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YEAR 12 *Trial Exam Paper*

2021

MATHEMATICAL METHODS

Written examination 1

Worked solutions

This book presents:

- worked solutions
- mark allocations
- tips.

Question 1a.

Worked solution

$$\frac{dy}{dx} = \cos(x) \times e^{\sin(x)}$$

Mark allocation: 1 mark

- 1 answer mark for the correct answer

Question 1b.

Worked solution

$$f'(x) = (2x+1) \times \frac{1}{x^2 + x + 1}$$

$$f'(1) = (3) \times \frac{1}{3}$$

$$f'(1) = 1$$

Mark allocation: 2 marks

- 1 answer mark for correctly evaluating the derivative $f'(x) = \dfrac{2x+1}{x^2 + x + 1}$
- 1 answer mark for the correct answer $f'(1) = 1$

Tip

- *You should expect the first question on the exam to be a differentiation question that requires you to apply the chain, product or quotient rules.*

Question 2

Worked solution

$$\begin{aligned}
\mathrm{E}(X) &= \int_0^1 x\left(-12x^3 + 12x^2\right)dx \\
&= \int_0^1 -12x^4 + 12x^3 \, dx \\
&= \left[-\frac{12}{5}x^5 + 3x^4\right]_0^1 \\
&= \left[-\frac{12}{5} + 3\right] - \left[-0 + 0\right] \\
&= \frac{3}{5}
\end{aligned}$$

Mark allocation: 2 marks

- 1 method mark for $\mathrm{E}(X) = \int_0^1 -12x^4 + 12x^3 \, dx$
- 1 answer mark for $\mathrm{E}(X) = \frac{3}{5}$

Tip

- *The probability density function is positive only between 0 and 1, so you should check that your final answer is between these values. Also, given that the probability density function has a $\cap$ type shape, you might expect that* $\mathrm{E}(X)$ *is somewhere in the middle of these two endpoints.*

Question 3a.

Worked solution

$$g\left(-\frac{1}{2}\right)=\log_2\left(-\frac{1}{2}+1\right)=\log_2\left(\frac{1}{2}\right)=-1$$

$$f\left(g\left(-\frac{1}{2}\right)\right)=f(-1)=2(-1)(-1-1)=2(-1)(-2)=4$$

Mark allocation: 2 marks

- 1 answer mark for evaluating $g\left(-\frac{1}{2}\right)=-1$
- 1 answer mark for the answer $f\left(g\left(-\frac{1}{2}\right)\right)=4$

Question 3b.

Worked solution

$g(x)$ is strictly increasing; hence, the minimum value of $g(f(x))$ will occur when $f(x)$ is at its minimum.

The minimum value of $f(x)$ occurs at $x=\frac{1}{2}$; that is, halfway between the two x-axis intercepts.

$$f\left(\frac{1}{2}\right)=2\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)=-\frac{1}{2}$$

$$g\left(-\frac{1}{2}\right)=\log_2\left(-\frac{1}{2}+1\right)=\log_2\left(\frac{1}{2}\right)=-1$$

The minimum value of $g(f(x))$ is -1.

Alternatively:

$$g\left(f\left(x\right)\right)=\log_2\left(2x(x-1)+1\right)$$
$$=\frac{1}{\log_e 2}\log_e\left(2x^2-2x+1\right)$$

$$\frac{d}{dx}g\left(f\left(x\right)\right)=\frac{1}{\log_e 2}\cdot\frac{1}{2x^2-2x+1}\cdot\left(4x-2\right)$$
$$=\frac{4x-2}{\log_e 2\left(2x^2-2x+1\right)}$$

The minimum will occur when $\frac{d}{dx}g\left(f\left(x\right)\right)=0$.

$$\frac{d}{dx}g\left(f\left(x\right)\right)=0=\frac{4x-2}{\log_e 2\left(2x^2-2x+1\right)}$$
$$4x-2=0$$
$$x=\frac{1}{2}$$

$$g\left(f\left(x\right)\right)=\log_2\left(2x(x-1)+1\right)$$
$$g\left(f\left(\frac{1}{2}\right)\right)=\log_2\left(2\cdot\frac{1}{2}\left(\frac{1}{2}-1\right)+1\right)$$
$$=\log_2\left(\frac{1}{2}\right)$$
$$=-1$$

Mark allocation: 2 marks

- 1 method mark for finding the minimum value of the parabola and substituting into $g\left(x\right)$

OR

- 1 method mark for differentiating $g\left(f\left(x\right)\right)$ and solving for $\frac{d}{dx}g\left(f\left(x\right)\right)=0$
- 1 answer mark for the solution -1

Tip

- *This question is much easier if you do not try to use differentiation to solve it.*

Question 4

Worked solution

$$2\log_4(x+6)-\log_4(x+3)=2$$
$$\log_4\left[(x+6)^2\right]-\log_4(x+3)=2$$
$$\log_4\left[\frac{(x+6)^2}{(x+3)}\right]=2$$
$$\frac{(x+6)^2}{(x+3)}=4^2=16$$
$$(x+6)^2=16(x+3)$$
$$x^2+12x+36=16x+48$$
$$x^2-4x-12=0$$
$$(x-6)(x+2)=0$$
$$x=-2,\ 6$$

Mark allocation: 3 marks

- 1 method mark for simplifying to $2\log_4(x+6)-\log_4(x+3)=\log_4\left[\frac{(x+6)^2}{(x+3)}\right]$
- 1 method mark for simplifying to $x^2-4x-12=0$ or equivalent
- 1 answer mark for $x=-2,\ 6$

Tip

- *Log laws are not on the formula sheet provided in the exam, you will need to learn them through practice.*

Question 5a.

Worked solution

$n = 12$

$\hat{p} = \frac{3}{12} = \frac{1}{4}$

$$
\begin{aligned}
z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} &= \frac{49}{25} \times \sqrt{\frac{\frac{1}{4}\left(1-\frac{1}{4}\right)}{12}} \\
&= \frac{49}{25} \times \sqrt{\frac{\frac{1}{4} \times \frac{3}{4}}{12}} \\
&= \frac{49}{25} \times \sqrt{\frac{1}{64}} \\
&= \frac{49}{200}
\end{aligned}
$$

$$\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \frac{1}{4} - \frac{49}{200} = \frac{1}{200}$$

$$\hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \frac{1}{4} + \frac{49}{200} = \frac{99}{200}$$

The approximate 95% confidence interval is $\left(\frac{1}{200}, \frac{99}{200}\right)$.

Mark allocation: 2 marks

- 1 answer mark for $z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \frac{49}{200}$ or equivalent
- 1 answer mark for the solution $\left(\frac{1}{200}, \frac{99}{200}\right)$

Tip

- *A lot of probability rules are included on the formula sheet that is included in the exam. You can use the format used for the approximate confidence interval that is shown on the formula sheet as a guide to how you should format your final answer in this question – you should write it using interval notation.*

Question 5b.

Worked solution

There are $10p$ red cubes in the bag.

$$\begin{aligned}\Pr(\text{RB})+\Pr(\text{BR}) &= \frac{10p}{9}\times\frac{8-(10p-1)}{8}+\frac{9-10p}{9}\times\frac{10p}{8}\\ &= 2\times\frac{10p}{9}\times\frac{9-10p}{8}\\ &= \frac{90p-100p^2}{36}\\ &= -\frac{25}{9}p^2+\frac{5}{2}p\end{aligned}$$

Mark allocation: 2 marks

- 1 method mark for $\frac{10p}{9}\times\frac{8-(10p-1)}{8}+\frac{9-10p}{9}\times\frac{10p}{8}$ or equivalent
- 1 answer mark for $-\frac{25}{9}p^2+\frac{5}{2}p$

Question 6a.

Worked solution

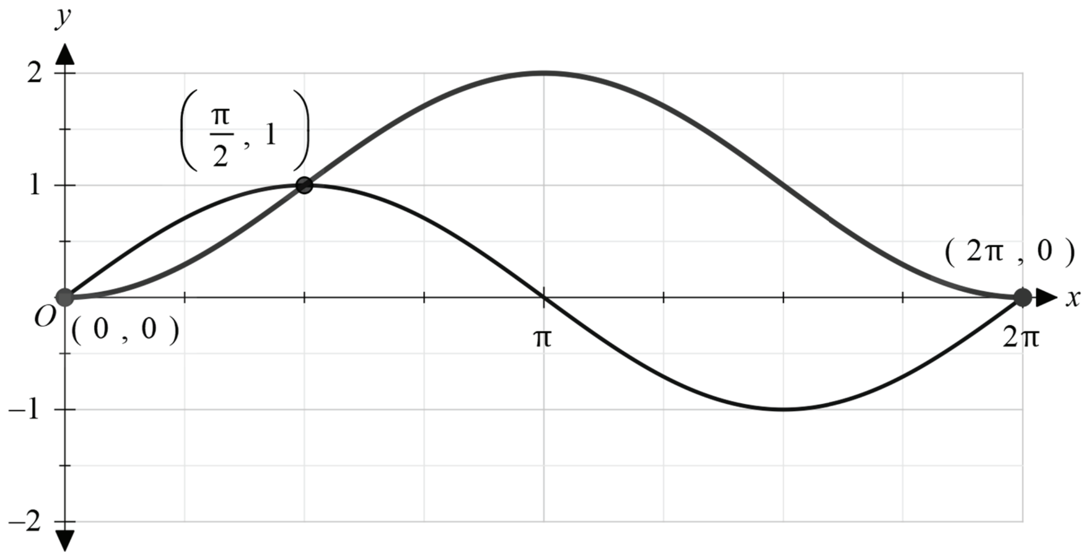
$$g(x) = f\left(x - \frac{\pi}{2}\right) + 1$$

$$= \sin\left(x - \frac{\pi}{2}\right) + 1$$

$$= -\cos(x) + 1$$

Alternatively:

Visualise the graph of g as a translation of f translated right by $\frac{\pi}{2}$ (a quarter of its period and translated up by 1).



Mark allocation: 2 marks

- 2 answer marks for a fully correct solution that has the curve correctly drawn and all three coordinates correctly labelled
- 1 answer mark for a partially correct solution, such as a correct shape with missing labels or some correctly located key features with a poorly drawn curve (e.g. the local maximum not being correctly located)

Tip

- *In graphing questions, when considering what features to label, you should pay careful attention to the features that are mentioned in the question.*

Question 6b.

Worked solution

From the graph in **part a.**, the graphs of *f* and *g* have a point of intersection at $x = \frac{\pi}{2}$.

$$A_{\text{left}} = \int_0^{\frac{\pi}{2}} \sin(x) - \left[\sin\left(x - \frac{\pi}{2}\right) + 1\right] dx$$

$$= \int_0^{\frac{\pi}{2}} \sin(x) - \left[-\cos(x) + 1\right] dx$$

$$= \int_0^{\frac{\pi}{2}} \sin(x) + \cos(x) - 1\, dx$$

$$= \left[-\cos(x) + \sin(x) - x\right]_0^{\frac{\pi}{2}}$$

$$= \left[-0 + 1 - \frac{\pi}{2}\right] - \left[-1 + 0 - 0\right]$$

$$= 2 - \frac{\pi}{2}$$

$$A_{\text{right}} = \int_{\frac{\pi}{2}}^{2\pi} -\sin(x) + \left[\sin\left(x - \frac{\pi}{2}\right) + 1\right] dx$$

$$= \int_{\frac{\pi}{2}}^{2\pi} -\sin(x) - \cos(x) + 1\, dx$$

$$= \left[\cos(x) - \sin(x) + x\right]_{\frac{\pi}{2}}^{2\pi}$$

$$= \left[1 - 0 + 2\pi\right] - \left[0 - 1 + \frac{\pi}{2}\right]$$

$$= 2 + \frac{3\pi}{2}$$

$$A_{\text{total}} = 2 - \frac{\pi}{2} + 2 + \frac{3\pi}{2}$$

$$= 4 + \pi$$

Mark allocation: 3 marks

- 1 method mark for

  $$\text{Area} = \int_0^{\frac{\pi}{2}} \sin(x) - \left[\sin\left(x - \frac{\pi}{2}\right) + 1\right] dx + \int_{\frac{\pi}{2}}^{2\pi} -\sin(x) + \left[\sin\left(x - \frac{\pi}{2}\right) + 1\right] dx$$

  or equivalent
- 1 method mark for evaluating the antiderivative

  $\int \sin(x) - \left[\sin\left(x - \frac{\pi}{2}\right) + 1\right] dx = -\cos(x) + \sin(x) - x$ or its negative
- 1 answer mark for $\text{Area} = 4 + \pi$

Question 6c.

Worked solution

The critical values of k will be where the two graphs touch each other. At these points, the graphs are tangential at their points of intersection.

$f'(x) = \cos(x)$

$g'(x) = \sin(x)$

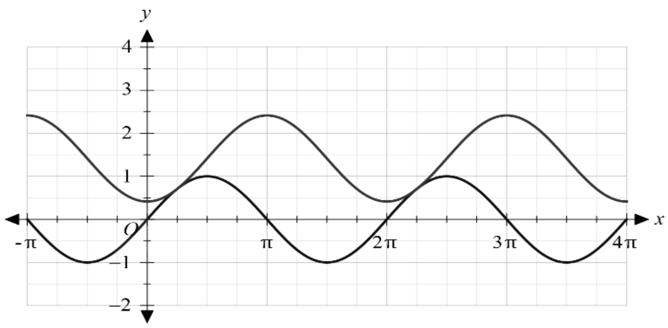
Find when the derivatives are equal (on the interval $0 \le x \le 2\pi$).

$\cos(x) = \sin(x)$

$x = \frac{\pi}{4}, \frac{5\pi}{4}$

$f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2},\ h\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4} - \frac{\pi}{2}\right) + k = -\frac{\sqrt{2}}{2} + k$

These functions only touch when $k = \sqrt{2}$. Below is a graph of the functions when $k = \sqrt{2}$.



$f\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2},\ h\left(\frac{5\pi}{4}\right) = \sin\left(\frac{5\pi}{4} - \frac{\pi}{2}\right) + k = \frac{\sqrt{2}}{2} + k$

These functions only touch when $k = -\sqrt{2}$

So *f* and *h* intersect when $-\sqrt{2} \le k \le \sqrt{2}$.

Mark allocation: 3 marks

- 1 answer mark for finding the critical values of $x = \frac{\pi}{4}, \frac{5\pi}{4}$
- 1 method mark for finding the difference in *y* values at $x = \frac{\pi}{4}, \frac{5\pi}{4}$ for each function
- 1 answer mark for the solution $-\sqrt{2} \le k \le \sqrt{2}$ or equivalent

Question 7a.

Worked solution

$f(x) = ae^x - x$, $f(1) = ae^1 - 1 = e \times a - 1$

$f'(x) = ae^x - 1$, $f'(1) = ae^1 - 1 = e \times a - 1$

Tangent is given by

$$
\begin{aligned}
y - (e \times a - 1) &= (e \times a - 1)(x - 1) \\
y &= (e \times a - 1)(x - 1) + (e \times a - 1) \\
&= (e \times a - 1)(x - 1 + 1) \\
&= (e \times a - 1)x
\end{aligned}
$$

Mark allocation: 2 marks

- 1 method mark for the calculation of the derivative
- 1 method mark for substituting $f(1)$ and $f'(1)$ to derive the given equation for the tangent

Question 7b.

Worked solution

$y = (e \times a - 1)x$ is a tangent of f.

This will be equal to $y = x$ when $e \times a - 1 = 1$, so

$$
\begin{aligned}
e \times a - 1 &= 1 \\
e \times a &= 2 \\
a &= 2e^{-1} = \frac{2}{e}
\end{aligned}
$$

Mark allocation: 1 mark

- 1 answer mark for $a = 2e^{-1} = \frac{2}{e}$

Question 7c.

Worked solution

$f(x) = ae^x - x$

$f'(x) = ae^x - 1$

Stationary point occurs when $f'(x) = 0$, so

$$\begin{aligned} ae^x - 1 &= 0 \\ ae^x &= 1 \\ e^x &= \frac{1}{a} \\ x &= \log_e\left(\frac{1}{a}\right) = -\log_e(a) \end{aligned}$$

Mark allocation: 1 mark

- 1 answer mark for $x = \log_e\left(\frac{1}{a}\right) = -\log_e(a)$

Question 7d.

Worked solution

The stationary point of f has a y-coordinate of $f\left(\log_e\left(\frac{1}{a}\right)\right)$.

$$f\left(\log_e\left(\frac{1}{a}\right)\right)=a\times e^{\log_e\left(\frac{1}{a}\right)}-\log_e\left(\frac{1}{a}\right)$$
$$=a\times\frac{1}{a}+\log_e(a)$$
$$=1+\log_e(a)$$

The stationary point (a local minimum) will be on the x-axis when $1+\log_e(a)=0$.

$$1+\log_e(a)=0$$
$$\log_e(a)=-1$$
$$a=e^{-1}$$

The stationary point will be below the axis. Hence, the graph will have two x-axis intercepts when $0<a<e^{-1}$.

Alternatively:

It might be recognised from the working out in **part a.** that $f(1)=f'(1)$.

Hence, when $f(1)=0$ the gradient will also be zero and there will be a stationary point on the x-axis.

To find the corresponding value of a, take the given tangent $y=(e\times a-1)x$ and solve to find the value of a that sets the gradient to zero.

$$e\times a-1=0$$
$$e\times a=1$$
$$a=\frac{1}{e}=e^{-1}$$

The stationary point will be below the axis. Hence, the graph will have two x-axis intercepts when $0<a<e^{-1}$.

Mark allocation: 2 marks

- 1 method mark for finding the value $a=\frac{1}{e}=e^{-1}$ that gives the stationary point on the x-axis
- 1 answer mark for the solution $0<a<e^{-1}$

Tip

- *When asked to find the range of values that a parameter can take to satisfy a particular condition, you must consider both the lower and upper boundaries of the parameter.*

Question 8a.i.

Worked solution

$f'(1)=2$

$$g'(x)=2a(x-p)$$
$$g'(1)=2a(1-p)$$
$$g'(1)=f'(1)=2$$

$2a(1-p)=2$

$$a=\frac{1}{1-p}=-\frac{1}{p-1}$$

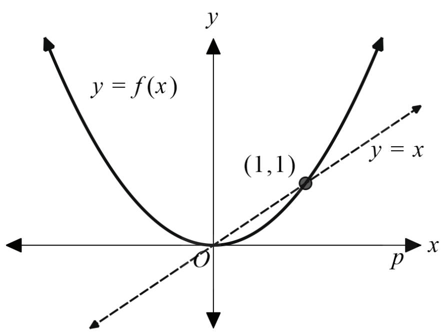
Mark allocation: 1 mark

- 1 method mark for evaluating the derivative of g and equating $g'(1)=f'(1)$ to show that $a=\dfrac{1}{1-p}$

Question 8a.ii.

Worked solution

The turning point of $g(x)$ is at the coordinate (p,p) and thus lies on the line $y=x$.



The line $y=x$ is greater than or equal to the parabola for $x\in[0,1]$.

From the original definition $p\neq 1$; therefore, $g(x)\geq f(x)$ for $p\in[0,1)$.

Note that when $p=0$, the two functions are equal.

Alternatively, from **part a.i.** the function g is the result of dilating f by a factor of $\dfrac{1}{1-p}$ from the x-axis.

For $g(x)\geq f(x)$ the vertical dilation factor must be greater than or equal to 1.

So $\dfrac{1}{1-p}\geq 1$ when $0\leq p<1$.

Alternatively, a comparison of the second derivatives at $x=1$ can be used.

Mark allocation: 2 marks

- 1 answer mark for the correct left endpoint, $p\geq 0$
- 1 answer mark for the correct right endpoint, $p<1$

Question 8b.i.

Worked solution

By substitution:

$f\left(\dfrac{q}{p}\right)=\left(\dfrac{q}{p}\right)^2=\dfrac{q^2}{p^2}$; hence, $\left(\dfrac{q}{p},\dfrac{q^2}{p^2}\right)$ is a point on f.

$$\begin{aligned}
h\left(\frac{q}{p}\right)&=\frac{q}{q-p^2}\left(\left(\frac{q}{p}\right)-p\right)^2+q\\
&=\frac{q}{q-p^2}\left(\frac{q-p^2}{p}\right)^2+q\\
&=\frac{q}{q-p^2}\cdot\frac{\left(q-p^2\right)^2}{p^2}+q\\
&=\frac{q\left(q-p^2\right)}{p^2}+q\\
&=\frac{q^2-p^2q}{p^2}+\frac{p^2q}{p^2}\\
&=\frac{q^2}{p^2}
\end{aligned}$$

Hence, $\left(\dfrac{q}{p},\dfrac{q^2}{p^2}\right)$ is a point on h.

Alternatively:

Solve $x^2=\dfrac{q}{q-p^2}(x-p)^2+q$.

$$\begin{aligned}
x^2&=\frac{q}{q-p^2}(x^2-2px+p^2)+q\\
&=\frac{q}{q-p^2}x^2-\frac{2pq}{q-p^2}x+\frac{p^2q}{q-p^2}+q\\
&=\frac{q}{q-p^2}x^2-\frac{2pq}{q-p^2}x+\frac{p^2q+q\left(q-p^2\right)}{q-p^2}
\end{aligned}$$

$$\begin{aligned}
\left(q-p^2\right)x^2&=qx^2-2pqx+q^2\\
0&=p^2x^2-2pqx+q^2\\
&=(px-q)^2\\
x&=\frac{q}{p}
\end{aligned}$$

$$f\left(\frac{q}{p}\right) = \left(\frac{q}{p}\right)^2 = \frac{q^2}{p^2}$$

Hence, $f(x)$ and $h(x)$ intersect at $\left(\frac{q}{p}, \frac{q^2}{p^2}\right)$.

Mark allocation: 2 marks

If shown by substitution:

- 1 method mark for showing that $\left(\frac{q}{p}, \frac{q^2}{p^2}\right)$ is a point on f
- 1 method mark for showing that $\left(\frac{q}{p}, \frac{q^2}{p^2}\right)$ is a point on h

If shown by solving simultaneous equations:

- 1 method mark for letting $x^2 = \frac{q}{q-p^2}(x-p)^2 + q$ and showing some steps towards its solution
- 1 method mark for showing the solution $x = \frac{q}{p}$ and then finding the *y* value of $\frac{q^2}{p^2}$

Tip

- *In 'show that' type questions, you must show every step of your method and that the final line of your response is the statement given in the question.*

Question 8b.ii.

Worked solution

The line passes through the origin and therefore takes the form $y = m \times x$.

The line passes through (p, q); therefore, $q = m \times p$ and $m = \dfrac{q}{p}$.

This gives $y = \dfrac{q}{p}x$.

Mark allocation: 1 mark

- 1 answer mark for $y = \dfrac{q}{p}x$

Question 8b.iii.

Worked solution

The point will be closer to the vertex of $f(x)$ when the horizontal distance between the vertex of $f(x)$ (the origin) and the point of intersection, $\dfrac{q}{p}$, is less than the distance between the vertex of $h(x)$ and the point of intersection, $p - \dfrac{q}{p}$.

$$\frac{q}{p} < p - \frac{q}{p} = \frac{p^2 - q}{p}$$

When $p > 0$,

$$\begin{aligned} q &< p^2 - q \\ 2q &< p^2 \\ q &< \frac{1}{2}p^2 \end{aligned}$$

When $p < 0$, $q > \dfrac{1}{2}p^2$.

Mark allocation: 2 marks

- 1 method mark for $\dfrac{q}{p} < p - \dfrac{q}{p}$ or equivalent
- 1 answer mark for $q < \dfrac{1}{2}p^2$ when $p > 0$; and $q > \dfrac{1}{2}p^2$ when $p < 0$

Question 8b.iv.

Worked solution

Change in x is 21. Change in y is 2100.

The point of intersection lies on the line $y = 100(x - 2000) + 21 = 100x - 199\,979$.

The y-coordinate of the point of intersection is
$100(2100 - 2000) + 21 = 100(100) + 21 = 10\,021$.

Alternatively:

The vertices of the two parabolas have been translated 2000 units in the positive x direction and 21 units in the positive y direction, as has the point of intersection.

Therefore, the x-coordinate of the point of intersection can be given by $\frac{q}{p} + 2000$.

$$\frac{q}{p} + 2000 = 2100$$

$$\frac{q}{p} = 100$$

The y-coordinate of the point of intersection is given by $\left(\frac{q}{p}\right)^2 + 21$.

$$\left(\frac{q}{p}\right)^2 + 21 = 100^2 + 21 = 10\,021$$

Mark allocation: 2 marks

- 1 method mark for determining the gradient between the two vertices or identifying that $\frac{q}{p} = 100$
- 1 answer mark for the y-coordinate being located at 10 021

Tip

- *The later parts of longer, multipart questions will often require you to use or apply information from earlier questions.*

END OF WORKED SOLUTIONS