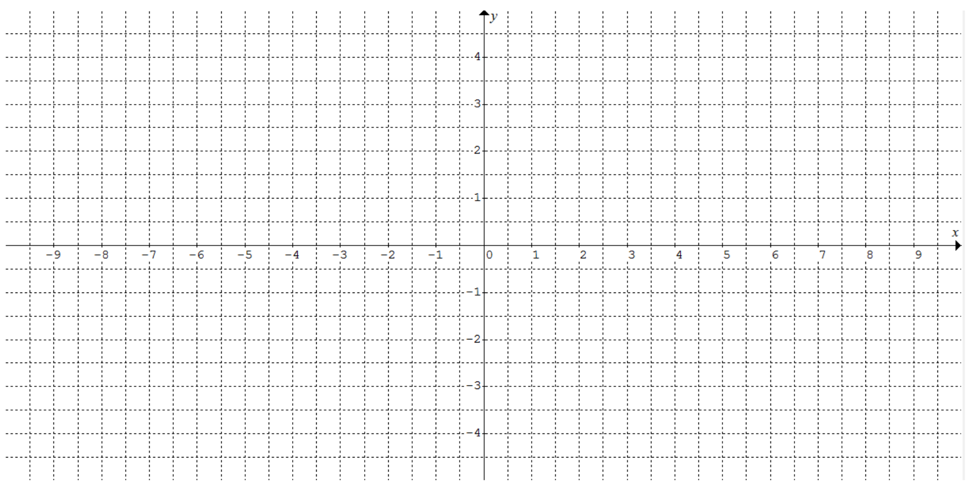
Rollercoaster track $h(x)$, $x\in[-9,9]$, consists of 3 sections and $h(x)$ is an odd function.

The middle section is modeled by $f(x)=\dfrac{x^3}{27}$ for $x\in[-3,3]$.

The section for $x\in(3,9]$ is modeled by $g(x)=2\sqrt{x-2}-1$.

d. Express $h(x)$ as a piece-wise function to model the rollercoaster track. 2 marks

e. Sketch the graph of the rollercoaster showing the coordinates of endpoints and joining points. 2marks



f. Line $y = mx + c$, $x \in [-9, 9]$, meets the graph of $h(x)$ at exactly two points.
The line is a common tangent to the graph of $h(x)$ at the two points.
Find the value of m. Correct answer to 4 decimal places. 3 marks

Question 5 (12 marks)
An orchard produces 2000000 pears. The probability that a pear from the orchard meeting supermarket requirements is 0.75.

a. Find the probability that more than 1501000 pears meet the supermarket requirements.
Round your answer to 2 decimal places 2 marks

Consider 25 random samples of 100 pears each from the orchard.

b. Using X, the number of pears meeting the supermarket requirements as the random variable, find the number of samples containing 78 to 86 pears meeting the supermarket requirements.
Correct answer to nearest whole number. 2 marks

c. Use sample proportion $\hat{P}$ and its approximate normal distribution to answer part b.
Explain any discrepancies between the two answers (parts b and c). 2 marks

There are only two supermarket requirements of pears. They are:
(1) The weight (grams) of a pear is $w \in (270, 285)$.
(2) The circumference (cm) of a pear is $c \in (22, 23)$.

The probability that a pear from the orchard meeting neither the weight requirement nor the circumference requirement is 0.05.
The probability that a pear meeting the supermarket circumference requirement only is 1.125 times the probability of meeting the supermarket weight requirement only.

d. Find the probability that a pear from the orchard meeting the supermarket circumference requirement but not the weight requirement.

3 marks

e. Find the probability that a pear from the orchard does **not** meet the circumference requirement OR does **not** meet the weight requirement.

1 mark

fi. Find the probability that a pear from the orchard meeting the circumference requirement among those pears meeting the weight requirement.

1 mark

fii. Explain whether or not the circumference requirement and the weight requirement are independent.

1 mark

End of Examination 2