THE SCHOOL FOR EXCELLENCE (TSFX)
UNIT 3 & 4 MATHEMATICAL METHODS 2020
WRITTEN EXAMINATION 2 – SOLUTIONS

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SECTION A – MULTIPLE-CHOICE QUESTIONS

1	2	3	4	5	6	7	8	9	10
D	B	A	E	E	D	A	D	C	B

11	12	13	14	15	16	17	18	19	20
C	B	B	E	A	E	D	D	C	E

Advice:

- Never leave a multiple choice question unanswered.
- There are a variety of methods for answering a multiple choice question:

  1. Calculate an answer and match the answer with an option.
  2. Eliminate wrong options (some questions are deliberately designed so that this is the only reasonable method).
  3. Guess (sometimes used in combination with Method 2).

  Identifying the most efficient method is important.

Question 1 Answer is D

Write the function in the standard form $f(x) = -2\cos\left(-\frac{x}{3} + \frac{2\pi}{5}\right) + 4$

Period $= \dfrac{2\pi}{\frac{1}{3}} = 6\pi$

Minimum value: $-2(1) + 4 = 2$

Maximum value: $-2(-1) + 4 = 6$

Range: $[2, \ 6]$

Question 2 Answer is B

Definition: $\overline{f} = \dfrac{\int_{-a/2}^{a/2} \frac{2}{x-a}\, dx}{\frac{a}{2} - \left(-\frac{a}{2}\right)} = \dfrac{\int_{-a/2}^{a/2} \frac{2}{x-a}\, dx}{a}$

Evaluate using a CAS: $\overline{f} = -\frac{1}{a}\log_e(9) \qquad = -\frac{2}{a}\log_e(3)$

Question 3 Answer is A

- $f(x) \to f(x-2) = 3 - 2\sqrt{1-3(x-2)} = 3 - 2\sqrt{7-3x}$

  Let $g(x) = 3 - 2\sqrt{7-3x}$

- $g(x) \to 2g(x) = 2\left(3 - 2\sqrt{7-3x}\right) = 6 - 4\sqrt{7-3x}$

  Let $h(x) = 6 - 4\sqrt{7-3x}$

- $h(x) \to h(-x) = 6 - 4\sqrt{7-3(-x)} = 6 - 4\sqrt{7+3x}$

Answer: $y = 6 - 4\sqrt{7+3x}$

Question 4 Answer is E

$$f(x) \xrightarrow[\text{from the } y\text{-axis}]{\text{Dilate by } \frac{1}{2}} f(2x) \xrightarrow[\text{along the } x\text{-axis}]{\text{Translate by } -2} f\left(2(x+2)\right) \xrightarrow[\text{the } x\text{-axis and reflect in the } x\text{-axis}]{\text{Dilate by } \frac{1}{2} \text{ from}} -\frac{1}{2} f\left(2(x+2)\right)$$

$$(-2, \ 4) \xrightarrow[\text{from the } y\text{-axis}]{\text{Dilate by } \frac{1}{2}} (-1, \ 4) \xrightarrow[\text{along the } x\text{-axis}]{\text{Translate by } -2} (-3, \ 4) \xrightarrow[\text{the } x\text{-axis and reflect in the } x\text{-axis}]{\text{Dilate by } \frac{1}{2} \text{ from}} (-3, \ -2)$$

Answer: $(-3, \ -2)$

Question 5 Answer is E

Complete the square: $y=-x^2+2bx-2 \qquad =-(x-b)^2+b^2-2$

Therefore, the turning point is at $\left(b,\ b^2-2\right)$

Let *D* be the distance of the turning point from the origin:

$$D=\sqrt{(b-0)^2+\left(b^2-2-0\right)^2}=\sqrt{b^2+\left(b^2-2\right)^2}$$

Use a CAS to solve $\frac{dD}{db}=0$: $b=\pm\frac{\sqrt{6}}{2}$

$$b=\pm\frac{\sqrt{6}}{2}$$
$$\Rightarrow D=\frac{\sqrt{7}}{2}$$

Question 6 Answer is D

Let $x=\frac{\sqrt{y+1}}{y-3}$ where $y=f^{-1}(x)$

Use a CAS to solve $x=\frac{\sqrt{y+1}}{y-3}$ for *y*: $y=\frac{6x^2+1\pm\sqrt{16x^2+1}}{2x^2}$

Use the fact that $f(a)=b\Rightarrow f^{-1}(b)=a$ (where *a* is a convenient element of $\mathrm{dom}(f)$) to choose between the two potential solutions for *y*.

$0\in\mathrm{dom}(f)$ and $f(0)=-\frac{1}{3}$. Therefore $\left(0,\ -\frac{1}{3}\right)$ is a point on the graph of $y=f(x)$.

Therefore, a point on the graph of $y=f^{-1}(x)$ is $\left(-\frac{1}{3},\ 0\right)$. Therefore $f^{-1}\left(-\frac{1}{3}\right)=0$.

$f^{-1}\left(-\frac{1}{3}\right)=0$ can be used to decide which solution for *y* to reject:

$$0=\frac{6\left(-\frac{1}{3}\right)^2+1\pm\sqrt{16\left(-\frac{1}{3}\right)^2+1}}{2\left(-\frac{1}{3}\right)^2}=\frac{\frac{5}{3}\pm\frac{5}{3}}{\frac{2}{9}}$$

Therefore, the negative root solution is required: $f^{-1}(x)=\frac{6x^2+1-\sqrt{16x^2+1}}{2x^2}$

$\text{dom}\left(f^{-1}\right) = \text{ran}(f)$

From a graph of $y = f(x)$ (use a CAS) it can be seen that $\text{ran}(f) = \left[f(0),\ f(2)\right]$.

Therefore: $\text{ran}(f) = \left[-\frac{1}{3},\ -\sqrt{3}\right]$

Answer: $f^{-1}: \left[-\frac{1}{3},\ -\sqrt{3}\right] \to R$, $f^{-1}(x) = \frac{6x^2 + 1 - \sqrt{16x^2 + 1}}{2x^2}$

Advice:

- Do not find the equation of an inverse function 'by hand', use a CAS to solve $x = f(y)$ for *y*.

- If there is more than one solution to $x = f(y)$ when finding the equation of the inverse function, it is often useful to use a known simple point that satisfies $y = f(x)$ to get a simple point that satisfies $y = f^{-1}(x)$, making it easier to choose between several potential solutions for *y*.

Question 7 Answer is A

$f: (a,\ b] \to R$, $f(x) = \frac{2}{x^2}$, where a and b are positive real numbers, is a decreasing function therefore its range is defined by the endpoints $f(a) = \frac{2}{a^2}$ and $f(b) = \frac{2}{b^2}$.

$a < b \Rightarrow f(b) < f(a)$

$(a,$ therefore $f(a)$ is not included.
$,b]$ therefore $f(b)$ is included.

Answer: $\left[\frac{2}{b^2},\ \frac{2}{a^2}\right)$

Question 8 Answer is D

The function has *x*-intercepts at $x = a$ and $x = b$ and an *x*-intercept that is also a turning point at $x = c$.

Model for the rule of function: $y = A(x-a)(x-b)(x-c)^2$

Since $y \to -\infty$ as $x \to +\infty$, *A* is negative. Let $A = -1$.

Therefore: $y = -(x-a)(x-b)(x-c)^2 = (x-a)(b-x)(x-c)^2$

Question 9   Answer is C

Use a CAS to solve $(2-p)x^2+3x=5x+p-1$:

$$x=\frac{-1\pm\sqrt{-p^2+3p-1}}{p-2}$$

No solution when $-p^2+3p-1<0 \qquad p^2-3p+1>0$

Question 10   Answer is B

- The area under the graph of *f* over $x\in[-1,\ 3]$ is given by $\int_{-1}^{3} f(x)\,dx$ since $f(x)>\frac{3}{2}$ for $x\in[-1,\ 3]$.

- $g(x)=x-4f(x)$

  $\Rightarrow f(x)=\frac{x-g(x)}{4}$

  $$\Rightarrow \int_{-1}^{3} f(x)\,dx \quad =\int_{-1}^{3}\frac{x-g(x)}{4}\,dx \quad =\frac{1}{4}\int_{-1}^{3} x-g(x)\,dx \quad =\frac{1}{4}\int_{-1}^{3} x\,dx-\frac{1}{4}\int_{-1}^{3} g(x)\,dx$$

  $$=1-\frac{1}{4}\int_{-1}^{3} g(x)\,dx$$

- The area between the graph of *g* and the *x*-axis over the interval $x\in[-1,\ 3]$ is eleven units.

  Therefore, the sign of *g* over $x\in[-1,\ 3]$ is needed so that this area can be expressed as a definite integral:

  $f(x)>\frac{3}{2} \qquad \Rightarrow 4f(x)>6 \qquad \Rightarrow -4f(x)<-6 \qquad \Rightarrow x-4f(x)=g(x)<x-6$

  Therefore $g(x)<0$ over the interval $-1\le x\le 3$ and so $\int_{-1}^{3} g(x)\,dx=-11$

- Therefore $\int_{-1}^{3} f(x)\,dx=1-\frac{1}{4}\int_{-1}^{3} g(x)\,dx=1-\frac{1}{4}(-11)=\frac{15}{4}$

Question 11   Answer is C

Maximum value: $p(1)+q=p+q$

Therefore, $f(x)<0\Rightarrow p+q<0\Rightarrow q<-p$

Question 12   Answer is B

Let $f(x)=\left(4x^2-2\right)^{3/2}=\left((2x)^2-2\right)^{3/2}\rightarrow\left(x^2-2\right)^{3/2}=f\left(\dfrac{x}{2}\right)$

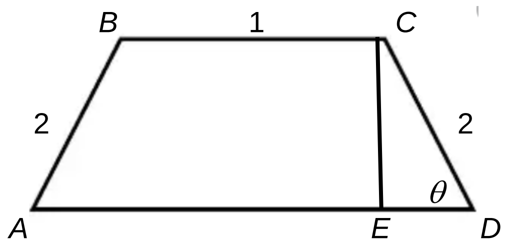
$$f(x)\xrightarrow[\text{from } y\text{-axis}]{\text{Dilation by 2}} f\left(\frac{x}{2}\right)$$

Question 13   Answer is B

- Horizontal asymptote at $y=-2$: $-C=-2\Rightarrow C=2$
- Vertical asymptote at $x=-3$:

  $A(-3)+B=0 \qquad \Rightarrow B=3A \qquad$ …. (1)
- $(0,\ -5)$ is a point on the curve: $-5=\dfrac{1}{0+B}-2 \qquad \Rightarrow B=-\dfrac{1}{3}$
- Substitute $B=-\dfrac{1}{3}$ into equation (1): $-\dfrac{1}{3}=3A \qquad \Rightarrow A=-\dfrac{1}{9}$

Question 14   Answer is E



- From the VCAA formula sheet: $A=\dfrac{1}{2}(a+b)h$
- $h=2\sin(\theta)$ (using triangle *CED*).
- $a=BC=1$
- $b=AD=BC+2ED$ (by symmetry)

  $=1+4\cos(\theta)$ (using triangle *CED*)
- Therefore:

  $A=\dfrac{1}{2}(a+b)h \qquad =\dfrac{1}{2}\left(1+4\cos(\theta)\right)2\sin(\theta) \qquad =\left(1+4\cos(\theta)\right)\sin(\theta)$
- Use a CAS to solve $\dfrac{dA}{d\theta}=0$: $\theta\approx 0.866676$ radians

Therefore, the maximum area is $\left(1+4\cos(0.866676)\right)\sin(0.866676)\approx 2.73582$

Question 15   Answer is A

$3\cos(2x) - k = 4 \qquad \Rightarrow 3\cos(2x) - 4 = k$

The solutions to $3\cos(2x) - k = 4$ are the *x*-intercepts of the points of intersection of the graphs of $y = 3\cos(2x) - 4$ and $y = k$.

From a graph of $y = 3\cos(2x) - 4$ over the interval $x \in \left(0,\ \frac{5\pi}{4}\right)$ it can be seen that there will be two intersection points when $-4 \le k < -7$.

Note:

The endpoints $x = 0$ and $x = \frac{5\pi}{4}$ are **not** included on the graph of $y = 3\cos(2x) - 4$ over the interval $x \in \left(0,\ \frac{5\pi}{4}\right)$. Therefore there is **one** intersection point when $k = -1$ and **two** intersection points when $k = -4$.

Question 16   Answer is E

$\overline{X} = np$ and $\text{sd}(X) = \sqrt{np(1-p)}$

$\overline{X} = 2\text{sd}(X) \qquad \Rightarrow np = 2\sqrt{np(1-p)}$

Use a CAS to solve for *p* under the restriction $0 < p < 1$: $p = \frac{4}{n+4}$

Use a CAS to solve $p = \frac{4}{n+4} < 0.1$: $n > 36$

Question 17   Answer is D

- Let *B* denote the random variable *number of shots that hit the bullseye*.
- *B* ~ Binomial($p = 0\cdot 3,\ n = ?$)
- The smallest value of *n* such that $\Pr(B \ge 6) > 0\cdot 9$ is required:

  $\Pr(X \ge 6) > 0\cdot 9 \Rightarrow 1 - \Pr(X \le 5) > 0\cdot 9 \Rightarrow \Pr(X \le 5) < 0\cdot 1$

  Solve this inequality using a CAS:

  1. Define the function $f(x) =$ binomcdf(X, 0.3, 5)

     Note: In this rule X represents the sample size variable. Scroll down a table of values until the first integer value of X satisfying $f(x)$ < 0.1 is found.

  2. X = 29

Question 18 Answer is D

$y = f(x) = g'(x) < 0$ on the interval $(a,\ b)$.

By definition *g* will therefore be a decreasing function on the interval $(a,\ b)$.

Question 19 Answer is C

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ -3 \end{bmatrix} = \begin{bmatrix} -x \\ -2y-3 \end{bmatrix}$$

$\Rightarrow x' = -x$ and $y' = -2y - 3$

$\Rightarrow x = -x'$ and $y = \dfrac{y'+3}{-2}$

$\Rightarrow y = f(x) \rightarrow \dfrac{y'+3}{-2} = f(-x')$

$\Rightarrow y = g(x) = -2f(-x) - 3$

$\Rightarrow \int_{-2}^{1} g(x)dx = \int_{-2}^{1} -2f(-x) - 3dx \quad = -2\int_{-2}^{1} f(-x)dx - \int_{-2}^{1} 3dx \quad = -2\int_{-1}^{2} f(x)dx - \int_{-2}^{1} 3dx$

$= -2(4) - 9$

$= -17$

Question 20 Answer is E

$\sigma^2 = 9 \Rightarrow \sigma = 3$

Substitute into the transformation formula $Z = \dfrac{X - \mu}{\sigma}$ from normal to standard normal:

$X = 13 \quad \Rightarrow Z = \dfrac{13-16}{3} = -1$

$X = 22 \quad \Rightarrow Z = \dfrac{22-16}{3} = 2$

Therefore:

$\Pr(13 < X < 22) \quad = \Pr(-1 < Z < 2) \quad = 1 - \Pr(Z < -1) - \Pr(Z > 2)$

Substitute $\Pr(Z < -1) = \Pr(Z > 1)$ (by the symmetry of the normal distribution):

$\Pr(13 < X < 22) = 1 - \Pr(Z > 1) - \Pr(Z > 2)$

SECTION B

Marking Legend:

- $\left(\mathrm{A}\,\frac{1}{2}\times 4\downarrow\right)$ means four answer half-marks rounded **down** to the next integer.
  Rounding occurs at the end of a part of a question.
- **M**1 = 1 **M**ethod mark.
- **A**1 = 1 **A**nswer mark (it **must** be this or its equivalent).
- **H**1 = 1 consequential mark (**H**is/**H**er mark…correct answer from incorrect statement or slip, arithmetic slip preventing an **A** mark).

Question 1

a. In a '*Solve* …' question worth more than 1 mark, algebra (NOT a CAS) must be used to show the working that leads to the solution.

Substitute $e^{2x}=\left(e^x\right)^2$:

$$\left(e^x\right)^2-5e^x+5=0$$

Substitute $w=e^x$:

$$w^2-5w+5=0$$ **M**1

Recognition of a quadratic in the variable e^x.
Hurdle requirement for receiving answer marks.

$$\Rightarrow w=\frac{5\pm\sqrt{5}}{2}$$

Case 1: $w=\frac{5+\sqrt{5}}{2}>0 \qquad \Rightarrow e^x=\frac{5+\sqrt{5}}{2}>0$

$$\Rightarrow x=\log_e\left(\frac{5+\sqrt{5}}{2}\right)$$ **A**1

Case 2: $w=\frac{5-\sqrt{5}}{2}>0 \qquad \Rightarrow e^x=\frac{5-\sqrt{5}}{2}>0$

$$\Rightarrow x=\log_e\left(\frac{5-\sqrt{5}}{2}\right)$$ **A**1

Advice: A CAS should be used to confirm answers found using algebra.

b. i. Although this question is worth two marks (and hence more than 1 mark), working is **not** required because only a statement of two answers is required:

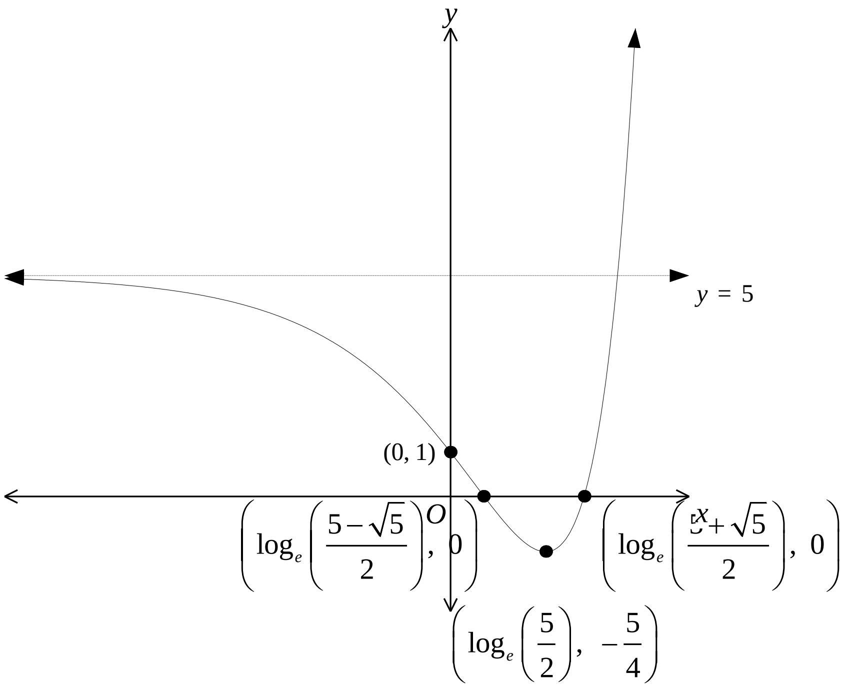
1 answer mark for stating the minimum value of the function *f*, and
1 answer mark for stating the value of *x* for which the minimum value occurs.

A CAS can therefore be used and both answers stated.

Answer: $x = \log_e\left(\frac{5}{2}\right)$ **A**1

Answer: $f(x) = -\frac{5}{4}$ **A**1

b. ii.



- Shape **A**1
- x-intercepts at $\left(\log_e\left(\frac{5+\sqrt{5}}{2}\right), 0\right)$ and $\left(\log_e\left(\frac{5-\sqrt{5}}{2}\right), 0\right)$

  y-intercept at $(0, 1)$ and turning point at $\left(\log_e\left(\frac{5}{2}\right), -\frac{5}{4}\right)$ **A**1
- Horizontal asymptote $y = 5$. **A**1

Calculations:

- Shape: Use a CAS.
- Horizontal asymptote: A horizontal is suggested by the shape and can be found from $\lim_{x \to -\infty} \left(e^{2x} - 5e^{x} + 5\right)$ either by inspection or by using a CAS.

  Note: It is possible for a graph to cross a horizontal asymptote. (A graph can never cross a vertical asymptote).

- *x*-intercepts: Solve $e^{2x} - 5e^{x} + 5 = 0$ (done in **part a.**).
- Turning point at $\left(\log_e\left(\frac{5}{2}\right), \ -\frac{5}{4}\right)$ (found in **part b.i.**).

Advice:

- Make sure that you label all of the features that the question asks you to label.
- Make sure that the asymptotes are labelled with an equation: $y = 5$ NOT 5.
- Since the question does not specify an accuracy for the coordinates of required points, exact values must be used.

c. Use a CAS to solve $e^{2x} + ke^{x} - k = 0$:

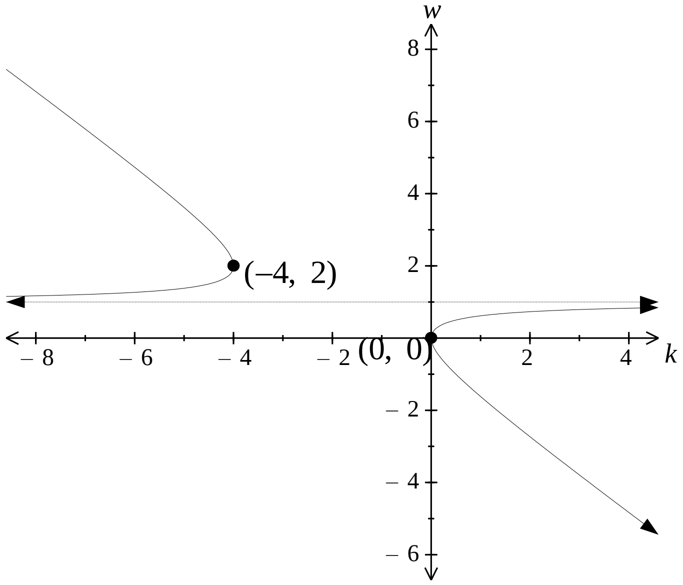
$$e^{x} = \frac{-k \pm \sqrt{k^2 + 4k}}{2}$$ **A**1

x has one real value when $w = \frac{-k \pm \sqrt{k^2 + 4k}}{2}$ has one positive value.

Use a graph of $w = \frac{-k \pm \sqrt{k^2 + 4k}}{2}$ to identify the values of *k* for which

$w = \frac{-k \pm \sqrt{k^2 + 4k}}{2}$ has one positive value.

From a CAS:



M1
Method

By inspection the values of *k* for which w has one positive value are $k > 0$ and $k = -4$.

Answer: $k > 0$ and $k = -4$ **A1**

Advice:

Using a graph to identify required values of a parameter can often be helpful and constitutes legitimate working for 'method marks'.

Question 2

a. "… period of 3":

$$\frac{2\pi}{b\pi}=3$$

$$\Rightarrow b=\frac{2}{3}$$ **A**1

"…range $[-1,\ 3]$":

Amplitude $=a=\dfrac{3-(-1)}{2}=2$ **A** $\frac{1}{2}$

Median $=d=\dfrac{3+(-1)}{2}=1$ **A** $\frac{1}{2}$

Update the model: $f(x)=2\sin\left(\dfrac{2}{3}x-c\right)+1$

"…passes through the point $(0,\ 2)$":

$$f(0)=2$$

$\Rightarrow 2=2\sin(-c)+1$ and $-\dfrac{\pi}{2}<c<\dfrac{\pi}{2}$

Use a CAS to solve for *c* over the domain $-\dfrac{\pi}{2}<c<\dfrac{\pi}{2}$:

$$c=-\frac{\pi}{6}$$ **H**1

Advice:

After the value of a parameter is calculated, it is usually a good idea to update the model with its value before trying to calculate the value of other parameters.

b. **Definition:** A function $g(x)$ is said to be strictly decreasing on an interval *I* if $g(b)<g(a)$ for all $b>a$, where $a,b\in I$.

If $g'(x)<0$ on an interval then $g(x)$ is strictly decreasing on that interval.

But $g'(x)<0$ does not always give the maximal set of points for which $g(x)$ is strictly decreasing. In particular, for a trigonometric function, values of *x* at the endpoints of maximal intervals where $g'(x)<0$ must also be included.

This is because if $g'(x)<0$ over the interval $x\in(x_1,\ x_2)$ and $g(x)$ has turning points at the endpoints of this interval then for $b\in(x_1,\ x_2)$:

$g(b) < g(x_1)$ and $g(x_2) < g(b)$

Therefore, the solution to $g'(x) \leq 0$ over the domain $x \in [-4, 4]$ is required. **M**1

Solve using a CAS.

Answer: $x \in \left[-\frac{7}{2}, -\frac{1}{2}\right] \cup \left[\frac{5}{2}, 4\right]$ **A**1

Advice:

When finding the maximal set of points where a function is strictly decreasing (or strictly increasing), the inclusion of stationary points must always be checked.

c. $g(0) = -2\sin\left(\frac{2\pi}{3}\right) + 3 = 3 - \sqrt{3}$

Therefore, the coordinates of the point are $\left(0, 3-\sqrt{3}\right)$. **A** $\frac{1}{2}$

Gradient of normal $= -\dfrac{1}{g'(0)}$

From a CAS: $g'(0) = \dfrac{\pi}{3}$

Therefore, the gradient of the normal is $-\dfrac{3}{\pi}$ **A** $\frac{1}{2}$

Substitute the point and the gradient into the model $y - y_1 = m(x - x_1)$ and re-arrange into the required form $y = mx + c$.

Answer: $y = -\dfrac{3}{\pi}x + 3 - \sqrt{3}$ **H**1

Advice:

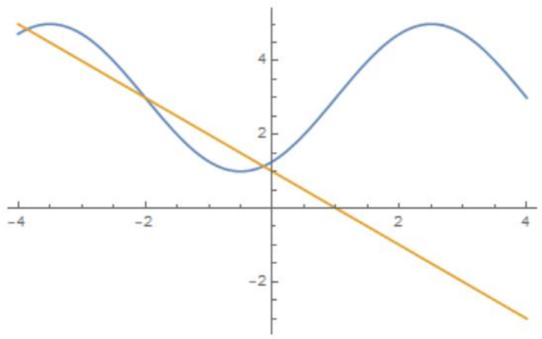
Always check that your final answer is in the form that is asked for in the question.

d. The line $y = -x + 1$ is tangent to *g* at points where $g'(x) = -1$.

Use a CAS to solve $g'(x) = -1$ over the domain $x \in [-4,\ 4]$:

$x = -3.0246739$, $x = -0.9753326$, $x = 2.975332$

Graphs of $y = -x + 1$ and $y = -2\sin\left(\frac{\pi}{3}(x+2)\right) + 3$ show that the minimum vertical translation upwards will occur when $y = -x + 1$ is tangent to $y = -2\sin\left(\frac{\pi}{3}(x+2)\right) + 3$ at the point where $x = -3.0246739$:



Therefore $y = -x + 1$ is required to be tangent at the point where $x = -3.0246739$ **A**1

Note:

Using more than four decimal places of accuracy during the calculation
is required so that the possibility of rounding error in the final answer is avoided.

Use a CAS to get the *y*-coordinate of the tangent point:

$y = g(-3.0246739) = 4.757302$ **A**$\frac{1}{2}$

The *y*-coordinate of the line $y = -x + 1$ at the point where $x = -3.0246739$ is

$x = 4.0246739$. **A**$\frac{1}{2}$

Therefore, the required minimum vertical translation upwards is

$4.757302 - 4.0246739 = 0.732628$

Answer: 0.7326 **H**1

e. Intersection points occur when:

$$p^2\cos\left(\frac{x}{5}\right)+4=(3p-1)\cos\left(\frac{x}{5}\right)+8$$

$$\Rightarrow p^2\cos\left(\frac{x}{5}\right)-(3p-1)\cos\left(\frac{x}{5}\right)=4$$

$$\Rightarrow \left(p^2-3p+1\right)\cos\left(\frac{x}{5}\right)=4$$ **M**1

Since $-1\le\cos\left(\frac{x}{5}\right)\le 1$, there will be no intersection points when

$$-4<p^2-3p+1<4$$ **M**1

The question says that $p\in R^+$ therefore the largest solution to $-4<p^2-3p+1<4$ **and** $p\in R^+$ is required.

Solve using a CAS: $0<w<\frac{1}{2}\left(3+\sqrt{21}\right)$

Answer: $w=\frac{1}{2}\left(3+\sqrt{21}\right)$ **A**1

Question 3

a. In a '*Show that* …' question worth more than 1 mark, algebra (NOT a CAS) must be used to show the given result.

"*Show that* …." questions usually state a result that will be useful or needed in a later part of the question.

In a "*Show that* …." question a student must clearly demonstrate that s/he could have obtained the given result without it being stated. Some relevant comments from past VCAA Examination Reports regarding "*Show that* …." questions include:

"There were some unconvincing arguments, often due to insufficient steps being shown."

[2015 Examination Report Question 8 part a.]

"As often happens in a 'show that' type question, some students were unable to do any convincing [working], yet still managed to obtain the result stated."

[2012 Examination Report Question 9 part c.]

A good way to ensure that all necessary working is given when answering a "*Show that* …." question is to treat the question as if the given result is unknown.

The question:

"Let $t = x + \frac{1}{x}$. Show that $t^2 - at + 5 - 2a = 0$."

should therefore be treated as asking

"By making the substitution $t = x + \frac{1}{x}$, ***express*** the equation $x^4 - ax^3 + (7-2a)x^2 - ax + 1 = 0$ as a quadratic equation in *t*".

$x^4 - ax^3 + (7-2a)x^2 - ax + 1 = 0$

$\Rightarrow x^2 - ax + (7-2a) - \frac{a}{x} + \frac{1}{x^2} = 0$

$\Rightarrow x^2 + \frac{1}{x^2} - a\left(x + \frac{1}{x}\right) + 7 - 2a = 0$ **M**1

Substitute $t = x + \frac{1}{x}$ $\Rightarrow t^2 = x^2 + 2 + \frac{1}{x^2}$ $\Rightarrow t^2 - 2 = x^2 + \frac{1}{x^2}$: **M**1

$x^2 + \frac{1}{x^2} - a\left(x + \frac{1}{x}\right) + 7 - 2a = 0$

$\Rightarrow t^2 - 2 - at + 7 - 2a = 0$

$\Rightarrow t^2 - at + 5 - 2a = 0$ which was to be shown.

b. **i.** There can only be real solutions for *x* if *t* has real solutions.
Use a CAS to solve $t^2 - at + 5 - 2a = 0$ for *t*:

$t = \frac{a \pm \sqrt{a^2 + 8a - 20}}{2}$ …. (1) **A1**

Real solutions for *t* when $a^2 + 8a - 20 \geq 0$
Use a CAS to solve $a^2 + 8a - 20 \geq 0$:

$a \geq 2$ or $a \leq -10$ **A1**

b. **ii.** It must be investigated whether or not all real values of *t* (and hence all values $a \geq 2$ and $a \leq -10$) give real values of *x*.

Use a CAS to solve $t = x + \frac{1}{x}$ for *x*: $x = \frac{t \pm \sqrt{t^2 - 4}}{2}$ …. (2)

Real solutions for *x* when $t^2 - 4 \geq 0$

Use a CAS to solve $t^2 - 4 \geq 0$: $t \leq -2$ or $t \geq 2$

Case 1:

Substitute $t \le -2$ into equation (1) and use a CAS to solve

$\frac{a \pm \sqrt{a^2 + 8a - 20}}{2} \le -2$ subject to the restriction $a \ge 2$ or $a \le -10$.

Answer: $a \le -10$ **A1**

Case 2:

Substitute $t \ge 2$ into equation (1) and use a CAS to solve

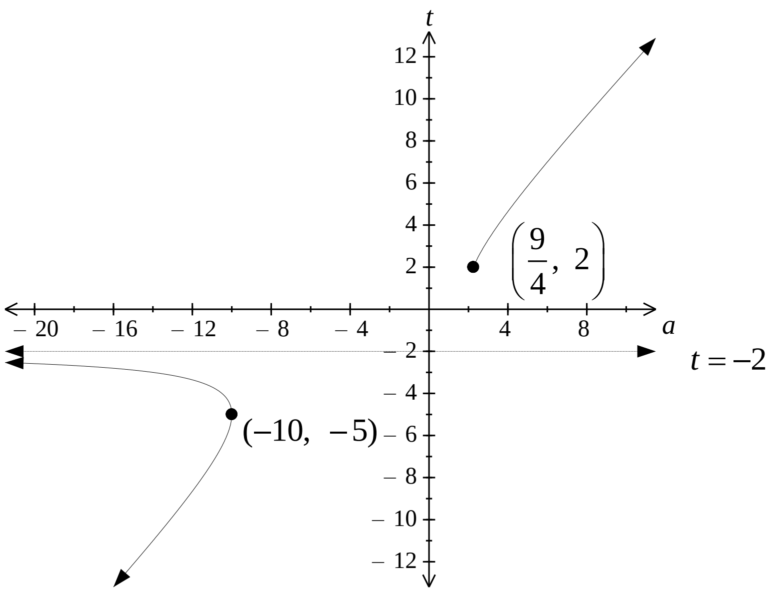
$\frac{a \pm \sqrt{a^2 + 8a - 20}}{2} \ge 2$ subject to the restriction $a \ge 2$ or $a \le -10$

$\frac{a - \sqrt{a^2 + 8a - 20}}{2} \ge 2$: No real solution

$\frac{a + \sqrt{a^2 + 8a - 20}}{2} \ge 2$: $a \ge \frac{9}{4}$ **A1**

c. Use a CAS to draw the graph of $t = \frac{a \pm \sqrt{a^2 + 8a - 20}}{2}$

From **part b. i.** the maximal domain is $a \le -10 \cup a \ge \frac{9}{4}$



From the shape it appears that $t = -2$ is a horizontal asymptote (behaviour as $a \to -\infty$). This can be confirmed using the Limit command on a CAS.

Therefore $t = -2$ is a horizontal asymptote.

Note: The range (t) is $t < -2 \cup t \ge 2$

Shape: **A1**

Coordinate of endpoints: **A1**

Horizontal asymptote $t = -2$**:** **A1**

d. **i.** *x* has four real values when *t* has two real values.

By inspection of the graph of *t* versus *a* in **part c.** it is seen that
t has two real values when $a < -10$

Answer: $a < -10$ **A**1

d. **ii.** *x* only has two real values when *t* only has one real value.

By either inspection of the graph of *t* versus *a* or equations (1) and (2) there are two cases to consider.

Case 1: $a = -10$ $(\Rightarrow t = -5)$

Use a CAS to factorise $x^4 - ax^3 + (7-2a)x^2 - ax + 1$ when $a = -10$:

$x^4 + 10x^3 + 27x^2 + 10x + 1 = \left(x^2 + 5x + 1\right)^2$

and $\left(x^2 + 5x + 1\right)^2 = 0$ obviously has two repeated solutions. **A**1

Case 2: $a \geq \frac{9}{4}$

Reject $a = \frac{9}{4} \Rightarrow t = 2$ because $x = \frac{t \pm \sqrt{t^2 - 4}}{2}$ (equation (2)) has only

one value. **M**$\frac{1}{2}$

Reject $a > \frac{9}{4}$ because $x^4 - ax^3 + (7-2a)x^2 - ax + 1 = 0$ has two

non-repeated solutions. **M**$\frac{1}{2}$

This is readily seen by substituting particular values of *a* into
$x^4 - ax^3 + (7-2a)x^2 - ax + 1 = 0$ and solving using a CAS.

Answer: $a = -10$

Note 1:

It is seen that $a = -10$ corresponds to a ‘special’ point (a vertex) on the graph.

The values $a > \frac{9}{4}$ do not correspond to any ‘special’ points on the graph.

Note 2: As an example, take $a = 3$.

$$x^4 - 3x^3 + x^2 - 3x + 1 = \left(x^2 - \left(\frac{3+\sqrt{13}}{2}\right)x + 1\right)\left(x^2 - \left(\frac{3-\sqrt{13}}{2}\right)x + 1\right)$$

and $x^2 - \left(\frac{3-\sqrt{13}}{2}\right)x + 1$ is an irreducible quadratic $\left(\Delta = -\frac{3}{2}\left(\sqrt{13} - 1\right)\right)$

e. **i.** **Answer:** $a > \frac{9}{4}$ **A**1

(See **part d. ii.** Case 2 for justification)

ii. **Answer:** $a = \frac{9}{4}$ **A**1

(See **part d. ii.** Case 2 for justification)

Question 4

a. *Define the random variable*:

Let *X* be the random variable

Amount of OFTC in mg required to achieve surgical anaesthesia in a 78 kg adult.

Define the distribution that the random variable follows:

$X \sim$ Normal($\mu = 60, \quad \sigma = 15$)

a. **i.** *Define the problem in terms of a probability statement*:

$\Pr(X < 68) = ?$

Use the Normal Cdf command on a CAS.

Answer: 0.7031 **A**1

Advice:

Even though the question is only worth 1 mark and therefore only an answer is required, it is important that you:

- Clearly define the random variable relevant to the question (use a different symbol for each new random variable).
- Define the distribution that the random variable follows.
- Define the problem in terms of a probability statement.

Doing these things is important because:

1. It helps clarify in your mind how to get the correct answer.

2. It is often helpful for showing appropriate working in questions that follow that are worth more than 1 mark.

- Make sure that you correctly round and give your answer correct to the required accuracy.

a. **ii.** *Define the problem in terms of a probability statement*:

$\Pr(X > a) = 0.65 \qquad \Rightarrow \Pr(X < a) = 0.35$

Use the Inverse Normal command on a CAS:

Answer: $a = 54.2$ **A1**

Check for reasonableness:

Since $\Pr(X > a) > 0.5$ the answer must be less than the mean of 60 ✓

Advice:

- Where possible and when time permits, it is useful to check an answer for its reasonableness.

- Even though the question is only worth 1 mark and therefore working is not required, you should nevertheless define the problem in terms of a probability statement using a clearly defined random variable (this variable may have been defined in a previous part of the question, as is the case here) because it helps clarify in your mind how to get the correct answer.

- Make sure that you give your answer to the required accuracy.

b. **i.** *Define the problem in terms of a probability statement*:

$\Pr(18 < X < b) = 0.94$

$\Rightarrow \Pr(X < b) - \Pr(X < 18) = 0.94$ **M1**

$\Rightarrow \Pr(X < b) = 0.94 + \Pr(X < 18)$

$\Rightarrow \Pr(X < b) = 0.94 + 0.002555 \qquad = 0.942555$

Note: To avoid the possibility of rounding error in the final answer, more accuracy than is required in the final answer should always be used during a calculation. Since the final answer must be correct to one decimal place, a value of $\Pr(X < 18)$ that is correct to six decimal places is used.

Use the Inverse Normal command on a CAS:

Answer: $b = 83.6$ **A1**

Check for reasonableness:

Use the Normal Cdf command on a CAS to calculate $\Pr(18 < X < 83.6)$:

$\Pr(18 < X < 83.6) = 0.94$ ✓

Advice:

- To avoid the possibility of rounding error in the final answer, more accuracy than is required in the final answer should always be used during a calculation.
- Make sure that you define the problem in terms of a probability statement using a clearly defined random variable.
- Be guided by the number of marks allotted for the question. In this case there is more than 1 mark (two marks), so appropriate working must be shown. VCAA does NOT accept CAS syntax as appropriate working, so the equation to be solved must be written down. Use a CAS to get the final answer.
- Where possible and when time permits, it is useful to check an answer for its reasonableness.
- Make sure you correctly round and give your answer correct to the required accuracy.

b. ii. *Define the problem in terms of a probability statement*:

The idea of conditional probability is required. The given condition (that is, the event which is known to have occurred) is that the adult is OLAL-*typical*:

$\Pr(18 < X < 60 \mid \text{OLAL-}typical) = ?$

Note:

ALOL-*typical* means $18 < X < b$ therefore the required event is $18 < X < 60$ NOT $X < 60$.

$$\Pr(18 < X < 60 \mid \text{OLAL-}typical) = \frac{\Pr(18 < X < 60)}{\Pr(18 < X < b)}$$

Substitute $\Pr(18 < X < b) = 0.94$:

$$= \frac{\Pr(18 < X < 60)}{0.94}$$ **M**1

Use the Normal Cdf command on a CAS:

$$= \frac{0.4974448}{0.94}$$

where greater than 4 decimal place accuracy is used during the calculation to avoid the possibility of rounding error in the final answer.

Answer: 0.5292 **A**1

Advice:

- To avoid the possibility of rounding error in the final answer, more accuracy than is required in the final answer should always be used during a calculation.

- Make sure that you define the problem in terms of a probability statement using a clearly defined random variable.
- Be guided by the number of marks allotted for the question. In this case there is more than 1 mark (two marks), so appropriate working must be shown. VCAA does NOT accept CAS syntax as appropriate working, so the equation to be solved must be written down. Use a CAS to get the final answer.
- Make sure you correctly round and give your answer correct to the required accuracy.

c. **i.** *Define the random variable*:

Let *Y* be the random variable.

Number of 78 kg adults that require between 46 mg and 57 mg of OLAL to achieve surgical anaesthesia.

Define the distribution that the random variable follows:

$Y \sim$ Binomial$(n = 9,\ \ p = \Pr(46 < X < 57))$ **M**1

Use the Normal Cdf command on a CAS to calculate $p = \Pr(46 < X < 57)$:

$Y \sim$ Binomial$(n = 9,\ \ p = 0.2454164)$

where greater than 4 decimal place accuracy is used to avoid the possibility of rounding error in the final answer.

Define the problem in terms of a probability statement: $\Pr(Y = 3) = ?$
Use the Binomial Pdf command on a CAS:

Answer: 0.2292 **A**1

Advice:

- Be ready for a 'switch' of random variable (and a possible link to the old random variable). Switching from a continuous random variable (such as normal) to a discrete random variable (such as binomial) is a common question type.

- Make sure that you clearly define a new random variable. A different symbol (such as *Y*) to the ones used in **part a.** and **part b.**. must be used.
- Make sure that you define the distribution that the random variable follows.
- Make sure that you define the problem in terms of a probability statement.
- To avoid the possibility of rounding error in the final answer, more accuracy than is required in the final answer should always be used during a calculation.

- Make sure that you give your answer to the required accuracy.
- Be guided by the number of marks allotted for the question. In this case there is more than 1 mark (two marks), so appropriate working must be shown. VCAA does NOT accept CAS syntax as appropriate working. Defining the random variable and its distribution and defining the problem in terms of a probability statement will often be recognised as appropriate working. Use a CAS to get the final answer.

c. **ii.** *Define the random variable*:

Let *W* be the random variable.

Number of 78 kg adults that are OLAL-sensitive.

Define the distribution that the random variable follows:

$W \sim \text{Binomial}(n = ?,\ \ p = \Pr(X < 18))$

Substitute $p = \Pr(X < 18) = 0.002555$ (found during the calculation in **part b.i.**):

$W \sim \text{Binomial}(n = ?\ \ p = 0.002555)$ **M**1

Define the problem in terms of a probability statement:

The **smallest** value of *n* such that $\Pr(W > 1) > 0.4$ is required.

$$\Pr(W > 1) > 0.4$$
$$\Rightarrow \Pr(W \le 1) < 0.6$$

Define the function $f1(x) =$ binomCdf(*x*, 0.002555, 1) on a CAS.

Scroll down the function TABLE until the first integer value of *x* satisfying $f1(x) < 0.6$ is found.

Answer: 539 **A**1

Check for reasonableness:

Use the Binomial Cdf command on a CAS to calculate $\Pr(W > 1) = \Pr(W \ge 2)$ where $W \sim \text{Binomial}(n = 539\ \ p = 0.002555)$:

$\Pr(W \ge 2) = 0.4004$ ✓

Advice:

- Finding a minimum sample size is a common question type. Make sure you can use your CAS in the way described in the solution to answer this type of question.

- Make sure that you clearly define a new random variable. A different symbol (such as *W*) to the ones used in **part a.**, **part b.** and **part c.i.** must be used.

- Make sure that you define the distribution that the random variable follows.

- Make sure that you define the problem in terms of a probability statement.

- Be guided by the number of marks allotted for the question. In this case there is more than 1 mark (two marks), so appropriate working must be shown. VCAA does NOT accept CAS syntax as appropriate working. Defining the random variable and its distribution and defining the problem in terms of a probability statement will often be recognised as appropriate working. Use a CAS to get the final answer.

- Where possible and when time permits, it is useful to check an answer for its reasonableness.

d. *Define the random variable*:

Let *L* be the random variable.

Numb tongue dose of OLAL in mg for 78 kg adults.

Define the distribution the random variable follows:

$L \sim$ Normal($\mu = 130, \quad \sigma = 14$)

Define the problem in terms of a probability statement:

$\Pr(L < x) = ?$ where $\Pr(X < x) = 0.90$

Note:

The proportion of adults that will get a numb tongue is the area to the left of *L* (because when the amount is larger a greater proportion of adults will get a numb tongue).

$\Pr(X < x) = 0.90$

$\Rightarrow x = 79.2232735$ **A**1

using the Inverse Normal command on a CAS.

$\Pr(L < x) = \Pr(L < 79.2232735) = 0.00014345$

Convert to a percentage:

Answer: 0.0143 % **A**1

Advice:

- Make sure that you clearly define a new random variable. A different symbol (such as *L*) to the ones used in previous parts must be used.
- Make sure that you define the distribution that the random variable follows.
- Make sure that you define the problem in terms of a probability statement.
- To avoid the possibility of rounding error in the final answer, more accuracy than is required in the final answer should always be used during a calculation.
- Make sure that you convert the probability into a percentage and that give your answer to the required accuracy.
- Be guided by the number of marks allotted for the question. In this case there is more than 1 mark (two marks), so appropriate working must be shown. VCAA does NOT accept CAS syntax as appropriate working. Defining the random variable and its distribution and defining the problem in terms of a probability statement will often be recognised as appropriate working. Use a CAS to get the final answer.

e. *Define the random variable*:

Let *C* be the random variable.

Amount of OLAL in mg required to achieve surgical anaesthia in a 30 kg child.

Define the distribution the random variable follows:

$C \sim \text{Normal}(\mu = 31, \ \ \sigma = ?)$

Define the problem in terms of a probability statement:

The value of σ such that $\Pr(C > 27) = 0.83$ is required.

$\Pr(Z > z^*) = 0.83 \qquad \Rightarrow \Pr(Z < z^*) = 0.17$

Use the Inverse Normal command on a CAS:

$z^* = -0.9541652$ **A**1

$$Z = \frac{C - \mu}{\sigma} \qquad \Rightarrow z^* = \frac{27 - 31}{\sigma} = -\frac{4}{\sigma} \qquad \Rightarrow -0.9541652 = -\frac{4}{\sigma}$$

Solve for σ using a CAS:

Answer: $\sigma = 4.19$ **H**1

Check for reasonableness:

Use the Normal Cdf command on a CAS to calculate $\Pr(C > 27)$ when $\sigma = 4.19$:

$\Pr(C > 27) = 0.8301$ ✓

Advice:

- Make sure that you clearly define a new random variable. A different symbol (such as *C*) to the ones used in previous parts must be used.
- Make sure that you define the distribution that the random variable follows.
- Make sure that you define the problem in terms of a probability statement.
- To avoid the possibility of rounding error in the final answer, more accuracy than is required in the final answer should always be used during a calculation.

- Make sure that you give your answer to the required accuracy.
- Be guided by the number of marks allotted for the question. In this case there is more than 1 mark (three marks), so appropriate working must be shown. VCAA does NOT accept CAS syntax as appropriate working, so the equations to be solved must be written down. Use a CAS to get the final answer.
- Where possible and when time permits, it is useful to check an answer for its reasonableness.

Question 5

a. The inverse function f^{-1} of *f* only exists if *f* is a one-to-one function.

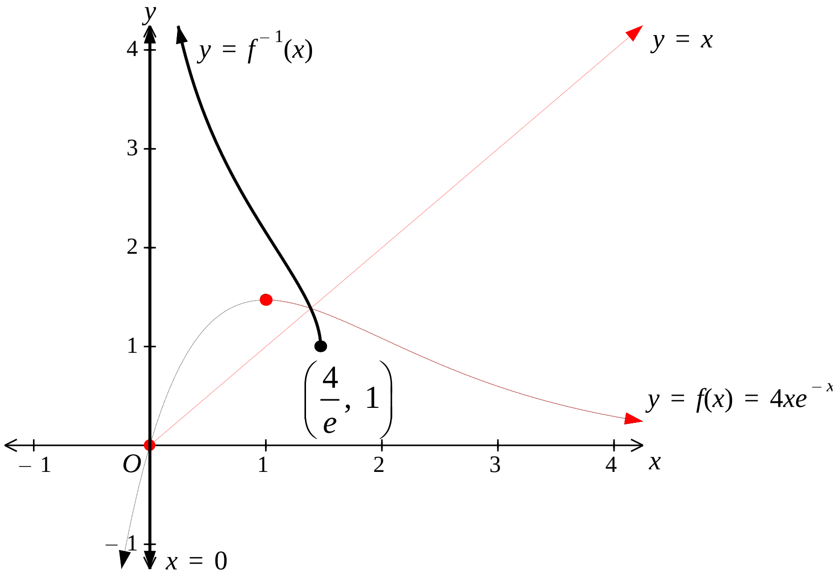
Therefore, the smallest value of *k* for which f^{-1} exists will be the *x*-coordinate of the turning point of $f(x)=4xe^{-x}$. **M**1

Solve $f'(x)=0$ using a CAS: $x=1$

Therefore, the smallest value of *k* such that *f* is a one-to-one function when $x\in[k,\ +\infty)$ is $k=1$.

Answer: $k=1$ **A**1

b. The graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ (draw using a CAS) in the line $y = x$.



- Endpoint at $\left(\frac{4}{e},\ 1\right)$ **A1**

- Vertical asymptote $x = 0$. **A1**

- Shape including location of endpoint $\left(\frac{4}{e},\ 1\right)$ consistent with the scale on each axis. **A1**

Calculations:

- Endpoint: $f(x) = 4xe^{-x}$ has an endpoint at $\left(1,\ \frac{4}{e}\right)$ (from **part a.**).

  Therefore, f^{-1} has an endpoint at $\left(\frac{4}{e},\ 1\right)$.

- Shape: Reflect the graph of $y = f(x) = 4xe^{-x}$ (draw using a CAS) in the line $y = x$.

  In particular:

  The point $\left(1,\ \frac{4}{e}\right)$ on *f* reflects to the point $\left(\frac{4}{e},\ 1\right)$ on f^{-1}.

  The horizontal asymptote $y = 0$ reflects to the vertical asymptote $x = 0$.

- f has a horizontal asymptote $y = 0$ therefore f^{-1} has a vertical asymptote $x = 0$.

Advice:

- Make sure that you label all of the features that the question asks you to label.
- Make sure that the asymptotes are labelled with an equation: $x = 0$ NOT 0.
- Since the question does not specify an accuracy for the coordinates of the endpoint, exact values must be used.

c. By definition: $x = 4ye^{-y}$ where $y = f^{-1}(x)$ **M**1

Substitute $x = \frac{2}{3}$:

$\frac{2}{3} = 4xe^{-x}$ where $x = f^{-1}\left(\frac{2}{3}\right)$ **M**1

Solve using a CAS: $x = 2.83315$ or $x = 0.20448$

Require the value of *x* such that $x \in \text{Ran}\left(f^{-1}\right) = \text{Dom}(f) = [1, +\infty)$

Alternatively, the graph in **part b.** can be used to decide which value of *x* is wanted.

Therefore $x = 2.83315$

Answer: $f^{-1}\left(\frac{2}{3}\right) = 2.8332$ **A**1

Advice:

Make sure that you give your answer to the required accuracy.