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YEAR 12 *Trial Exam Paper*

2021

MATHEMATICAL METHODS

Written examination 2

Worked solutions

This book presents:

- worked solutions
- mark allocations
- tips.

SECTION A – Multiple-choice questions

Question	Answer
1	*A*
2	*D*
3	*B*
4	*E*
5	*C*
6	*D*
7	*A*
8	*A*
9	*B*
10	*E*
11	*C*
12	*D*
13	*B*
14	*B*
15	*B*
16	*B*
17	*B*
18	*C*
19	*A*
20	*E*

Question 1

Answer: A

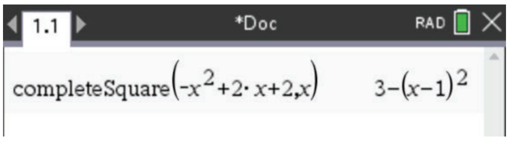
Explanatory notes

Note that

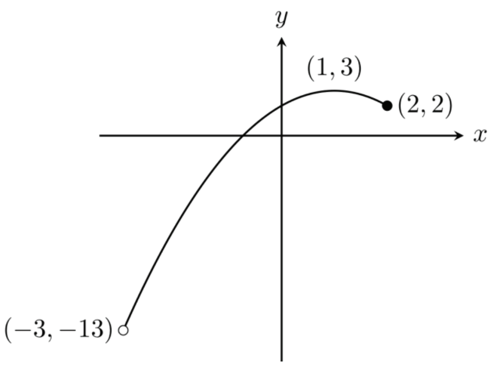
$$\begin{aligned} f(x) &= -x^2 + 2x + 2 \\ &= -\left[x^2 - 2x\right] + 2 \\ &= -\left[(x-1)^2 - 1\right] + 2 \\ &= -(x-1)^2 + 3 \end{aligned}$$

and so the turning point of the quadratic occurs at the point with coordinates $(1, 3)$.

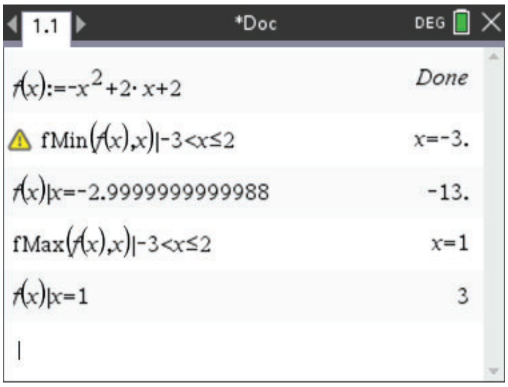
This can also be found using CAS:



The function is plotted below:



Alternatively, the minimum and maximum can be found using CAS:



The range of f is $(-13, 3]$.

Tips

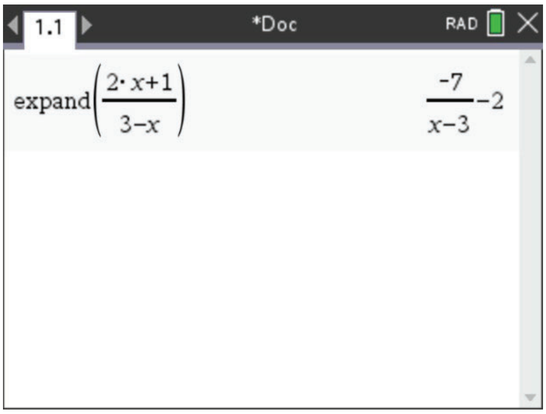
- *Consider drawing a quick sketch.*
- *fMax and fMin could be used to find the extrema of the function.*
- *Note that the minimum occurs at an open endpoint and that the maximum does not occur at an endpoint.*

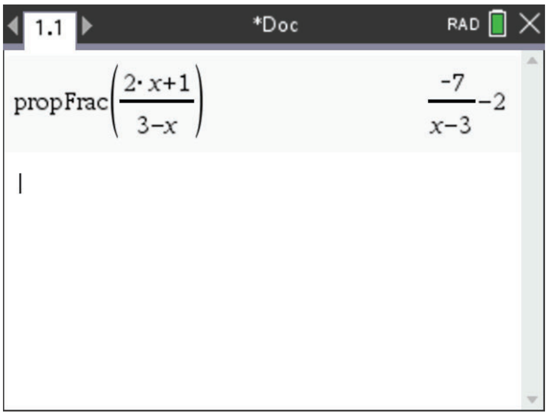
Question 2

Answer: D

Explanatory notes

Use CAS (either the *expand* or *propFrac* commands) to find that $f(x)=\dfrac{2x+1}{3-x}=-2-\dfrac{7}{x-3}$.





The asymptotes are $x=3$ (the vertical asymptote) and $y=-2$ (the horizontal asymptote).

Tip

- *Use CAS to simplify algebraic fractions. Long division can be used but is time-consuming and may lead to errors.*

Question 3

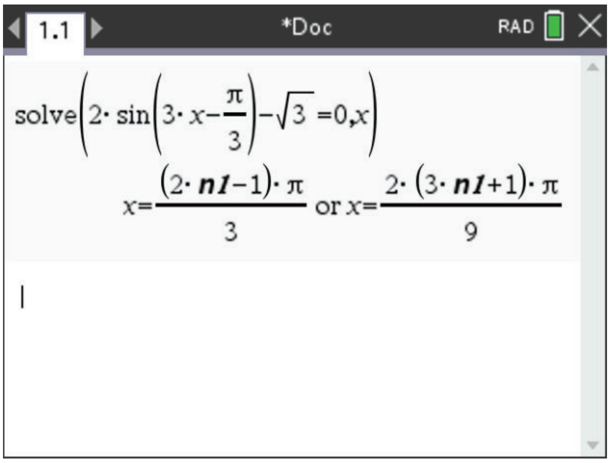
Answer: B

Explanatory notes

Note that

$$\begin{aligned}
2\sin\left(3x-\frac{\pi}{3}\right)-\sqrt{3}&=0\\
\sin\left(3x-\frac{\pi}{3}\right)&=\frac{\sqrt{3}}{2}\\
3x-\frac{\pi}{3}&=\frac{\pi}{3}+2k\pi,\frac{2\pi}{3}+2k\pi\\
3x&=\frac{2\pi}{3}+2k\pi,\pi+2k\pi\\
&=\frac{2\pi+6k\pi}{3},\frac{\pi+2k\pi}{3}\\
x&=\frac{2\pi(1+3k)}{9},\frac{\pi(2k+1)}{3}
\end{aligned}$$

This can also be found using CAS:



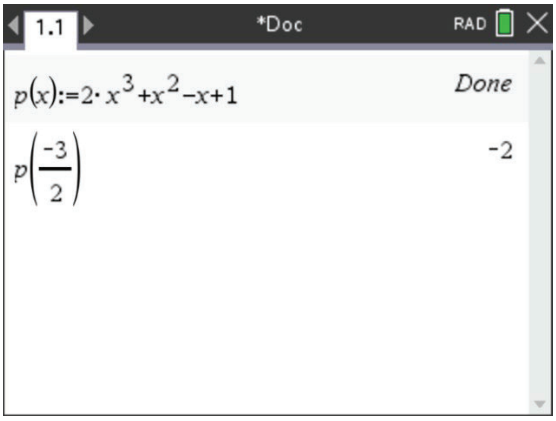
Note that the form of one solution is different to how it's given by the CAS. The CAS gives $\frac{(2\times n1-1)\times\pi}{3}$ which matches the solution $\frac{\pi(1+2k)}{3}$ in option B. The reason these are equivalent is that they both give odd numbers times $\frac{\pi}{3}$. The CAS uses even numbers, subtracting one each time $(\ldots,2-1,4-1,6-1,\ldots)$ while the written form adds one each time $(\ldots,0+1,2+1,4+1,\ldots)$.

Question 4

Answer: E

Explanatory notes

Use the remainder theorem, $p\left(-\frac{3}{2}\right) = -2$, and CAS to find the remainder:



Tip

- *Recall that the remainder when $p(x)$ is divided by $ax+b$ is $p\left(-\frac{b}{a}\right)$. This is best calculated using CAS to avoid errors.*

Question 5

Answer: C

Explanatory notes

Let $F(x)$ be an antiderivative of $f(x)$. The gradient of $F(x)$ is

- positive if $x<0$
- zero when $x=0$
- positive if $x>0$
- tends to zero as x becomes larger.

Graph C is the only graph which meets all four conditions.

Question 6

Answer: D

Explanatory notes

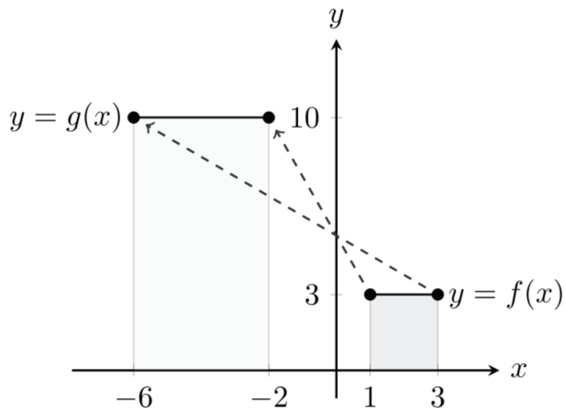
The average value of $f(x)$ on the interval $[1,3]$ is $\frac{1}{3-1}\int_1^3 f(x)dx = \frac{1}{2}\times 6 = 3$.

Suppose that $f(x)=3$ is a constant function. Then $g(x)$ is also a constant function.

Now note that point $(1, 3)$ is mapped to point $(-2, 10)$ and point $(3, 3)$ is mapped to point $(-6, 10)$:

$$T\left(\begin{bmatrix}1\\3\end{bmatrix}\right)=\begin{bmatrix}-2 & 0\\0 & 3\end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}-2\\10\end{bmatrix}$$

$$T\left(\begin{bmatrix}3\\3\end{bmatrix}\right)=\begin{bmatrix}-2 & 0\\0 & 3\end{bmatrix}\begin{bmatrix}3\\3\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}-6\\10\end{bmatrix}$$



Therefore

$$\int_{-6}^{-2} g(x)dx = 4\times 10 = 40.$$

Alternatively, consider that the new integral with the transformed function has the same shape but is twice as wide, three times as tall, reflected in the *y*-axis and moved up 1. So its area is six times the original area plus the area of a rectangle that is 1 high and 4 wide:
$A=(6\times 6)+(4\times 1)=40$.

Question 7

Answer: A

Explanatory notes

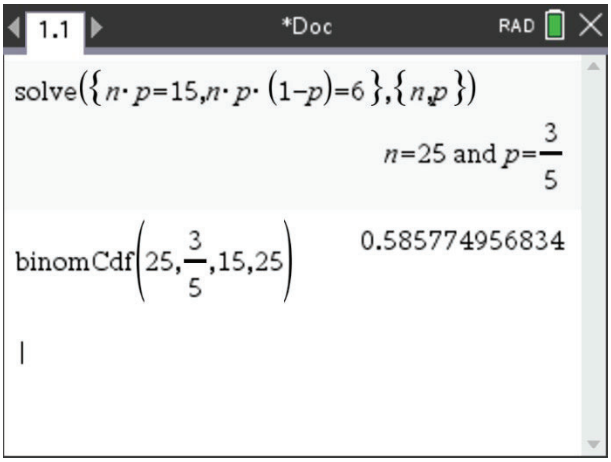
For a binomial distribution, $E(X) = np$ and $\text{Var}(X) = np(1-p)$.

Since $E(X) = 15$ and $sd(X) = \sqrt{6}$ we have two equations:

$$np = 15$$
$$np(1-p) = 6$$

These can be solved by hand or using CAS to give $n = 25$ and $p = \frac{3}{5}$.

Therefore $\Pr(X \geq 15) = 0.5858$ (correct to four decimal places).



Question 8

Answer: A

Explanatory notes

Let $Z \sim N(0,1)$.

Thus $\Pr(X > c) = \Pr\left(Z > \frac{c-75}{4}\right) = \Pr\left(Z < \frac{75-c}{4}\right)$.

Therefore

$$\frac{75-c}{4} = -1.5$$
$$75 - c = -6$$
$$c = 81$$

Question 9

Answer: B

Explanatory notes

First find the value of a.

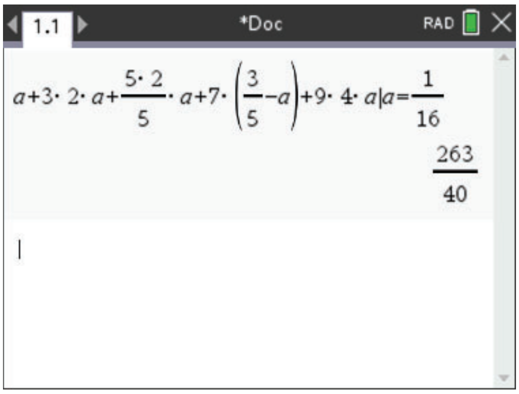
$$a+2a+\frac{2}{5}a+\frac{3}{5-a}+4a=1$$

$$a=\frac{1}{16}$$

Then

$$E(2X+1)=1\times a+3\times 2a+5\times\frac{2}{5}a+7\times\left(\frac{3}{5}-a\right)+9\times 4a$$

$$=\frac{263}{40}$$

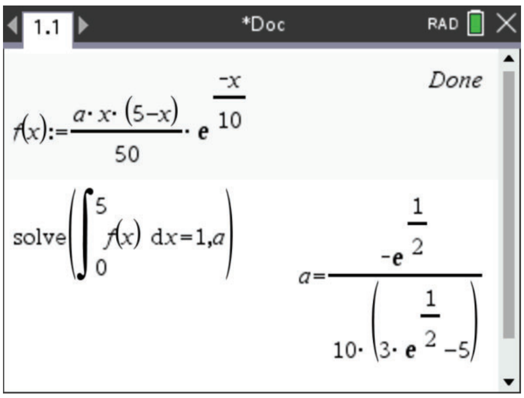


Question 10

Answer: E

Explanatory notes

Solve $\int_0^5 f(x)dx=1$ using CAS:



This is equivalent to $a=\dfrac{\sqrt{e}}{50-30\sqrt{e}}$.

Question 11

Answer: C

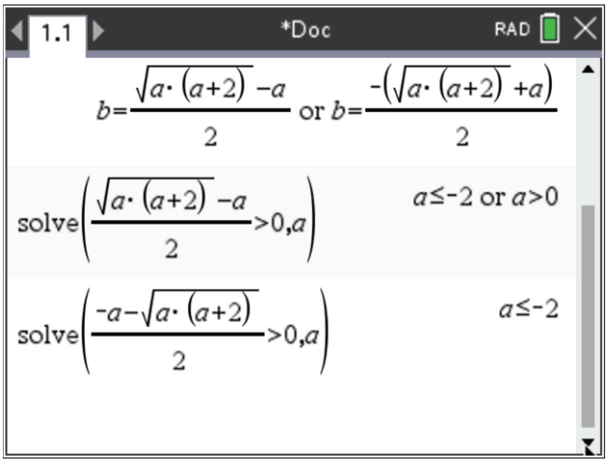
Explanatory notes

Let $b = e^x$.

Then the equation can be written $b^2 + ab - \frac{a}{2} = 0$.

This has two solutions: $b = \frac{-a \pm \sqrt{a(a+2)}}{2}$ when $a < -2$ and $a > 0$.

However, we also require that $b > 0$ since the equation $b = e^x$ has no solutions otherwise.



Checking with the CAS, it can be seen that both solutions for b are positive only when $a \le -2$. Since the solutions are equal when $a = -2$, the equation has two solutions when $a < -2$.

Question 12

Answer: D

Explanatory notes

The coordinates of the intercepts are $A(-b, 0)$ and $B\left(0, \sqrt{b}\right)$. The gradient of line AB is therefore $\frac{\sqrt{b}}{b} = \frac{1}{\sqrt{b}}$.

At any point $(a, f(a))$, the gradient of the tangent to f is $\frac{1}{2\sqrt{a+b}}$.

Equate the gradient of the tangent to f with the gradient of line AB at any point:

$$\frac{1}{2\sqrt{a+b}} = \frac{1}{\sqrt{b}} \text{ if } x = -\frac{3b}{4}.$$

Question 13

Answer: B

Explanatory notes

The period of the function is π and so $\frac{2\pi}{b} = \pi \Rightarrow b = 2$.

The amplitude is 2 and so $a = \pm 2$.

Since the y-intercept occurs when $y = -\sqrt{3}$ it could be the case that

$f(x) = 2\cos\left(2x - \frac{5\pi}{6}\right)$ or $f(x) = -2\cos\left(2x + \frac{\pi}{6}\right)$.

Only the first option matches any of the answers provided.

Question 14

Answer: B

Explanatory notes

Use the quotient and the product rules:

$$f'(x) = \frac{\left(g(x) + xg'(x)\right)h(x) - xg(x)h'(x)}{\left(h(x)\right)^2}$$

$$\begin{aligned} f'(3) &= \frac{\left(g(3) + 3g'(3)\right)h(3) - 3g(3)h'(3)}{\left(h(3)\right)^2} \\ &= \frac{(2-3)\times 5 - 3\times 2\times 2}{5^2} \\ &= \frac{-5-12}{25} \\ &= -\frac{17}{25} \end{aligned}$$

Question 15

Answer: B

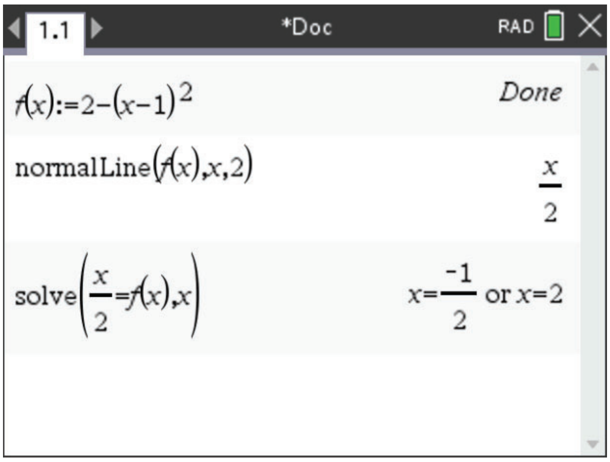
Explanatory notes

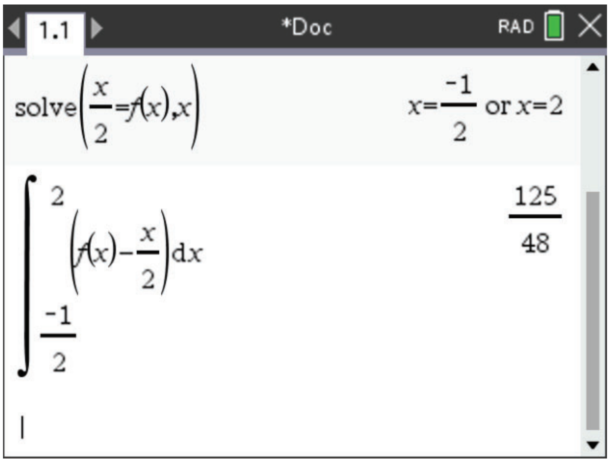
The equation of the perpendicular line is $y = \frac{x}{2}$.

The perpendicular line intersects the graph of $y = f(x)$ when $x = -\frac{1}{2}$ and $x = 2$.

Using CAS, the area of the shaded region is

$$\int_{-\frac{1}{2}}^{2} \left(f(x) - \frac{x}{2} \right) dx = \frac{125}{48}$$



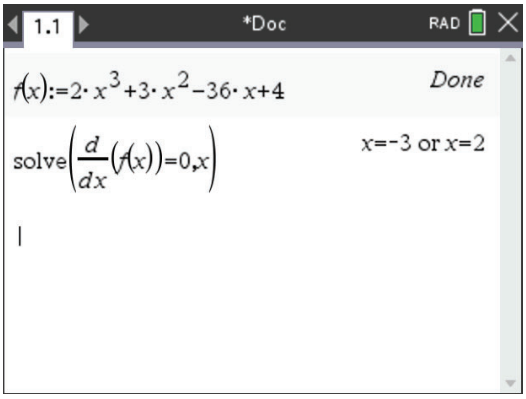


Question 16

Answer: B

Explanatory notes

The turning points of $f(x) = 2x^3 + 3x^2 - 36x + 4$ occur when $x = -3$ and $x = 2$.



Therefore f will have an inverse function if D is a subset of $(-\infty, -3]$, $[-3, 2]$ or $[2, \infty)$.

Of the options given, only $[0, 2]$ is a subset of $[-3, 2]$.

Question 17

Answer: B

Explanatory notes

The probability that all three marbles are red given that at least two of them are red is

$$\Pr(3 \text{ Red} | \text{At least } 2 \text{ Red}) = \frac{\Pr(3 \text{ Red})}{\Pr(\text{At least } 2 \text{ Red})}$$

$$= \frac{\Pr(RRR)}{\Pr(RRB, RBR, BRR, RRR)}$$

We can calculate these probabilities:

$$\Pr(RRR) = \frac{6}{10} \times \frac{5}{9} \times \frac{4}{8} = \frac{1}{6}$$

$$\Pr(RRB) = \frac{6}{10} \times \frac{5}{9} \times \frac{4}{8} = \frac{1}{6}$$

$$\Pr(RBR) = \frac{6}{10} \times \frac{4}{9} \times \frac{5}{8} = \frac{1}{6}$$

$$\Pr(BRR) = \frac{4}{10} \times \frac{6}{9} \times \frac{5}{8} = \frac{1}{6}$$

Therefore $\dfrac{\Pr(RRR)}{\Pr(RRB, RBR, BRR, RRR)} = \dfrac{\frac{1}{6}}{4 \times \frac{1}{6}} = \dfrac{1}{4}$.

Alternatively, consider that once two red marbles are removed there are an even number of red and blue marbles. All four possible outcomes with at least two red marbles must have the same probability. So the answer is one quarter.

Question 18

Answer: C

Explanatory notes

Since A and B are independent, $\Pr(A \cap B) = \Pr(A) \times \Pr(B) = ab$ and so $\Pr(A \cap B') = a - ab$.

Therefore

$$\begin{aligned}\Pr(A \cup B') &= \Pr(A) + \Pr(B) - \Pr(A \cap B') \\ &= a + 1 - b - a + ab \\ &= 1 - b + ab\end{aligned}$$

Question 19

Answer: A

Explanatory notes

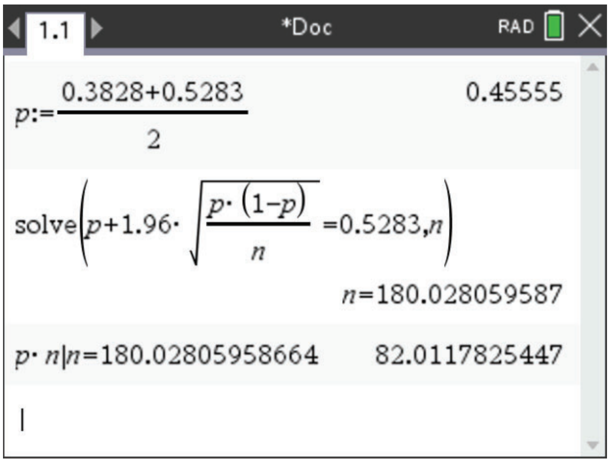
Find the value of $\hat{p}$ first:

$$\hat{p} = \frac{0.3828 + 0.5283}{2} = 0.45555.$$

Now solve the equation $\hat{p} + 1.96\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}} = 0.5283$ for n.

Thus $n = 180$ (to the nearest integer). This is the number of students in the sample.

The number of students who study chemistry from this random sample is $180 \times 0.4555 = 82$ (to the nearest integer).



Tip

- *Read the question carefully. The question asks for the number of students studying chemistry, not the number of students in the sample.*

Question 20

Answer: E

Explanatory notes

If $F(t)$ is an antiderivative of $\dfrac{1}{\sqrt{t^2+1}}$, then

$$f(x)=\int_1^x \frac{1}{\sqrt{t^2+1}}dt=F(x)-F(1).$$

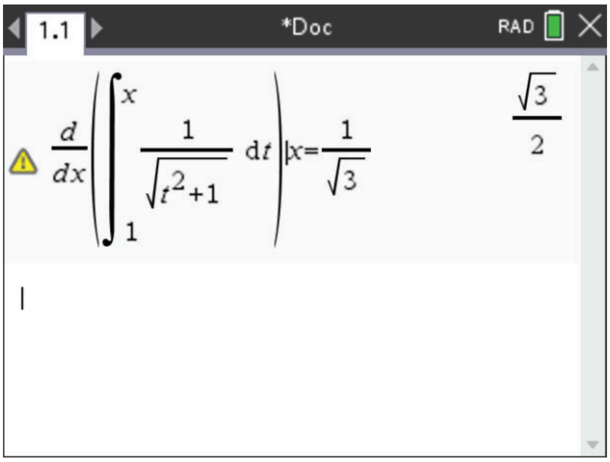
Since $F(1)$ is a constant,

$$f'(x)=F'(x)=\frac{1}{\sqrt{x^2+1}}.$$

Therefore

$$\begin{aligned} f'\left(\frac{1}{\sqrt{3}}\right)&=\frac{1}{\sqrt{\frac{1}{3}+1}}\\ &=\frac{1}{\sqrt{\frac{4}{3}}}\\ &=\frac{\sqrt{3}}{2}\end{aligned}$$

Alternatively, use CAS:



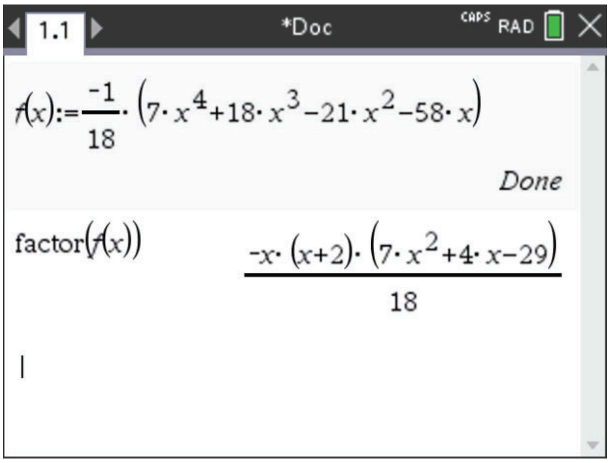
SECTION B

Question 1a.

Worked solution

Use the factor command in CAS to find that

$$f(x) = -\frac{1}{18}x(x+2)\left(7x^2+4x-29\right)$$



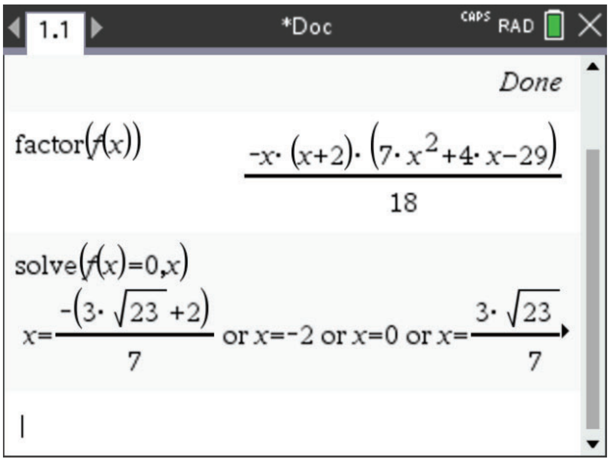
Mark allocation: 1 mark

- 1 mark for the correct answer

Question 1b.

Worked solution

Use CAS to solve the equation $f(x)=0$.



Therefore $x=0$, $x=-2$ or $x=\frac{-2\pm3\sqrt{23}}{7}$.

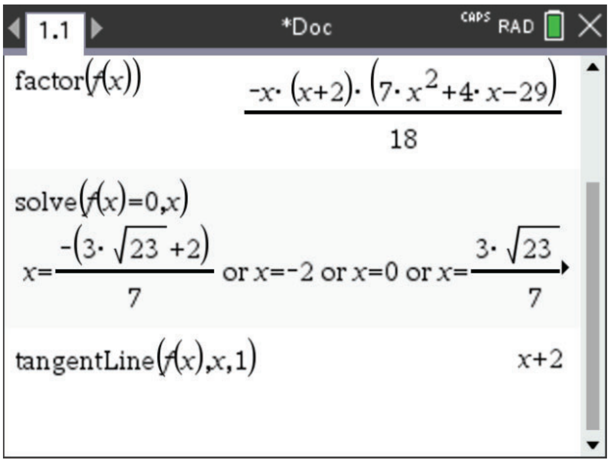
Mark allocation: 1 mark

- 1 mark for the correct answer

Question 1c.

Worked solution

Use CAS to find the equation of the tangent line:



Therefore $l_1(x) = x + 2$.

Mark allocation: 1 mark

- 1 mark for the correct answer

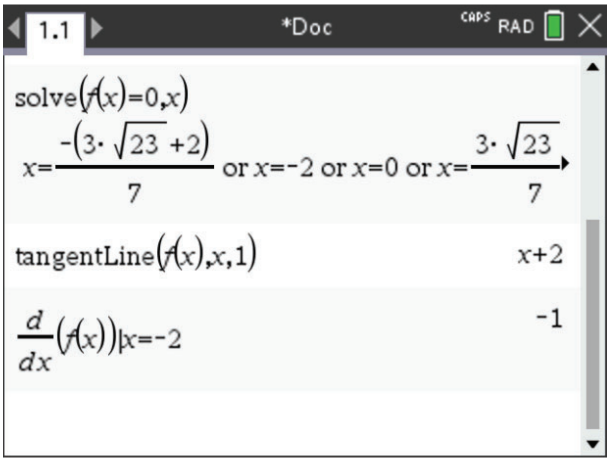
Question 1d.

Worked solution

The coordinates of B are $(-2, 0)$.

The gradient of the tangent to f at the point $B(-2, 0)$ is -1. The gradient from **part b.** is $+1$. So $m_1 \times m_2 = 1 \times -1 = -1$.

Therefore the tangent to the graph of f at A is the perpendicular to the graph of f at B.



Mark allocation: 1 mark

- 1 mark for finding that the gradient of f at $(-2, 0)$ is -1, from which the conclusion follows

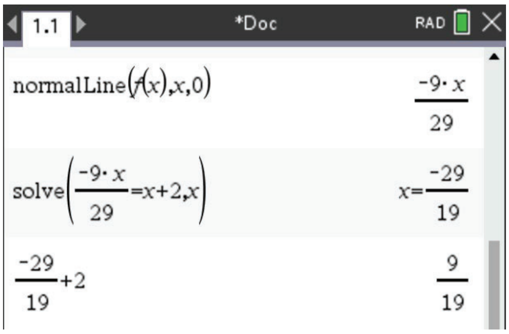
Question 1e.

Worked solution

The equation of the perpendicular to the graph of f at the origin is

$$l_2(x) = -\frac{9x}{29}.$$

The tangent and perpendicular meet when $l_1(x) = l_2(x) \Rightarrow x = -\frac{29}{19}$. Therefore, the coordinates of the point of intersection are $C\left(-\frac{29}{19}, \frac{9}{19}\right)$.



Mark allocation: 2 marks

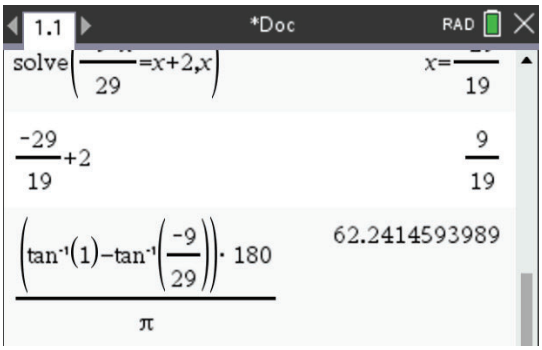
- 1 mark for finding the equation of the normal: $l_2(x) = -\frac{9x}{29}$
- 1 mark for finding the point of intersection: $C\left(-\frac{29}{19}, \frac{9}{19}\right)$

Question 1f.

Worked solution

The acute angle between lines l_1 and l_2 is

$$\begin{aligned} \tan^{-1}(1) - \tan^{-1}\left(-\frac{9}{29}\right) &= 45^\circ + 17.24^\circ \\ &= 62.24^\circ \end{aligned}$$



Mark allocation: 2 marks

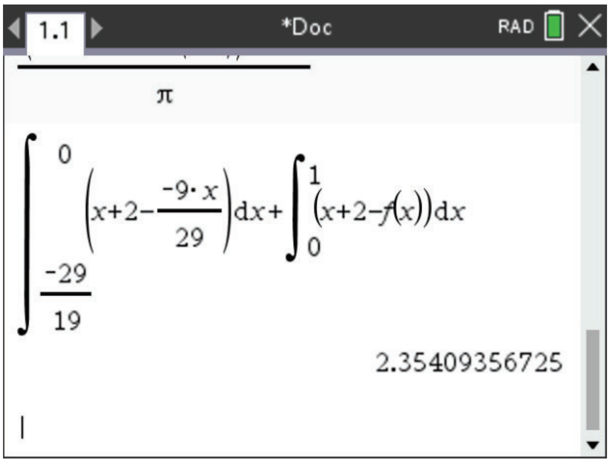
- 1 mark for using the difference of two inverse tangent functions
- 1 mark for the correct answer

Question 1g.

Worked solution

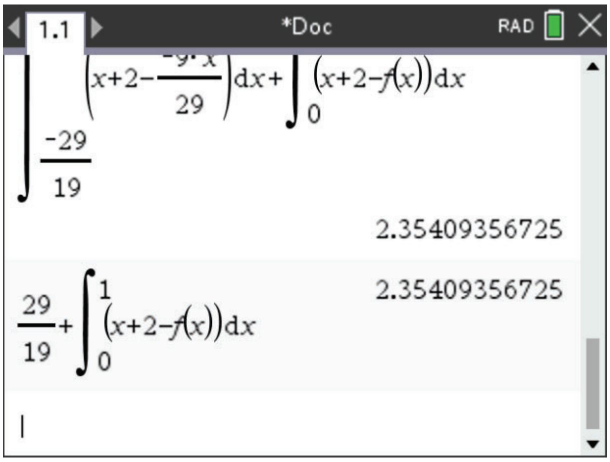
The area of the shaded region is

$$\int_{-\frac{29}{19}}^{0}\left(l_1(x)-l_2(x)\right)dx+\int_{0}^{1}\left(l_1(x)-f(x)\right)dx=2.35$$



Alternatively, the area can be described as the sum of the area of a triangle and an integral:

$$\frac{1}{2}\times 2\times\frac{29}{19}+\int_{0}^{1}\left(l_1(x)-f(x)\right)dx=2.35$$



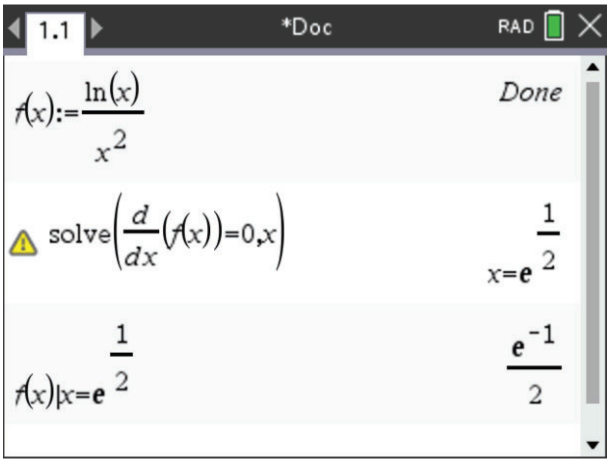
Mark allocation: 2 marks

- 1 mark for the appropriate integrals (or a triangle plus an integral)
- 1 mark for the correct answer

Question 2a.

Worked solution

Using CAS, the coordinates of the turning point are $\left(\sqrt{e}, \dfrac{1}{2e}\right)$.



Mark allocation: 1 mark

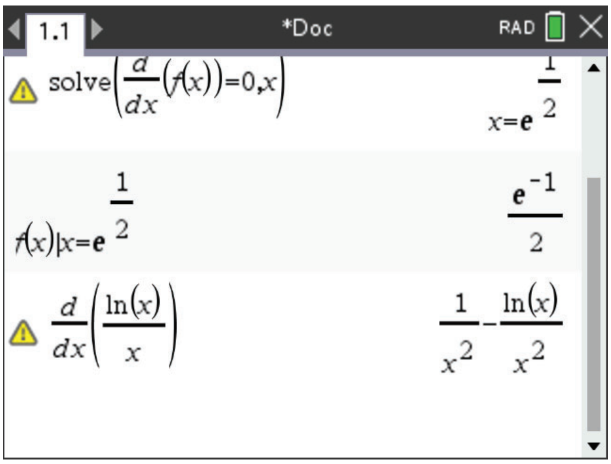
- 1 mark for the correct answer

Question 2b.i.

Worked solution

Use CAS or find the derivative of $\dfrac{\log_e(x)}{x}$ by hand:

$$\frac{d}{dx}\left(\frac{\log_e(x)}{x}\right) = \frac{1}{x^2} - \frac{\log_e(x)}{x^2}$$



Mark allocation: 1 mark

- 1 mark for the correct answer

Question 2b.ii.

Worked solution

From **part b.i.** we have

$$\frac{\log_e(x)}{x^2} = \frac{1}{x^2} - \frac{d}{dx}\left(\frac{\log_e(x)}{x}\right)$$

and so

$$\begin{aligned}\int \frac{\log_e(x)}{x^2}dx &= \int\left(\frac{1}{x^2} - \frac{d}{dx}\left(\frac{\log_e(x)}{x}\right)\right)dx \\ &= -\frac{1}{x} - \frac{\log_e(x)}{x} + c\end{aligned}$$

Mark allocation: 2 marks

- 1 mark for rearranging and attempting to integrate both sides
- 1 mark for the correct answer (the constant of integration, *c*, may be omitted)

Question 2b.iii.

Worked solution

Using the result from **part b.ii.** we have

$$\begin{aligned}\int_1^{e^2} f(x)dx &= \left[-\frac{1}{x} - \frac{\log_e(x)}{x}\right]_1^{e^2} \\ &= \left(-\frac{1}{e^2} - \frac{\log_e\left(e^2\right)}{e^2}\right) - \left(-\frac{1}{1} - \frac{\log_e(1)}{1}\right) \\ &= \left(-\frac{1}{e^2} - \frac{2}{e^2}\right) - (-1) \\ &= 1 - \frac{3}{e^2}\end{aligned}$$

Mark allocation: 2 marks

- 1 mark for substituting values from the previous result
- 1 mark for the correct answer

Question 2b.iv.

Worked solution

The average value of f on the interval $\left[1, e^2\right]$ is

$$\frac{1}{e^2-1}\int_1^{e^2} f(x)dx = \frac{1}{e^2-1}\left(1-\frac{3}{e^2}\right)$$
$$= \frac{1}{e^2-1}\left(\frac{e^2-3}{e^2}\right)$$
$$= \frac{e^2-3}{e^2\left(e^2-1\right)}$$

Mark allocation: 1 mark

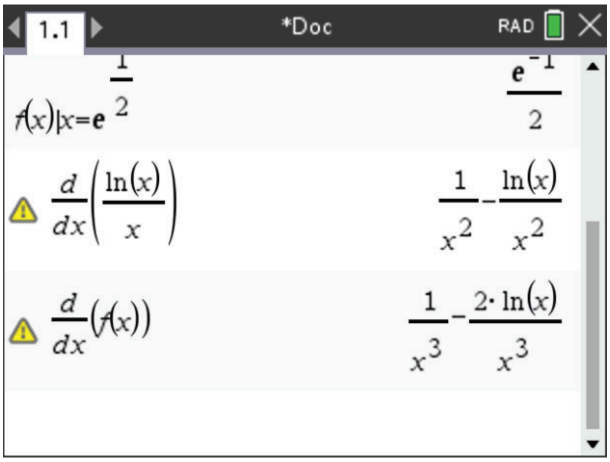
- 1 mark for the correct answer

Question 2c.i.

Worked solution

$$f'(x) = \frac{1}{x^3} - \frac{2\log_e(x)}{x^3}$$

This can be found using CAS:



Mark allocation: 1 mark

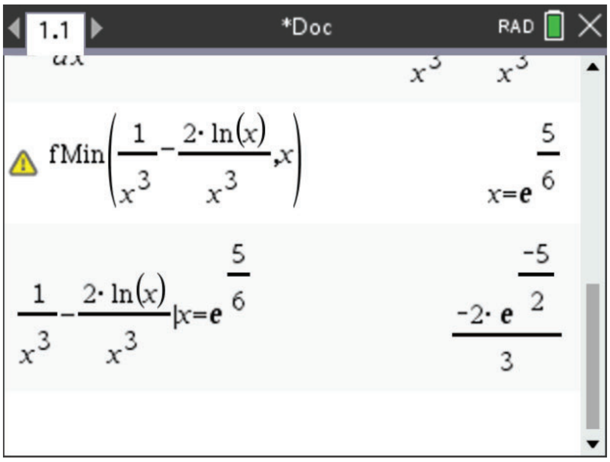
- 1 mark for the correct answer

Question 2c.ii.

Worked solution

The minimum value occurs when $x = e^{\frac{5}{6}}$ and is $-\frac{2}{3}e^{-\frac{5}{2}}$ or $-\frac{2}{3e^{\frac{5}{2}}}$.

This can be found using CAS:



Mark allocation: 1 mark

- 1 mark for the correct answer

Tip

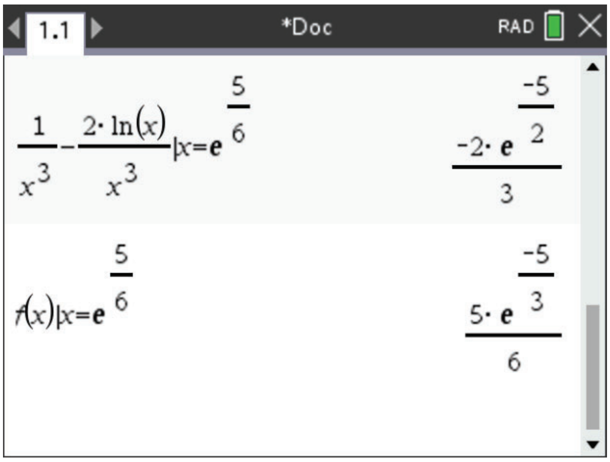
- *Read the question carefully – it is not asking you to find the minimum of f.*

Question 2d.i.

Worked solution

From the previous question we know that the gradient is a minimum when $x = e^{\frac{5}{6}}$. Therefore the coordinates of the point on the graph of f where the gradient is a minimum are $A\left(e^{\frac{5}{6}}, \frac{5}{6}e^{-\frac{5}{3}}\right)$.

This can be found using CAS:



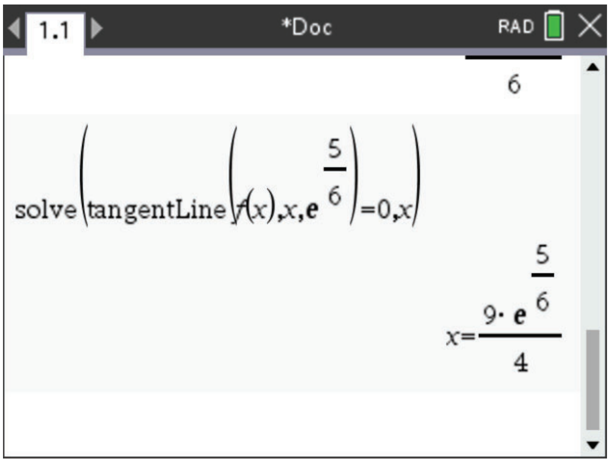
Mark allocation: 1 mark

- 1 mark for the correct answer

Question 2d.ii.

Worked solution

Use CAS to find the point where the tangent to f at A intersects the x-axis:



Therefore, $b = \frac{9}{4}e^{\frac{5}{6}}$

Mark allocation: 1 mark

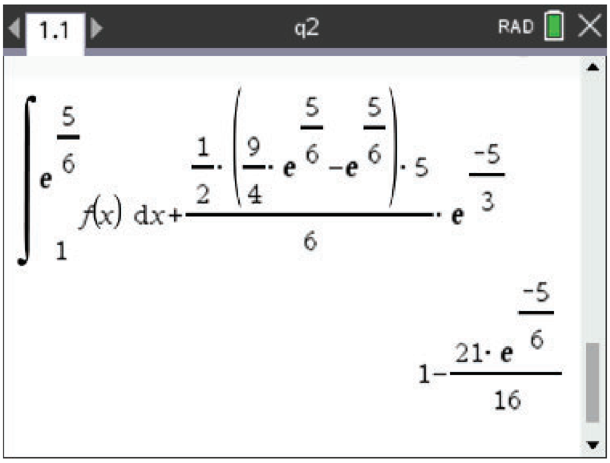
- 1 mark for the correct answer

Question 2e.

Worked solution

The area of the shaded region is made up of an area under the curve and a triangle under the tangent line:

$$\int_{1}^{e^{\frac{5}{6}}} f(x)dx + \frac{1}{2}\left(\frac{9}{4}e^{\frac{5}{6}} - e^{\frac{5}{6}}\right) \times \frac{5}{6}e^{-\frac{5}{3}} = 1 - \frac{21}{16}e^{-\frac{5}{6}}$$



Mark allocation: 3 marks

- 1 mark for determining the integral with correct terminals
- 1 mark for determining the area of the triangle
- 1 mark for the correct answer

Question 3a.

Worked solution

The amplitude is $\frac{1}{2}$ and so $a=\frac{1}{2}$.

The period is 4 and so $\frac{2\pi}{b}=4 \Rightarrow b=\frac{\pi}{2}$.

The vertical asymptote is $c=\frac{3+2}{2}=\frac{5}{2}$.

Therefore $f(x)=\frac{1}{2}\cos\left(\frac{\pi}{2}x\right)+\frac{5}{2}$.

Mark allocation: 1 mark

- 1 mark for the correct values of *a*, *b* and *c*

Question 3b.i.

Worked solution

$$\begin{aligned} d(x) &= f(x)-g(x) \\ &= \frac{1}{2}\cos\left(\frac{\pi}{2}x\right)+\frac{5}{2}-\left(-\sin(\pi x)+1\right) \\ &= \frac{1}{2}\cos\left(\frac{\pi}{2}x\right)+\sin(\pi x)+\frac{3}{2} \end{aligned}$$

Mark allocation: 1 mark

- 1 mark for showing the result

Question 3b.ii.

Worked solution

The period of $d(x)$ is 4. This can be seen from the graph, or from the lowest common multiple of the periods of the two parts of the graph.

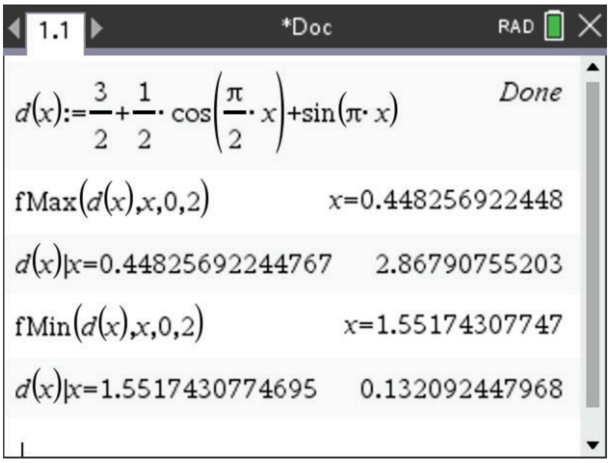
Mark allocation: 1 mark

- 1 mark for the correct answer

Question 3b.iii.

Worked solution

Use CAS to show that the range of $d(x)$ is $[0.132, 2.868]$:



Mark allocation: 2 marks

- 1 mark for each endpoint (up to 2 marks)

Question 3c.

Worked solution

The average value of $d(x)$ over one period is

$$\frac{1}{4}\int_0^4\left(\frac{3}{2}+\frac{1}{2}\cos\left(\frac{\pi}{2}x\right)+\sin(\pi x)\right)dx=\frac{1}{4}\left[\frac{3}{2}x+\frac{1}{\pi}\sin\left(\frac{\pi}{2}x\right)-\frac{1}{\pi}\cos(\pi x)\right]_0^4$$

$$=\frac{1}{4}\left(6-\frac{1}{\pi}\right)-\frac{1}{4}\left(-\frac{1}{\pi}\right)$$

$$=\frac{3}{2}$$

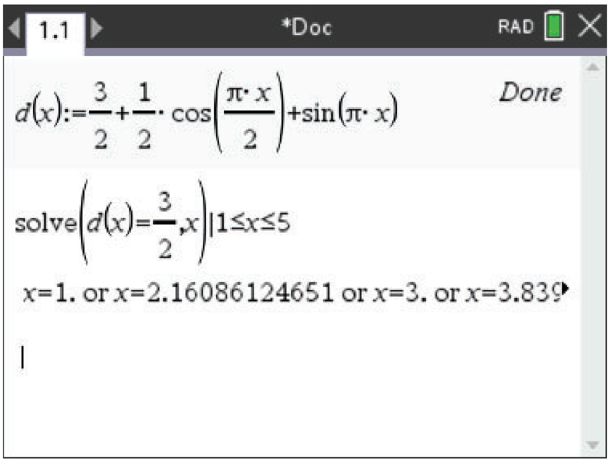
Mark allocation: 1 mark

- 1 mark for the correct answer

Question 3d.

Worked solution

Using CAS we find that $d(x) = \frac{3}{2}$ when x = 1, 2.161, 3, 3.839 and 5.



Mark allocation: 2 marks

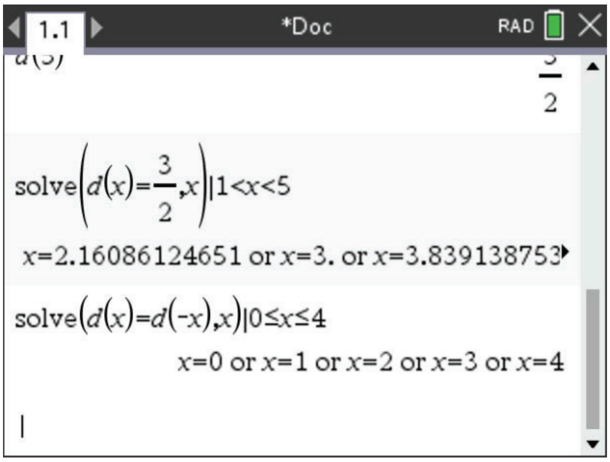
- 1 mark for the correct integer values
- 1 mark for the correctly rounded non-integer values

Question 3e.i.

Worked solution

Use CAS to calculate that the graphs of $d(x)$ and $d(-x)$ meet at points

$(0, 2)$, $\left(1, \frac{3}{2}\right)$, $(2, 1)$, $\left(3, \frac{3}{2}\right)$ and $(4, 2)$.



Mark allocation: 2 marks

- 1 mark for the coordinates $(0, 2)$, $(2, 1)$, $(4, 2)$
- 1 mark for the coordinates $\left(1, \frac{3}{2}\right)$, $\left(3, \frac{3}{2}\right)$

Question 3e.ii.

Worked solution

The area bounded by the graphs of $d(x)$ and $d(-x)$ is

$$\int_0^1 d(x)-d(-x)dx+\int_1^2 d(-x)-d(x)dx+\int_2^3 d(x)-d(-x)dx+\int_3^4 d(-x)-d(x)dx=\frac{16}{\pi}$$

Alternatively, using the symmetry of the graph

$$2\left(\int_0^1 d(x)-d(-x)dx+\int_1^2 d(-x)-d(x)dx\right)=\frac{16}{\pi}$$

Mark allocation: 2 marks

- 1 mark for appropriate integrals with terminals
- 1 mark for the correct answer

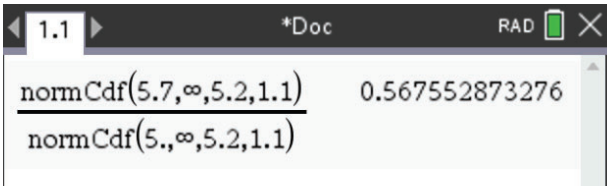
Question 4a.

Worked solution

Given that $X \sim N\left(5.2, 1.1^2\right)$, then

$$\begin{aligned}\Pr\left(X > 5.7 | X > 5.0\right) &= \frac{\Pr\left(X > 5.7 \cap X > 5.0\right)}{\Pr\left(X > 5.0\right)} \\ &= \frac{\Pr\left(X > 5.7\right)}{\Pr\left(X > 5.0\right)} \\ &= 0.5676\end{aligned}$$

This is found using CAS:



Mark allocation: 2 marks

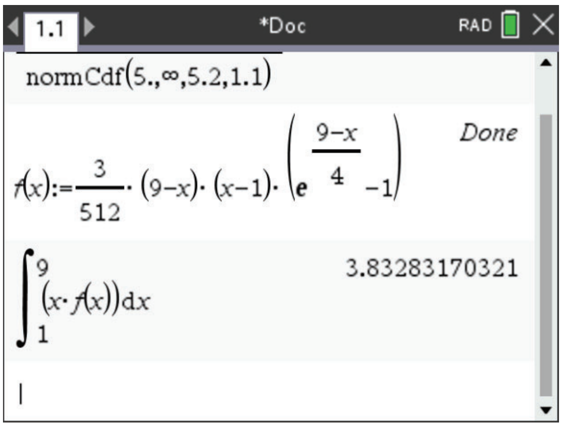
- 1 mark for correctly setting up conditional probability
- 1 mark for the correct answer

Question 4b.

Worked solution

Use CAS to find that

$$\begin{aligned}E(X) &= \int_1^9 x \times f(x)dx \\ &= 3.8328\end{aligned}$$



Mark allocation: 2 marks

- 1 mark for the correct formula for the expected value
- 1 mark for the correct answer

Question 4c.

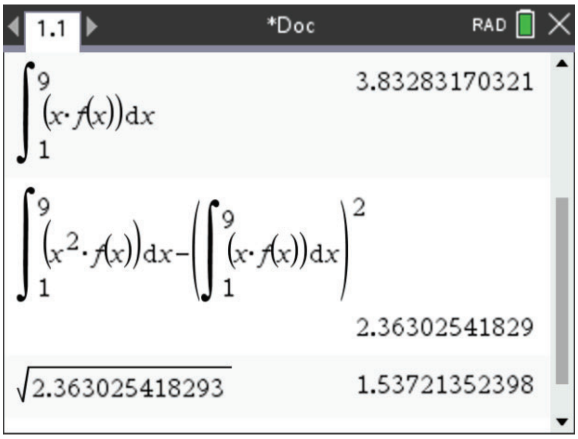
Worked solution

One approach is to use the formula for variance:

$$\text{Var}(X) = E\left(X^2\right) - \left(E(X)\right)^2$$
$$= 2.3630$$

Therefore $\text{sd}(X) = 1.5372$.

This is found using CAS:



Mark allocation: 2 marks

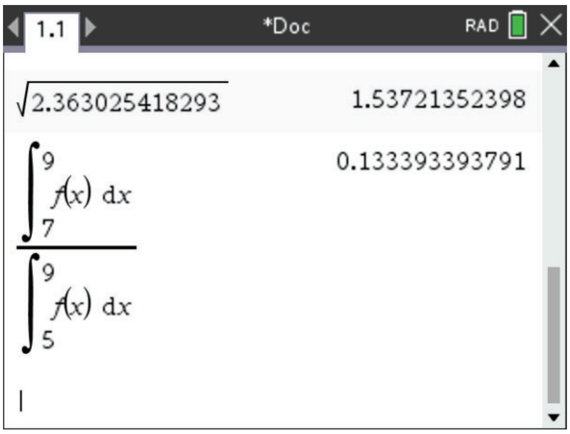
- 1 mark for finding the variance
- 1 mark for the correct answer

Question 4d.

Worked solution

$$\Pr\left(X > 7 | X > 5\right) = \frac{\Pr\left(X > 7\right)}{\Pr\left(X > 5\right)}$$
$$= 0.1334$$

This is found using CAS:



Mark allocation: 1 mark

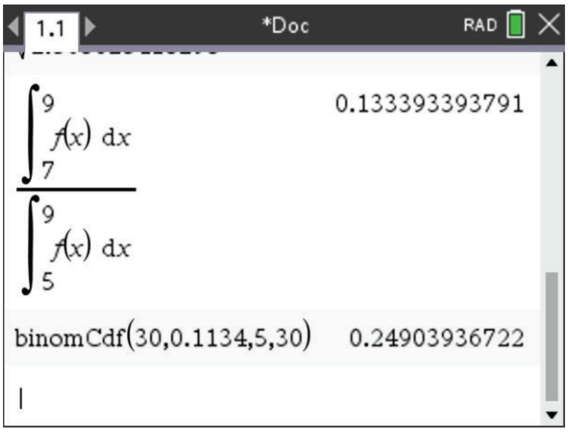
- 1 mark for the correct answer

Question 4e.

Worked solution

Let $W \sim \text{Bi}(30,\ 0.1134)$.

Then $\Pr(W \geq 5) = 0.25$, correct to two decimal places.



Mark allocation: 1 mark

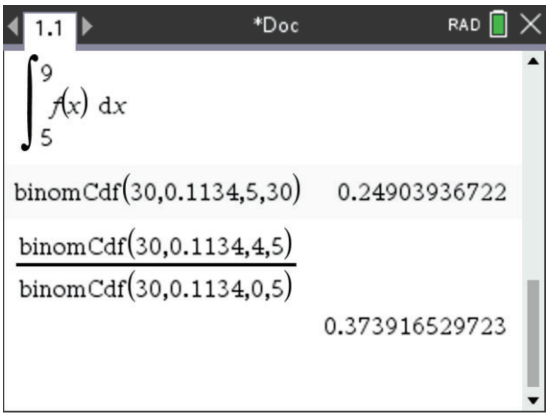
- 1 mark for the correct answer

Question 4f.

Worked solution

$$\begin{aligned}\Pr\left(\hat{P} > 0.1 | \hat{P} < 0.2\right) &= \Pr(W > 3 | W < 6) \\ &= \frac{\Pr(3 < W < 6)}{\Pr(W < 6)} \\ &= \frac{\Pr(4 \leq W \leq 5)}{\Pr(W \leq 5)} \\ &= 0.37\end{aligned}$$

correct to two decimal places.



Mark allocation: 2 marks

- 1 mark for expressing the conditional probability statement in terms of a random variable (W, for example)
- 1 mark for the correct answer

Question 4g.

Worked solution

First determine the value of $\hat{p}$:

$$\hat{p} = \frac{0.10641 + 0.29395}{2}$$
$$= 0.20018$$

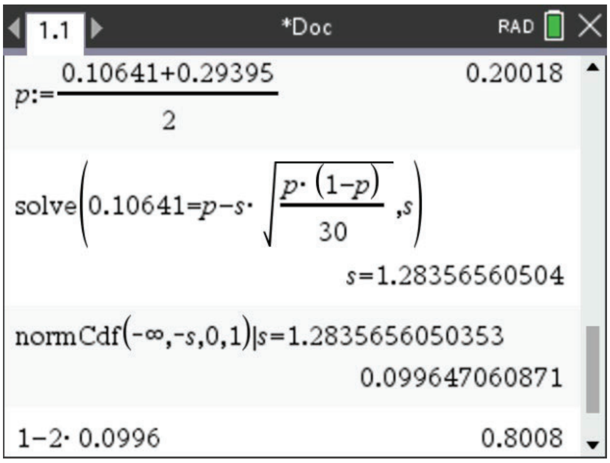
Using the left-hand value of the confidence interval we have

$$0.10641 = \hat{p} - s\sqrt{\frac{\hat{p}(1-\hat{p})}{30}}$$

giving $s = 1.2836$.

Let $Z \sim N(0,1)$. Thus $\Pr(Z < -s) = 0.0996$.

Then c = $1 - 2 \times 0.0996 = 0.8008$ = 80 to the nearest integer.



Alternatively, c = $\Pr(-s < Z < s) = 0.8007$

Therefore, c = 80 to the nearest integer.

Mark allocation: 2 marks

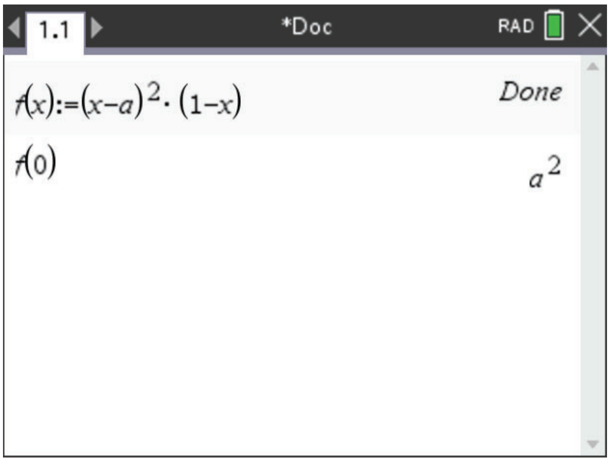
- 1 mark for attempting to find s
- 1 mark for the correct answer

Question 5a.

Worked solution

Since $f(0)=a^2$, the gradient of the line $g(x)$ is $-a^2$ and so $g(x)=-a^2x+a^2$.

The CAS screen below shows that the function $f(x)$ has been defined and the value of $f(0)$ found.



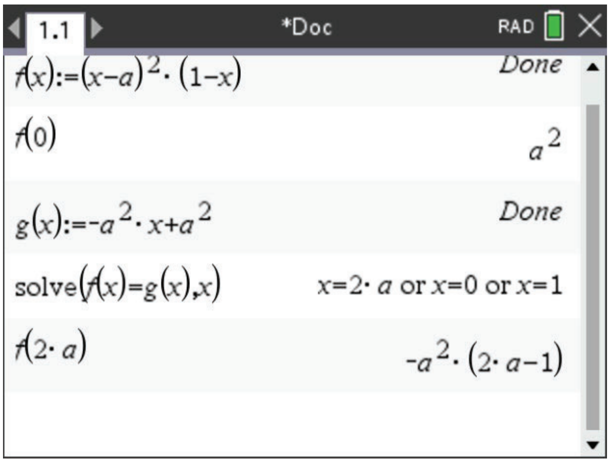
Mark allocation: 1 mark

- 1 mark for the correct answer

Question 5b.

Worked solution

$f(x)=g(x)$ when $x=0$, $x=1$ or $x=2a$. Therefore the coordinates of the point A are $A\left(2a,-a^2(2a-1)\right)=A\left(2a,a^2(1-2a)\right)$.



Mark allocation: 1 mark

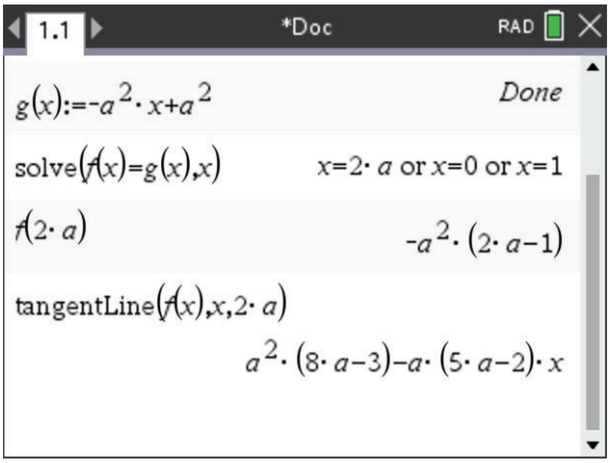
- 1 mark for the correct answer

Question 5c.i.

Worked solution

$h(x)$ is the tangent to $f(x)$ at $x = 2a$. The equation is therefore

$h(x) = a^2(8a-3) - a(5a-2)x$.



Mark allocation: 1 mark

- 1 mark for the correct answer

Question 5c.ii.

Worked solution

The *x*-intercept occurs when $x = b$. Therefore

$$h(b) = 0$$

$$b = \frac{a^2(8a-3)}{a(5a-2)}$$

$$= \frac{a(8a-3)}{5a-2}$$

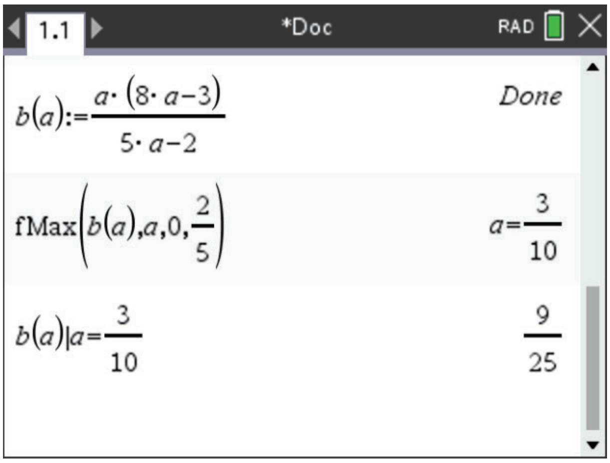
Mark allocation: 1 mark

- 1 mark for showing the required result

Question 5c.iii.

Worked solution

The maximum value of b is $\frac{9}{25}$ which occurs when $a = \frac{3}{10}$.



Mark allocation: 2 marks

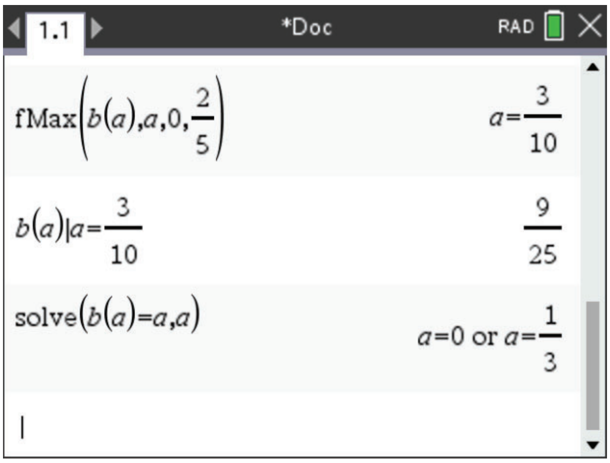
- 1 mark for calculating the value of *a*
- 1 mark for calculating the value of *b*

Question 5c.iv.

Worked solution

Solving $a = \frac{a(8a-3)}{5a-2}$ gives $a = 0$ or $a = \frac{1}{3}$.

By considering the graph of $b(a)$, it is seen that $0 \le a \le b$ if $0 \le a \le \frac{1}{3}$.



Mark allocation: 1 mark

- 1 mark for the correct answer

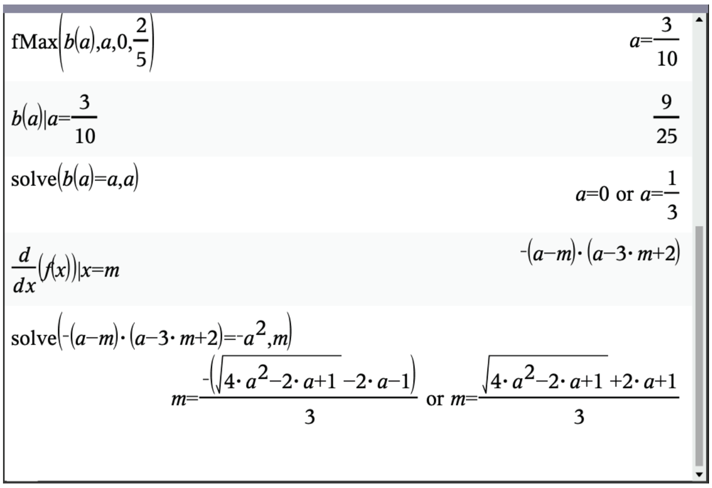
Question 5d.i.

Worked solution

The gradient of the line segment through $f(0)$ and $(1, 0)$ is $-a^2$.

$\frac{d}{dx}\big(f(x)\big) = -(a-x)(a-3x+2)$, so the gradient of the tangents to f is equal to $-a^2$ when

$$x = \frac{2a+1 \pm \sqrt{4a^2-2a+1}}{3}$$



Mark allocation: 2 marks

- 1 mark for equating $\frac{d}{dx}\big(f(x)\big)$ to $-a^2$
- 1 mark for finding $x = \frac{2a+1 \pm \sqrt{4a^2-2a+1}}{3}$

Question 5d.ii.

Worked solution

Let $g_m(x)$ be the tangent to f when $x = \frac{1}{3}\left(1 + 2a - \sqrt{4a^2 - 2a + 1}\right)$.

Let $g_n(x)$ be the tangent to f when $x = \frac{1}{3}\left(1 + 2a + \sqrt{4a^2 - 2a + 1}\right)$.

Then $\frac{g_n(0) + g_m(0)}{2} = \frac{1}{27}\left(16a^3 + 15a^2 - 6a + 2\right)$.

$\text{solve}\left(-(a-m)\cdot(a-3\cdot m+2) = -a^2, m\right)$

$$m = \frac{-\left(\sqrt{4\cdot a^2 - 2\cdot a + 1} - 2\cdot a - 1\right)}{3} \text{ or } m = \frac{\sqrt{4\cdot a^2 - 2\cdot a + 1} + 2\cdot a + 1}{3}$$

$gm(x) := \text{tangentLine}(f(x), x, m) \mid m = \frac{-\left(\sqrt{4\cdot a^2 - 2\cdot a + 1} - 2\cdot a - 1\right)}{3}$ *Done*

$gn(x) := \text{tangentLine}(f(x), x, m) \mid m = \frac{\sqrt{4\cdot a^2 - 2\cdot a + 1} + 2\cdot a + 1}{3}$ *Done*

$\frac{gm(0) + gn(0)}{2}$ $\quad \frac{16\cdot a^3 + 15\cdot a^2 - 6\cdot a + 2}{27}$

Mark allocation: 1 mark

- 1 mark for finding $\frac{g_n(0) + g_m(0)}{2} = \frac{1}{27}\left(16a^3 + 15a^2 - 6a + 2\right)$

Question 5d.iii.

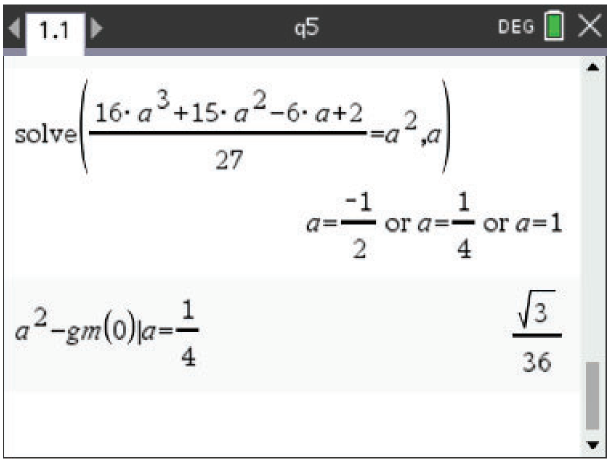
Worked solution

$d_m = d_n$ when $\dfrac{g_n(0) + g_m(0)}{2} = a^2$.

That is

$\dfrac{1}{27}\left(16a^3 + 15a^2 - 6a + 2\right) = a^2$ giving $a = \dfrac{1}{4}$.

The vertical distance is $d_m = a^2 - g_m(0) = \dfrac{\sqrt{3}}{36}$ when $a = \dfrac{1}{4}$.



Mark allocation: 2 marks

- 1 mark for finding the equation $\dfrac{1}{27}\left(16a^3 + 15a^2 - 6a + 2\right) = a^2$
- 1 mark for finding that $a = \dfrac{1}{4}$ and $d_m = \dfrac{\sqrt{3}}{36}$

END OF WORKED SOLUTIONS