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THE SCHOOL FOR EXCELLENCE

THE SCHOOL FOR EXCELLENCE (TSFX)
UNIT 3 & 4 MATHEMATICAL METHODS 2020
WRITTEN EXAMINATION 1 – SOLUTIONS

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Marking Legend:

- $\left(\text{A}\frac{1}{2}\times 4\downarrow\right)$ means four answer half-marks rounded **down** to the next integer.
  Rounding occurs at the end of a part of a question.
- **M**1 = 1 **M**ethod mark.
- **A**1 = 1 **A**nswer mark (it **must** be this or its equivalent).
- **H**1 = 1 consequential mark (**H**is/**H**er mark…correct answer from incorrect statement or slip, arithmetic slip preventing an **A** mark).

Question 1

a. Use the quotient rule:

$u=\cos(\pi x)$ $\qquad$ $v=\log_e(-3x)$ $\qquad$ **Note:** $\frac{dv}{dx}\neq\frac{1}{-3x}$.

$\Rightarrow\frac{du}{dx}=-\pi\sin(\pi x)$ $\qquad$ $\Rightarrow\frac{dv}{dx}=\frac{1}{x}$ (from the chain rule)

$$h'(x)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

$$=\frac{\log_e(-3x)\left(-\pi\sin(\pi x)\right)-\cos(\pi x)\frac{1}{x}}{\left(\log_e(-3x)\right)^2}.$$ **M**1

$$h'(-1)=\frac{\log_e(3)\left(-\pi\sin(-\pi)\right)-\cos(-\pi)(-1)}{\left(\log_e(3)\right)^2}=\frac{-1}{\left(\log_e(3)\right)^2}.$$

Answer: $\frac{-1}{\left(\log_e(3)\right)^2}$. **A**1

b. $h(x)=\dfrac{\cos(\pi x)}{\log_e(-3x)}$ is undefined when $\log_e(-3x)=0$:

$\log_e(-3x)=0 \quad \Rightarrow -3x=e^0=1 \quad \Rightarrow x=-\dfrac{1}{3}$.

Therefore $h(x)=\dfrac{\cos(\pi x)}{\log_e(-3x)}$ is defined for $x<-\dfrac{1}{3}$ and $x>-\dfrac{1}{3}$.

Answer: $a=-\dfrac{1}{3}$. **A**1

Question 2

a. $f(x)=\dfrac{1}{2}+\dfrac{1}{\sqrt{2-5x}} \quad =\dfrac{1}{2}+(2-5x)^{-1/2} \quad =\dfrac{1}{2}+(-5x+2)^{-1/2}$.

$f'(x)=-\dfrac{1}{2}(-5x+2)^{-3/2}(-5) \quad =\dfrac{5}{2}(2-5x)^{-3/2}$.

Answer: $f'(x)=\dfrac{5}{2}(2-5x)^{-3/2}$. **A**1

b. $g(x)=\int f(x)\,dx=\int \dfrac{1}{2}+(-5x+2)^{-1/2}dx$

$=\dfrac{x}{2}+\int(-5x+2)^{-1/2}dx$. **M**1

To calculate $\int(-5x+2)^{-1/2}dx$ substitute $a=-5$ and $n=-\dfrac{1}{2}$ into $\int(ax+b)^n dx=\dfrac{1}{a(n+1)}(ax+b)^{n+1}+c$ (from the VCAA formula sheet):

$\int(-5x+2)^{-1/2}dx=\dfrac{1}{-5\left(-\frac{1}{2}+1\right)}(-5x+2)^{1/2}+c \quad =-\dfrac{2}{5}(-5x+2)^{1/2}+c$. **M**1

$g(x)=\dfrac{x}{2}-\dfrac{2}{5}(-5x+2)^{1/2}+c$

$=\dfrac{x}{2}-\dfrac{2}{5}\sqrt{2-5x}+c$.

Substitute $g(0)=\sqrt{2}$ into $g(x)=\dfrac{x}{2}-\dfrac{2}{5}\sqrt{2-5x}+c$ and solve for *c*:

$\sqrt{2}=-\dfrac{2}{5}\sqrt{2}+c \quad \Rightarrow c=\sqrt{2}+\dfrac{2}{5}\sqrt{2}=\dfrac{7\sqrt{2}}{5}$

Answer: $g(x)=\dfrac{x}{2}-\dfrac{2}{5}\sqrt{2-5x}+\dfrac{7\sqrt{2}}{5}$. **A**1

Question 3

a. Let $x = y^2 - 2y$ where $y = f^{-1}(x)$.

$x = y^2 - 2y \quad = (y-1)^2 - 1 \quad \Rightarrow x + 1 = (y-1)^2 \quad \Rightarrow \pm\sqrt{x+1} = y - 1$

$\Rightarrow y = \pm\sqrt{x+1} + 1$. **M**1

Use the fact that $f(a) = b \Rightarrow f^{-1}(b) = a$ to choose between the two potential solutions for *y*.

$0 \in \left(-\infty, \frac{1}{2}\right)$ and $f(0) = 0$ therefore a point on the graph of $y = f(x)$ is $(0, 0)$.

Therefore a point on the graph of $y = f^{-1}(x)$ is $(0, 0)$ and so $f^{-1}(0) = 0$.

This point can be used to decide which solution for *y* to reject: $0 = \pm\sqrt{1} + 1$.

Therefore, the negative root solution is required: $y = -\sqrt{x+1} + 1$. **M**$\frac{1}{2}$

Answer: $f^{-1}(x) = -\sqrt{x+1} + 1$. **A**$\frac{1}{2}$

b. $\text{dom}\left(f^{-1}\right) = \text{ran}(f)$.

By inspection of a simple sketch graph of $f : \left(-\infty, \frac{1}{2}\right) \to R$, $f(x) = x^2 - 2x$:

$\text{ran}(f) = \left(-\frac{1}{4}, +\infty\right)$.

Answer: $\left(-\frac{1}{4}, +\infty\right)$. **A**1

c. $\begin{bmatrix} x' \\ y' \end{bmatrix} = T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ c \end{bmatrix} = \begin{bmatrix} ax \\ by+c \end{bmatrix}$

$\Rightarrow x' = ax$ and $y' = by + c$

$\Rightarrow \frac{x'}{a} = x$ and $\frac{y'-c}{b} = y$.

$y = x^2 - 2x \quad \rightarrow \frac{y'-c}{b} = \left(\frac{x'}{a}\right)^2 - 2\left(\frac{x'}{a}\right)$ **A**1

$\Rightarrow y' = b\left(\frac{x'}{a}\right)^2 - 2b\left(\frac{x'}{a}\right) + c$.

$y = b\left(\frac{x}{a}\right)^2 - 2b\left(\frac{x}{a}\right) + c$.

Compare $y = b\left(\frac{x}{a}\right)^2 - 2b\left(\frac{x}{a}\right) + c$ with $y = -2x^2 + 8x - 1$:

Constant term: $c = -1$.

Coefficient of x^2: $\frac{b}{a^2} = -2$. …. (1)

Coefficient of *x*: $-\frac{2b}{a} = 8 \quad \Rightarrow \frac{b}{a} = -4$. …. (2)

Solve equations (1) and (2) simultaneously.

Substitute equation (2) into equation (1): $\frac{1}{a}(-4) = -2 \quad \Rightarrow a = 2$.

Substitute $a = 2$ into equation (2): $b = -8$.

Answers: $a = 2$, $b = -8$, $c = -1$. **A**1

Question 4

a. $h(x) = g(f(x)) \quad = \log_e(8 - f(x)) \quad = \log_e\left(8 - \left(x^2 - 1\right)\right) \quad = \log_e\left(9 - x^2\right)$.

Answer: $h(x) = \log_e\left(9 - x^2\right)$. **A1**

b. Require $\text{ran}(f) \subseteq \text{dom}(g)$.

$\text{ran}(f) = f(x) = x^2 - 1$.

$\text{dom}(g) = (-\infty,\ 3]$.

Therefore, $x^2 - 1 \le 3$ **M1**

$\Rightarrow x^2 \le 4 \quad \Rightarrow x^2 - 4 \le 0$.

The inequality can be solved by inspecting a simple graph of $y = x^2 - 4$:

$-2 \le x \le 2$.

Answer: $-2 \le x \le 2$. **A1**

WARNING:

Do **not** calculate the maximal domain from the rule $h(x) = \log_e\left(9 - x^2\right)$ found in **part a. The rule for $g(f(x))$ can only be used to find the maximal domain of $g(f(x))$ when $g(x)$ has its maximal domain.** In this question the given domain of $g(x)$ is $(-\infty,\ 3]$ which is **not** its maximal domain (the maximal domain of $g(x)$ is $(-\infty,\ 8]$). Therefore, using the rule will **not** give the correct answer.

Question 5

Average rate of change:

$$-3 = \frac{f(1)-f(-1)}{1-(-1)} = \frac{-2+a+b-(2+a-b)}{2} = \frac{-4+2b}{2} = -2+b$$

$\Rightarrow b = -1$. **A**1

Substitute $b=-1$: $f(x) = -2x^3 + ax^2 - x$.

Average value:

$$-6 = \frac{\int_{-1}^{1} -2x^3 + ax^2 - x \, dx}{1-(-1)} = \frac{\left[\frac{1}{2}x^4 + \frac{a}{3}x^3 - \frac{1}{2}x^2\right]_{-1}^{1}}{2}$$ **M**1

$$\Rightarrow -12 = \left[\frac{1}{2}x^4 + \frac{a}{3}x^3 - \frac{1}{2}x^2\right]_{-1}^{1} = \frac{1}{2}+\frac{a}{3}-\frac{1}{2}-\left(\frac{1}{2}-\frac{a}{3}-\frac{1}{2}\right) = \frac{2a}{3}$$

$\Rightarrow a = -18$. **A**1

Question 6

It is first required to get both inequalities in the same direction.

$\Pr\left(Z > \frac{b}{4}\right) = \Pr\left(Z < -\frac{b}{4}\right)$ by symmetry around the mean.

Therefore: $\Pr(X<b) = \Pr\left(Z > \frac{b}{4}\right)$ $\Rightarrow \Pr(X<b) = \Pr\left(Z < -\frac{b}{4}\right)$.

$$X = b \Rightarrow Z = \frac{b-24}{5}$$

Therefore: $\Pr(X<b) = \Pr\left(Z < -\frac{b}{4}\right)$

$$\Rightarrow \Pr\left(Z < \frac{b-24}{5}\right) = \Pr\left(Z < -\frac{b}{4}\right)$$ **M**1

$$\Rightarrow \frac{b-24}{5} = -\frac{b}{4} \quad \Rightarrow 4b - 96 = -5b \quad \Rightarrow 9b = 96 \quad \Rightarrow b = \frac{32}{3}.$$

Answer: $b = \frac{32}{3}$ **A**1

Question 7

a. *Define the random variable*:

Let X denote the random variable

Number of batteries in a packet that will last for more than 100 hours.

Define the distribution the random variable follows:

$X \sim \text{Binomial}\left(n = 16,\ p = \frac{11}{12}\right)$.

Define the problem in terms of a probability statement:

$\Pr(X \geq 15) = ?$

$\Pr(X \geq 15) = {}^{16}C_{15}\left(\frac{11}{12}\right)^{15}\left(\frac{1}{12}\right) + {}^{16}C_{16}\left(\frac{11}{12}\right)^{16}$ **M**1

$$= 16\left(\frac{11}{12}\right)^{15}\left(\frac{1}{12}\right) + \left(\frac{11}{12}\right)^{16}$$

Re-arrange into the required "form $\frac{a}{4}\left(\frac{b}{12}\right)^{n}$ where a, b and n are integers":

$$= \left(\frac{11}{12}\right)^{15}\left(\frac{16}{12} + \frac{11}{12}\right)$$

$$= \left(\frac{11}{12}\right)^{15}\frac{27}{12}$$

$$= \frac{9}{4}\left(\frac{11}{12}\right)^{15}$$

Answer: $\frac{9}{4}\left(\frac{11}{12}\right)^{15}$ **A**1

b. *Define the problem in terms of a probability statement*:

$$\Pr(X \geq 15 \mid X \geq 1) = ?$$

$$\Pr(X \geq 15 \mid X \geq 1) = \frac{\Pr(X \geq 15)}{\Pr(X \geq 1)}$$

$$= \frac{\frac{9}{4}\left(\frac{11}{12}\right)^{15}}{\Pr(X \geq 1)}$$ **H**1

Consequential on answer to **part a.**

$$= \frac{\frac{9}{4}\left(\frac{11}{12}\right)^{15}}{1 - \Pr(X = 0)}$$

$$= \frac{\frac{9}{4}\left(\frac{11}{12}\right)^{15}}{1 - \left(\frac{1}{12}\right)^{16}}$$

Re-arrange into the required "form $\frac{k(11)^p}{12^m - 1}$ where k and m are positive integers":

$$= \frac{\frac{9}{4}(12)(11)^{15}}{(12)^{16} - 1}$$

$$= \frac{27(11)^{15}}{(12)^{16} - 1}.$$

Answer: $\frac{27(11)^{15}}{(12)^{16} - 1}$. **A**1

Question 8

a. Let $y = x^2 \log_e(x)$.

Use the product rule:

$u = x^2$ $\qquad\qquad$ $v = \log_e(x)$

$\Rightarrow \dfrac{du}{dx} = 2x$ $\qquad\qquad$ $\Rightarrow \dfrac{dv}{dx} = \dfrac{1}{x}$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \qquad = x^2\left(\frac{1}{x}\right) + 2x\log_e(x) \qquad = x + 2x\log_e(x).$$

Answer: $x + 2x\log_e(x)$. **A**1

b. $x\log_e(x) > 0$ for $1 \le x \le e$

Therefore area $= \int_1^e x\log_e(x)\, dx$.

Use integration by recognition to find $\int x\log_e(x)\, dx$.

From **part a.**:

$\dfrac{dy}{dx} = x + 2x\log_e(x)$ where $y = x^2\log_e(x)$.

Integrate both sides with respect to *x*:

$$y = \int x + 2x\log_e(x)\, dx$$

$$\Rightarrow x^2\log_e(x) = \int x + 2x\log_e(x)\, dx$$ **M**1

$$\Rightarrow x^2\log_e(x) = \int x\, dx + \int 2x\log_e(x)\, dx \qquad \Rightarrow x^2\log_e(x) = \frac{1}{2}x^2 + 2\int x\log_e(x)\, dx$$

$$\Rightarrow \int x\log_e(x)\, dx = \frac{1}{2}x^2\log_e(x) - \frac{1}{4}x^2$$

Therefore area $= \int_1^e x\log_e(x)\, dx$

$$= \left[\frac{1}{2}x^2\log_e(x) - \frac{1}{4}x^2\right]_1^e$$ **M**1

$$= \frac{1}{2}e^2\log_e(e) - \frac{1}{4}e^2 - \left(\frac{1}{2}\log_e(1) - \frac{1}{4}\right) = \frac{1}{2}e^2 - \frac{1}{4}e^2 + \frac{1}{4} = \frac{1}{4}e^2 + \frac{1}{4}.$$

Answer: $\frac{1}{4}e^2 + \frac{1}{4}$. **A**1

Question 9

a. $2\cos^2(\theta)=3\sin(\theta) \quad \Rightarrow 2\left(1-\sin^2(\theta)\right)=3\sin(\theta)$

$\Rightarrow 2\sin^2(\theta)+3\sin(\theta)-2=0$ **M**1

$\Rightarrow (2\sin(\theta)-1)(\sin(\theta)+2)=0$.

Case 1: $\sin(\theta)+2=0 \quad \Rightarrow \sin(\theta)=-2$. No real solutions.

Case 2: $2\sin(\theta)-1=0$ **M**1

$\Rightarrow \sin(\theta)=\dfrac{1}{2}$

$\Rightarrow \theta=\dfrac{\pi}{6}+2n\pi$ or $\theta=\dfrac{5\pi}{6}+2n\pi$, $n\in R$.

Answer: $\theta=\dfrac{\pi}{6}+2n\pi$ or $\theta=\dfrac{5\pi}{6}+2n\pi$, $n\in R$. **A**1

b. Let $D=2\cos^2(\theta)-3\sin(\theta)$.

$\dfrac{dD}{d\theta}=-4\cos(\theta)\sin(\theta)-3\cos(\theta)$. **M**1

$\dfrac{dD}{d\theta}=0 \Rightarrow -4\cos(\theta)\sin(\theta)-3\cos(\theta)=0$

$\Rightarrow \cos(\theta)(-4\sin(\theta)-3)=0$.

Case 1: $\cos(\theta)=0$. **M**$\frac{1}{2}$

$\cos(\theta)=0 \quad \Rightarrow \sin(\theta)=\pm 1$.

Substitute $\cos(\theta)=0$ and $\sin(\theta)=\pm 1$ into $D=2\cos^2(\theta)-3\sin(\theta)$:

$D=\pm 3$.

Case 2: $-4\sin(\theta)-3=0$

$\Rightarrow \sin(\theta)=-\dfrac{3}{4}$. **M**$\frac{1}{2}$

$\sin(\theta)=-\dfrac{3}{4} \quad \Rightarrow \cos^2(\theta)=1-\left(-\dfrac{3}{4}\right)^2=\dfrac{7}{16}$.

Substitute $\sin(\theta) = -\frac{3}{4}$ and $\cos^2(\theta) = \frac{7}{16}$ into $D = 2\cos^2(\theta) - 3\sin(\theta)$:

$D = 2\left(\frac{7}{16}\right) - 3\left(-\frac{3}{4}\right) = \frac{14}{16} + \frac{9}{4} = \frac{50}{16} = \frac{25}{8}$.

Answer: $\frac{25}{8}$. **A**1

Question 10

a. Solve $3^{2x} = 3^{x+1} + 4$.

Implied restriction: There is no implied restriction on *x*.

$3^{2x} = 3^{x+1} + 4 \quad \Rightarrow 3^{2x} - 3^{x+1} - 4 = 0$

$\Rightarrow \left(3^x\right)^2 - 3\left(3^x\right) - 4 = 0$ **M**1

$\Rightarrow \left(3^x - 4\right)\left(3^x + 1\right) = 0$.

Case 1: $3^x + 1 = 0 \Rightarrow 3^x = -1$. No real solutions.

Case 2: $3^x - 1 = 0 \Rightarrow 3^x = 4$.

$3^x = 4 \quad \Rightarrow x = \log_3(4)$. **A**1

This solution provides two intervals that are candidate solutions to $3^{2x} < 3^{x+1} + 4$:

$x < \log_3(4)$ and $x > \log_3(4)$.

A convenient value of *x* can be used to test $3^{2x} < 3^{x+1} + 4$ on each interval.

$x < \log_3(4)$. Test $x = \log_3(3) = 1$: $3^2 < 3^{1+1} + 4$ ✓

$x > \log_3(4)$. Test $x = \log_3(9) = 2$: $3^4 < 3^{2+1} + 4$ X

Answer: $x < \log_3(4)$. **H**1

b. **Implied restrictions on *x*:** The base of the logarithm cannot be negative or equal to 1. Therefore, only values of *x* that lie in the intervals

$0 < 2x < 1 \Rightarrow 0 < x < \frac{1}{2}$. **M**$\frac{1}{2}$

$2x > 1 \Rightarrow x > \frac{1}{2}$ **M**$\frac{1}{2}$

are acceptable as solutions.

Method 1: Solve $\log_{2x}(16) = -2$.

$\log_{2x}(16) = -2 \qquad \Rightarrow 16 = (2x)^{-2} = \frac{1}{(2x)^2} = \frac{1}{4x^2} \qquad \Rightarrow 64 = \frac{1}{x^2}$

$\Rightarrow x^2 = \frac{1}{64} \qquad \Rightarrow x = \pm\frac{1}{8}$.

$x = -\frac{1}{8}$ is rejected because of the implied restriction on *x*.

Therefore $x = \frac{1}{8}$. **A**1

The implied restriction on *x* and the solution $x = \frac{1}{8}$ provide three intervals that are candidate solutions to $\log_{2x}(16) < -2$:

$0 < x < \frac{1}{8}, \ \frac{1}{8} < x < \frac{1}{2}, \ x > \frac{1}{2}$.

A convenient value of *x* (use a power of 2) can be used to test $\log_{2x}(16) < -2$ on each interval.

Case 1: $0 < x < \frac{1}{8}$. Test $x = \frac{1}{32}$: $\log_{\frac{1}{16}}(16) = -1$ X

Case 2: $\frac{1}{8} < x < \frac{1}{2}$. Test $x = \frac{1}{4}$: $\log_{\frac{1}{2}}(16) = -4$ ✓

Case 3: $x > \frac{1}{2}$. Test $x = 1$: $\log_2(16) = 4$ X

Answer: $\frac{1}{8} < x < \frac{1}{2}$. **H**1

Method 2:

Case 1: $0 < 2x < 1 \Rightarrow 0 < x < \frac{1}{2}$.

By inspection there is a solution because $\log_{2x}(16) < 0$ when $0 < 2x < 1$:

$\log_{2x}(16) < -2 \qquad \Rightarrow 16 > (2x)^{-2}$ (you need to realise that the direction of the inequality sign must be reversed)

$\Rightarrow 16 > \frac{1}{(2x)^2} \qquad \Rightarrow 16 > \frac{1}{4x^2} \qquad \Rightarrow 64 > \frac{1}{x^2} \qquad \Rightarrow \frac{1}{64} < x^2 \qquad \Rightarrow x^2 > \frac{1}{64}$

$\Rightarrow x > \frac{1}{8} \cup x < -\frac{1}{8}$.

$x < -\frac{1}{8}$ is rejected because of the implied restriction on *x*.

Therefore $x > \frac{1}{8}$. **A1**

Apply the case 1 restriction $0 < x < \frac{1}{2}$: $\frac{1}{8} < x < \frac{1}{2}$.

Case 2: $2x > 1 \Rightarrow x > \frac{1}{2}$.

By inspection there is no solution because $\log_{2x}(16) > 0$ when $2x > 1$.

Answer: $\frac{1}{8} < x < \frac{1}{2}$. **H1**

WARNING:

$\log_{2x}(16) < -2 \Rightarrow 16 < (2x)^{-2}$ is **NOT** correct and **cannot** be used to solve the inequation.

For example, if $x = \frac{1}{4}$ then $\log_{\frac{1}{2}}(16) = -4 < -2$ but $16 < \left(\frac{1}{2}\right)^{-2}$ is **not** correct.

The inequality sign must be reversed in order to get a correct inequality: $16 > \left(\frac{1}{2}\right)^{-2}$.

Why is this ….? Consider

$16 > (2x)^{-2}$. …. (1)

Note that both sides of (1) are positive since $x > 0$.

If you simply take the logarithm to the base $2x$ of both sides of (1) you get

$\log_{2x}(16) > -2$. … (2)

But in taking the logarithm of both sides of (2), we see that both sides become negative if $0 < 2x < 1$.

The direction of the inequality must therefore be reversed in this case so as to maintain a correct inequality:

$16 > (2x)^{-2} \Rightarrow \log_{2x}(16) < -2$.

Therefore using $\log_{2x}(16) < -2 \Rightarrow 16 < (2x)^{-2}$ to solve the inequality **is not valid**.