The Mathematical Association of Victoria

Trial Examination 2020

MATHEMATICAL METHODS

Trial Written Examination 2 - SOLUTIONS

SECTION A: Multiple Choice

Question	Answer	Question	Answer
1	B	**11**	C
2	B	**12**	D
3	A	**13**	B
4	D	**14**	E
5	E	**15**	A
6	C	**16**	D
7	C	**17**	E
8	E	**18**	B
9	C	**19**	B
10	A	**20**	A

Question 1 **Answer B**

$$y=\frac{a}{x-5}+6$$

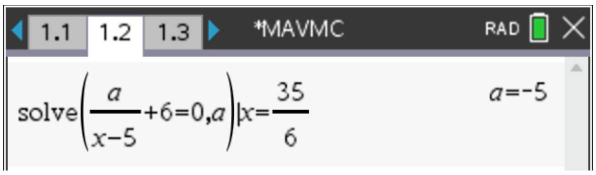
$$\frac{a}{\frac{35}{6}-5}+6=0$$

$$\frac{6a}{35-30}=-6$$

$$a=-5$$

$$y=\frac{-5}{x-5}+6$$

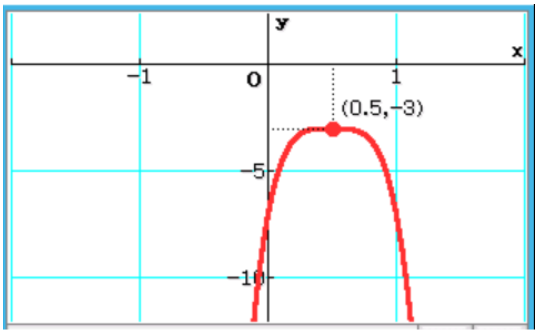
$$y=\frac{5}{5-x}+6$$



Question 2 **Answer B**

The graph of $y = -4(2x-1)^4 - 3$ is quartic reflected in the x-axis.

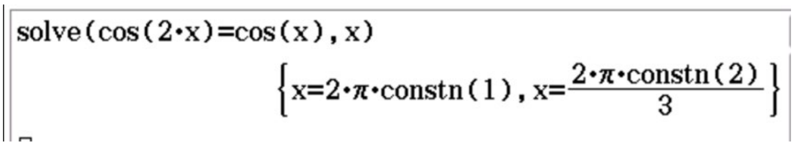
$\frac{dy}{dx} = -32(2x-1)^3 = 0$ gives stationary point at $x = \frac{1}{2}$ but **not** a point of inflexion



At the point $\left(\frac{1}{2}, -3\right)$ the gradient is zero.

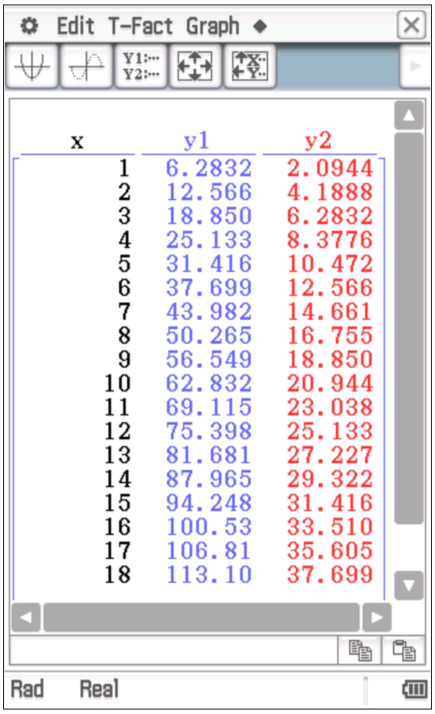
Question 3 **Answer A**

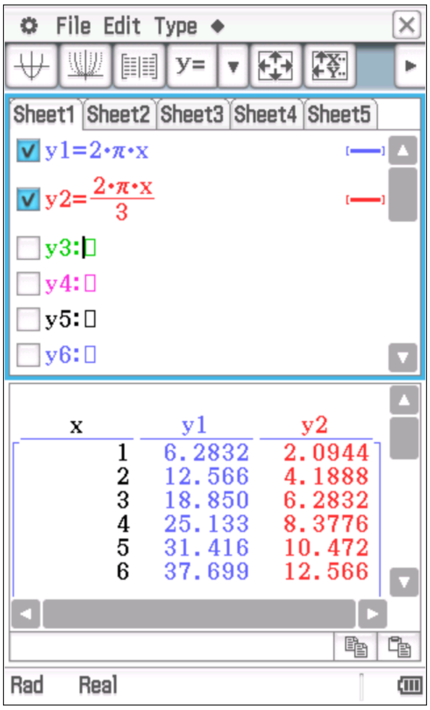
Solving $\cos(2x) = \cos(x)$



We read this as $x = 2\pi k, x = \frac{2\pi k}{3}$ where $k \in Z$

Option A : $x = \frac{2\pi k}{3}$ where $k \in Z$, also includes the set of solutions $x = 2\pi k$

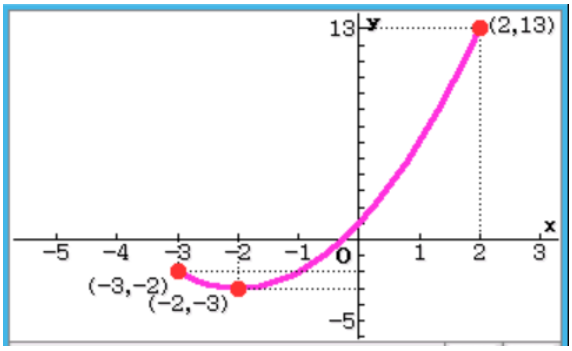




Question 4 **Answer D**

$f:[-3,4)\to R, f(x)=2x+1$ and $g:[-4,2]\to R, g(x)=x^2+2x$.

The domain of $h(x)=f(x)+g(x)$ is the intersection of the domains of f and $g=[-3,2]$



The range of $h(x)=f(x)+g(x)$ is $[-3,13]$

Question 5 **Answer E**

$y=\frac{1}{2x}$ to be transformed to $y_T=-\frac{3}{x-1}+6$.

Step 1: to change $y=\frac{1}{2x}$ to $y=\frac{1}{x}$ we can dilate from the x-axis by a factor of 2.

$y=\frac{1}{2x}\Rightarrow y_1=\frac{2}{2x}\Rightarrow y_1=\frac{1}{x}$

Step 2: to change $y_1=\frac{1}{x}$ to $y_2=\frac{3}{x}$ we can dilate from the y-axis by a factor of 3.

$y_1=\frac{1}{\left(\frac{x}{3}\right)}\Rightarrow y_2=\frac{3}{x}$

Step 3: to change $y_2=\frac{3}{x}$ to $y_3=-\frac{3}{x}$ we reflect in the x-axis

Step 4: to change $y_3=-\frac{3}{x}$ to $y_4=-\frac{3}{x-1}+6$ we translate in the positive direction of the x-axis by 1 unit and the y-axis by 6 units.

This is the required image graph: $y_T=-\frac{3}{x-1}+6$

Question 6 **Answer C**

$2x + ky = a$
$kx + 3y = 7$

Method 1 (Using ratios)

The simultaneous equations will have no solutions when $\frac{k}{2} = \frac{3}{k} \neq \frac{7}{a}$ or $\frac{2}{k} = \frac{k}{3} \neq \frac{a}{7}$

$\frac{k}{2} = \frac{3}{k}$

$k^2 = 6$

$k = \pm\sqrt{6}$

When $k = \sqrt{6}$

$\frac{\sqrt{6}}{2} \neq \frac{7}{a}$

$a \neq \frac{14}{\sqrt{6}}$

When $k = -\sqrt{6}$

$\frac{-\sqrt{6}}{2} \neq \frac{7}{a}$

$a \neq -\frac{14}{\sqrt{6}}$

$k = \sqrt{6}$ and $a \in R \setminus \left\{\frac{14}{\sqrt{6}}\right\}$ or $k = -\sqrt{6}$ and $a = R \setminus \left\{-\frac{14}{\sqrt{6}}\right\}$

Method 2 (using gradient and intercept)

Simultaneous equations will have no solutions when the gradients are equal and the y-intercepts are different.

$2x+ky=a,\ y=-\frac{2}{k}x+\frac{a}{k},$

$kx+3y=7,\ y=-\frac{k}{3}x+\frac{7}{3}$

$m_1=m_2$

$-\frac{2}{k}=-\frac{k}{3}$

$k^2=6$

$k=\pm\sqrt{6}$

$c_1\neq c_2$

$\frac{a}{k}\neq\frac{7}{3}$, $a\neq\frac{7k}{3}$

If $k=\sqrt{6},\ a\neq\frac{7\sqrt{6}}{3}$ i.e. $a\neq\frac{14}{\sqrt{6}}$

If $k=-\sqrt{6},\ a\neq\frac{-7\sqrt{6}}{3}$ i.e. $a\neq\frac{-14}{\sqrt{6}}$

Question 7 **Answer C**

$f(x)=2\log_e(1-4x)+1$

Let $y=2\log_e(1-4x)+1$

Inverse swap x and y

$x=2\log_e(1-4y)+1$

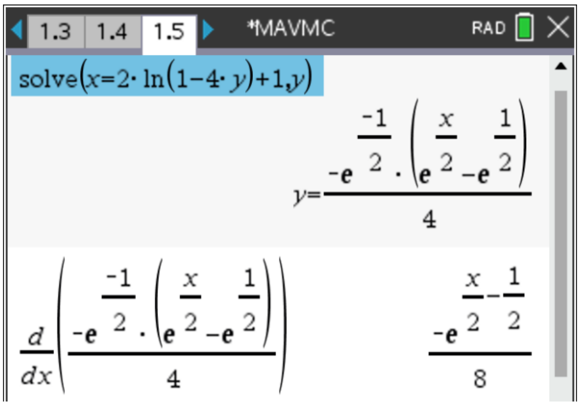
$\log_e(1-4y)=\frac{x-1}{2}$

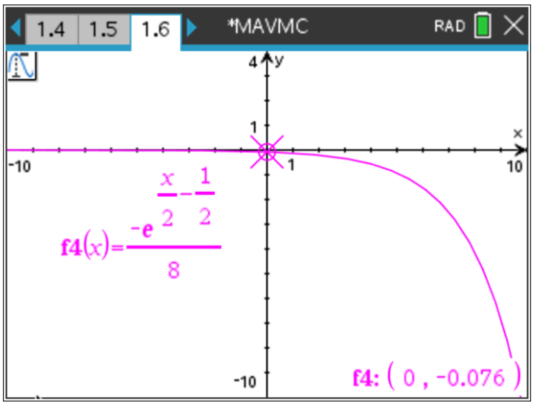
$1-4y=e^{\frac{x-1}{2}}$

$y=f^{-1}(x)=\frac{1}{4}\left(1-e^{\frac{x-1}{2}}\right)$

$\frac{dy}{dx}=-\frac{1}{8}e^{\frac{x-1}{2}}$

$\frac{dy}{dx}=-\frac{1}{8}e^{\frac{x-1}{2}}$ has an asymptote with equation $y=0$ and a y-axis intercept at $-\frac{1}{8}e^{-\frac{1}{2}}$.





Question 8 **Answer E**

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}+\begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$x' = y + 2$

$y' = x - 3$

$y' = x'^{\frac{1}{3}} + 1$

$x - 3 = (y+2)^{\frac{1}{3}} + 1$

$y = (x-4)^3 - 2$

$f(x) = a(x-b)^3 + c$

$a = 1,\ b = 4,\ c = -2$

OR

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}+\begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$x' = y + 2$

$y' = x - 3$

$y = x' - 2$ and $x = y' + 3$

$y = a(x-b)^3 + c$ and so $x' - 2 = a(y' + 3 - b)^3 + c$

$\dfrac{x' - 2 - c}{a} = (y' + 3 - b)^3$ and so $(y' + 3 - b)^3 = \dfrac{x' - 2 - c}{a}$

Now $y' = x'^{\frac{1}{3}} + 1$ and so $a = 1,\ c = -2,\ 3 - b = -1$

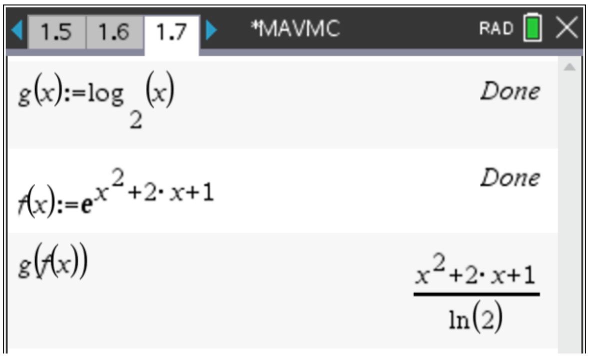
$a = 1,\ b = 4,\ c = -2$

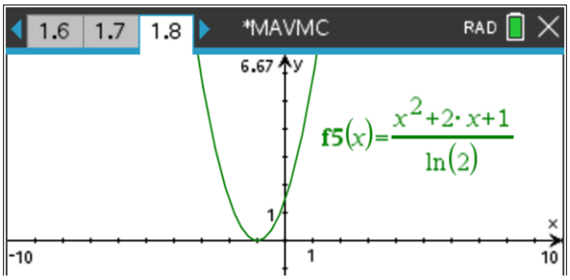
Question 9 **Answer C**

$f(x) = e^{x^2+2x+1}$ and $g(x) = \log_2(x)$

$g(f(x)) = \log_2\left(e^{(x+1)^2}\right)$

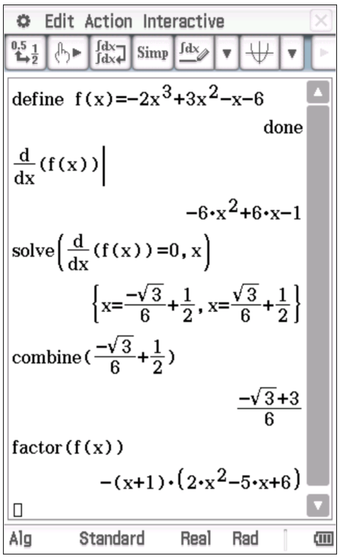
$g(f(x)) = \dfrac{(x+1)^2}{\log_e(2)}$, range is $[0, \infty)$

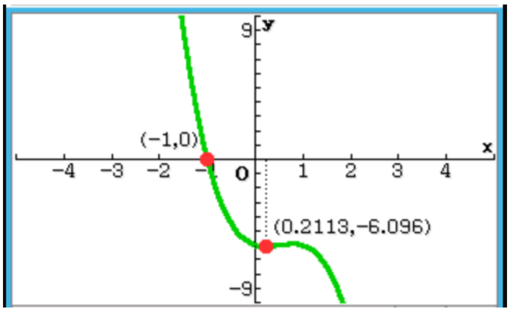




Question 10 **Answer A**

$Q(x) = -2x^3 + 3x^2 - x - 6$





One of the two stationary points of the graph is at $x = \frac{3-\sqrt{3}}{6}$.

Question 11 **Answer C**

Let $y = f\left(\log_e(2x)\right)$

Using the Chain rule $\frac{dy}{dx} = \frac{1}{x} f'\left(\log_e(2x)\right)$

Question 12 **Answer D**

$f(x) = ax^6 + bx^5 + x^4 - 3$

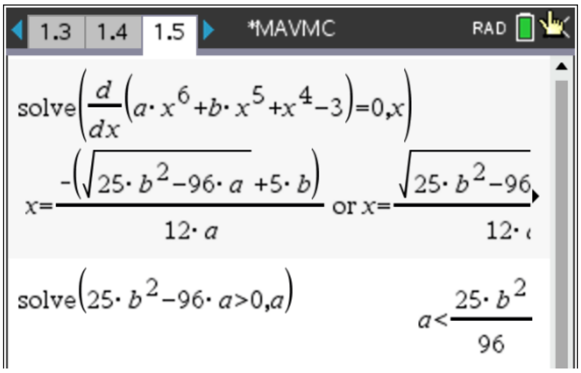
$f'(x) = 6ax^5 + 5bx^4 + 4x^3$

$x^3\left(6ax^2 + 5bx + 4\right) = 0$

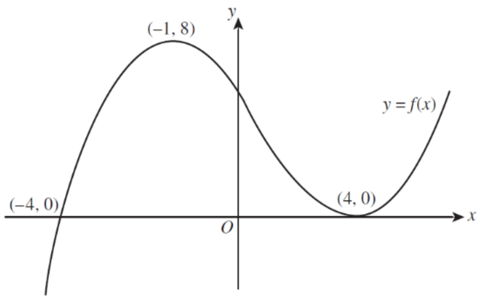
$x = 0,\ x = \frac{-5b \pm \sqrt{25b^2 - 96a}}{12a}$

Two more stationary points when $25b^2 - 96a > 0$

$a < \dfrac{25b^2}{96}$



Question 13 **Answer B**



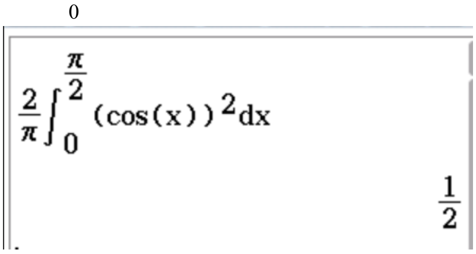
Options A, C, D and E are correct.
It is **incorrect** to state that the graph of the derivative of *f* is strictly increasing for $x \in (-\infty, -1] \cup [4, \infty)$.
These values relate to the graph of *f*, **not** the graph of the derivative of *f*.

Question 14 **Answer E**

$$\text{Average value } = \frac{1}{b-a}\int_a^b f(x)dx$$

Average value of $y = \cos^2(x)$ over the interval $\left[0, \frac{\pi}{2}\right]$ is $= \dfrac{1}{\frac{\pi}{2} - 0}\displaystyle\int_0^{\frac{\pi}{2}} \left(\cos^2(x)\right)dx$.

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \left(\cos^2(x)\right)dx = \frac{1}{2}$$



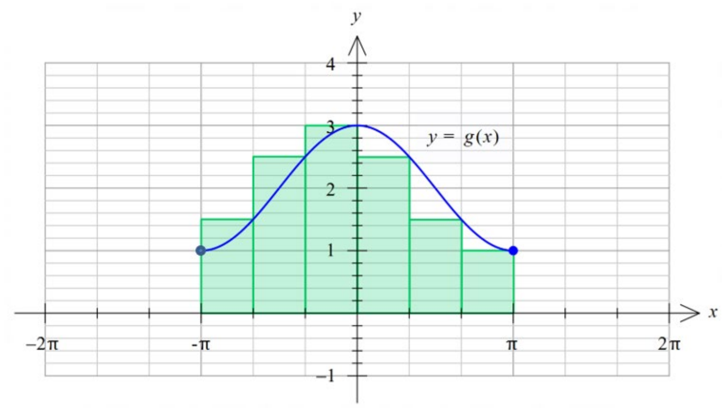
Question 15 **Answer A**

Given $\int_{-1}^{6} f(x) = 3$, swap limits to get $\int_{-1}^{6} -1 + 2f(x)\,dx$

$$\int_{-1}^{6} -1 + 2f(x)\,dx = \int_{-1}^{6} -1dx + 2\int_{-1}^{6} f(x)\,dx$$

$$= -7 + 2\int_{-1}^{6} f(x)\,dx = -7 + 2\times 3 = -1$$

Question 16 **Answer D**



$$\text{Area} = \frac{\pi}{3}\left(g\left(-\frac{2\pi}{3}\right) + g\left(-\frac{\pi}{3}\right) + g(0) + g\left(\frac{\pi}{3}\right) + g\left(\frac{2\pi}{3}\right) + g(\pi)\right)$$

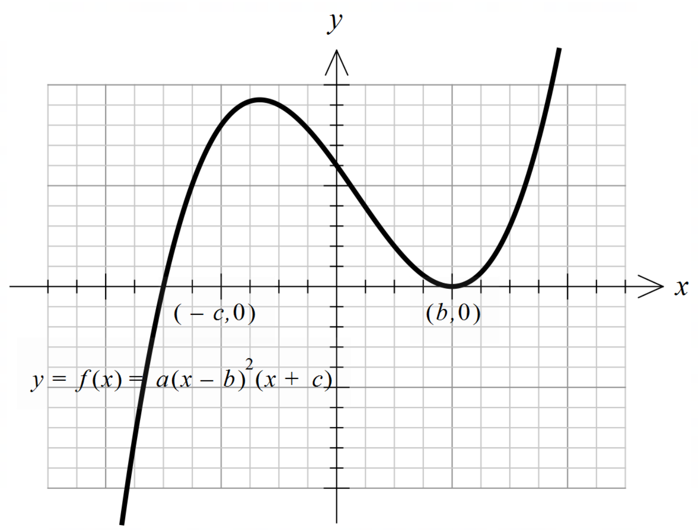
$$= \frac{\pi}{3}\left(2g\left(-\frac{2\pi}{3}\right) + 2g\left(-\frac{\pi}{3}\right) + g(0) + g(\pi)\right)$$

Question 17 **Answer E**

$f(x) = a(x-b)^2(x+c)$ where a, b and c are positive real constants.

The graph has x-intercepts at $(-c, 0)$ and $(b, 0)$.

Area = $\int_{-c}^{b} f(x)dx$



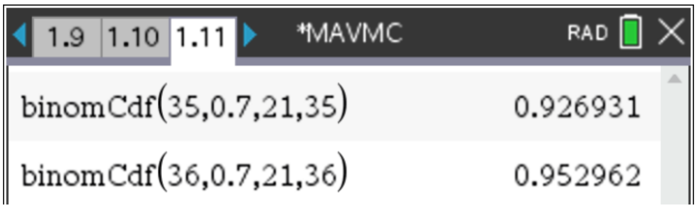
Question 18 **Answer B**

$X \sim \text{Bi}(n, 0.7)$

$\Pr(X > 20) > 0.95$

$\Pr(21 \le X \le 36) = 0.953$ correct to three decimal places

More than 35.



Question 19 **Answer B**

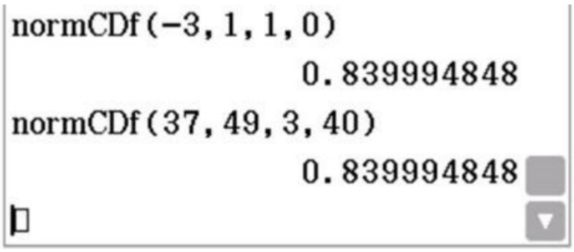
$X \sim \text{N}(40, 9)$

$Z \sim \text{N}(0, 1)$

$\Pr(-3 < Z < 1) = \Pr(-1 < Z < 3)$

$\Pr(40 - 3 < X < 40 + 3 \times 3)$

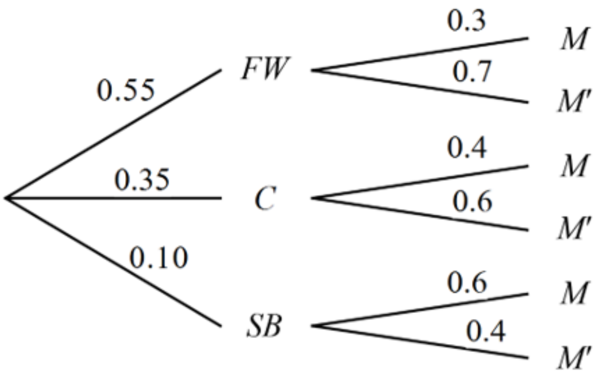
$= \Pr(37 < X < 49)$

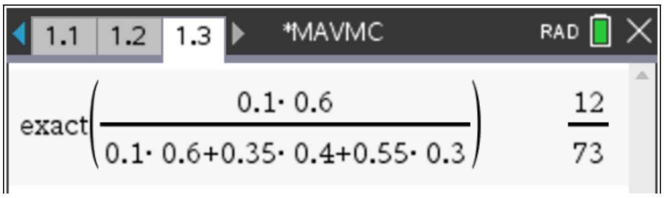


Question 20 **Answer A**

Let *FW* be flat white, *C* cappuccino, *SB* short black and *M* muffin.

$$\Pr(SB \mid M) = \frac{0.1\times 0.6}{0.1\times 0.6 + 0.35\times 0.4 + 0.55\times 0.3} = \frac{12}{73}$$





SECTION B

Question 1

$f(x)=a(x-b)^3(x-c)$

a. $f'(x)=a(x-b)^2(4x-b-3c)$ **1A in fully factorised form**

b. $f(x)=a(x-b)^3(x-c)$

Substituting $a=-\frac{1}{2},\ b=-3,\ c=1$ gives $f(x)=-\frac{1}{2}(x+3)^3(x-1)$.

By letting $y=0$, we get x-intercepts at $x=1$ and $x=-3$.

By using $f'(x)$ from **part a** we get $f'(x)=-\frac{1}{2}(x+3)^2(4x+3-3)=-\frac{1}{2}(x+3)^2(4x)$.

It follows that if $f'(x)=-\frac{1}{2}(x+3)^2(4x)$ we get

$f'(0)=-\frac{1}{2}(0+3)^2\times 0=0$ as required **1M Verify (2 parts)**

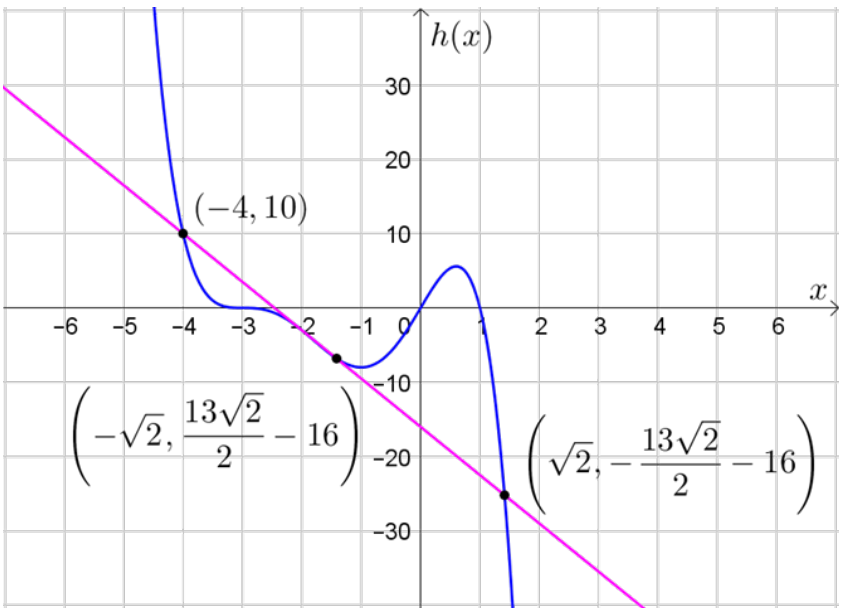
c. Stationary points are

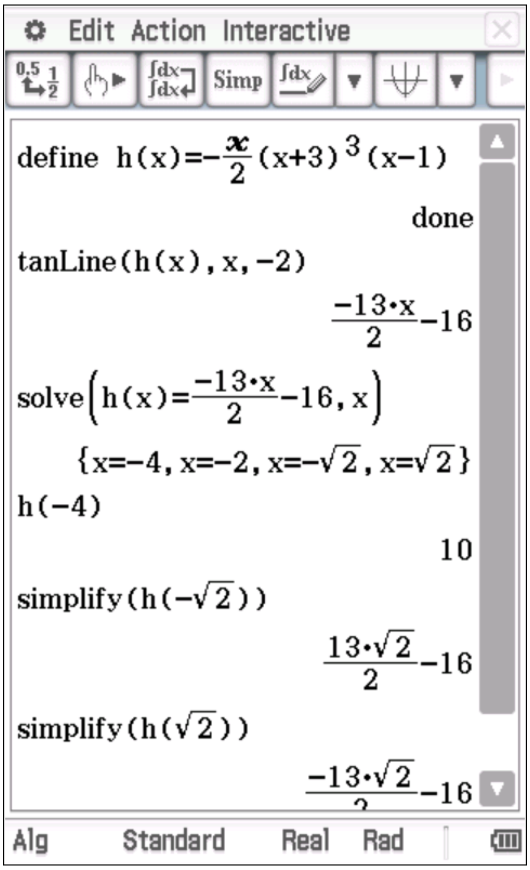
$\left(0,\frac{27}{2}\right)$: a local maximum turning point **1A**

$(-3,0)$: a stationary point of inflexion **1A**

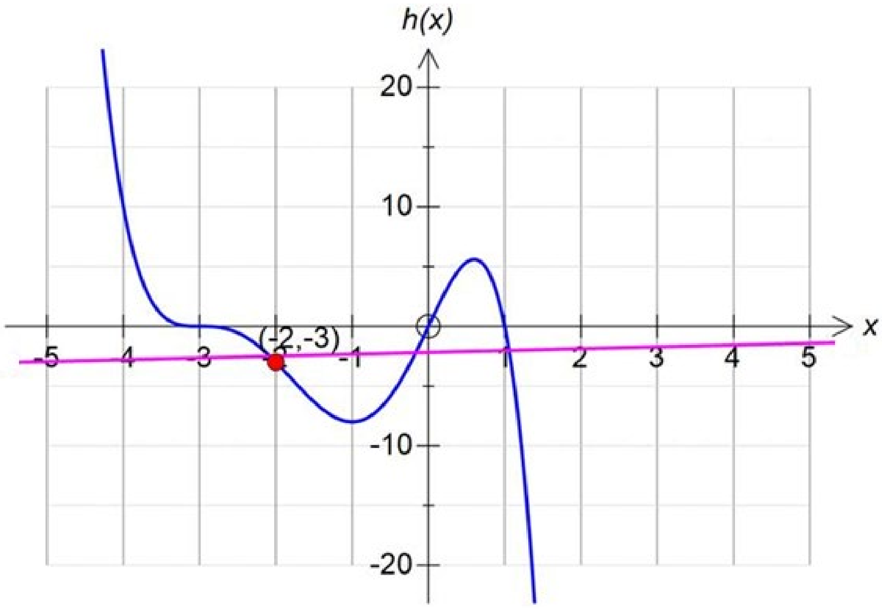
d. $(-4,10)$, $\left(-\sqrt{2},\frac{13\sqrt{2}}{2}-16\right)$, $\left(\sqrt{2},-\frac{13\sqrt{2}}{2}-16\right)$ **1A**

Correctly graphed **1A**





e. Correct line **1A**



f. $(-2,-3)$, $(-0.21,-2.72)$, $(1.07,-2.53)$ correct to two decimal places **2A**

$\text{solve}\left(h(x)=\frac{2\cdot x}{13}-\frac{35}{13},x\right)$

$\{x=-2, x=-0.2071888265, x=1.070070347\}$

h(−2)

−3

h(−0.2071)

−2.723074425

h(1.07007)

−2.52766751

g.i. Area $=\int_{-2}^{-0.21}\left(\frac{2}{13}x-\frac{35}{13}\right)-h(x)dx+\int_{-0.21}^{1.07}h(x)-\left(\frac{2}{13}x-\frac{35}{13}\right)dx$ **2A**

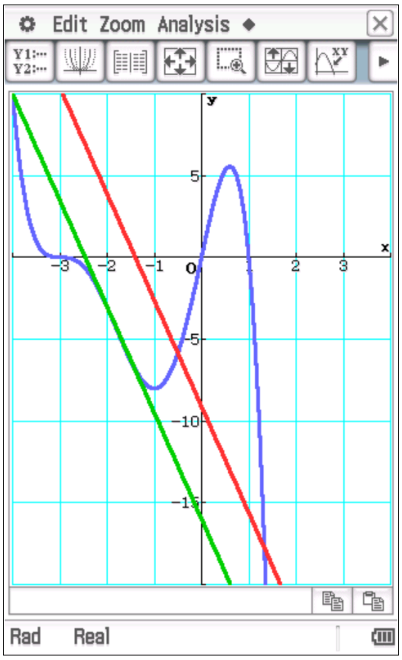
ii. Area =12.4 square units, correct to one decimal place. **1A**

h.i. Point on curve $\left(-\frac{1}{2},-\frac{375}{64}\right)$ **1A**

Gradient of tangent $=-\frac{13}{2}$.

Equation of the line parallel to the tangent going through point $\left(-\frac{1}{2},-\frac{375}{64}\right)$ is

$y=-\frac{13}{2}x-\frac{583}{64}$. **1A**



tanLine(h(x), x, −2)

$\frac{-13\cdot x}{2}-16$

$\text{solve}\left(y+\frac{375}{64}=-\frac{13}{2}\cdot\left(x+\frac{1}{2}\right),y\right)$

$\left\{y=\frac{-13\cdot x}{2}-\frac{583}{64}\right\}$

ii. Let A be a point on the tangent and B be a point on the parallel line.
Choose a point on one of the lines and let the point on the other line, $y = f(x)$, be $(a, f(a))$.

$A\left(a, -\frac{13a}{2} - 16\right)$ and $B\left(-\frac{1}{2}, -\frac{375}{64}\right)$

$$d(AB) = \sqrt{\left(-\frac{375}{64} - \left(-\frac{13a}{2} - 16\right)\right)^2 + \left(-\frac{1}{2} - a\right)^2}$$ **1M**

Find the minimum distance.
$d = 1.05$ correct to two decimal places **1A**

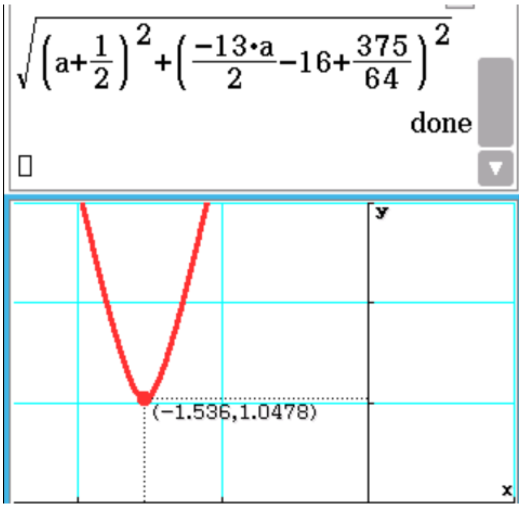
OR

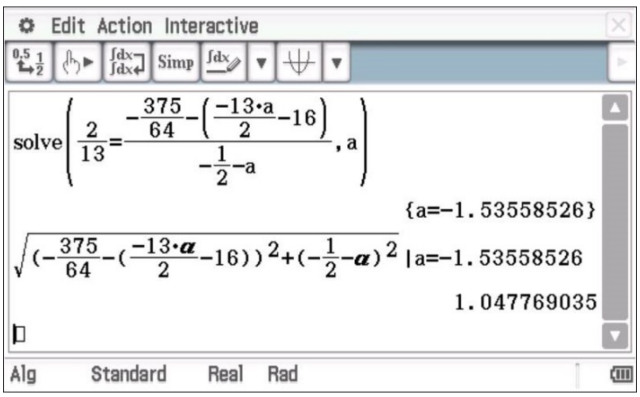
Use the perpendicular line. $m_p = \frac{2}{13}$

$$\frac{-\frac{375}{64} - \left(-\frac{13a}{2} - 16\right)}{-\frac{1}{2} - a} = \frac{2}{13},\ a = -1.53...$$ **1M**

$$d(AB) = \sqrt{\left(-\frac{375}{64} - \left(-\frac{13 \times -1.53...}{2} - 16\right)\right)^2 + \left(-\frac{1}{2} - (-1.53...)\right)^2}$$

$d = 1.05$ correct to two decimal places **1A**





Question 2

a. $C(t)=\dfrac{1000}{1+e^{-\frac{1}{5}(t-10)}}=\dfrac{1000}{1+\dfrac{1}{e^{\frac{1}{5}(t-10)}}}=\dfrac{1000e^{\frac{1}{5}(t-10)}}{e^{\frac{1}{5}(t-10)}+1}$

As required $C(t)=\dfrac{1000e^{\frac{1}{5}(t-10)}}{1+e^{\frac{1}{5}(t-10)}}$ **1M Show that**

b. $C(t)=\dfrac{1000e^{\frac{1}{5}(t-10)}}{1+e^{\frac{1}{5}(t-10)}}$

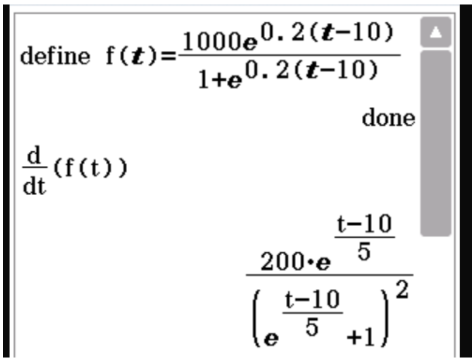
Using the quotient rule

$$C'(t)=\frac{\left(1+e^{\frac{1}{5}(t-10)}\right)200e^{\frac{1}{5}(t-10)}-1000e^{\frac{1}{5}(t-10)}\left(\frac{1}{5}e^{\frac{1}{5}(t-10)}\right)}{\left(1+e^{\frac{1}{5}(t-10)}\right)^2}$$

Simplify

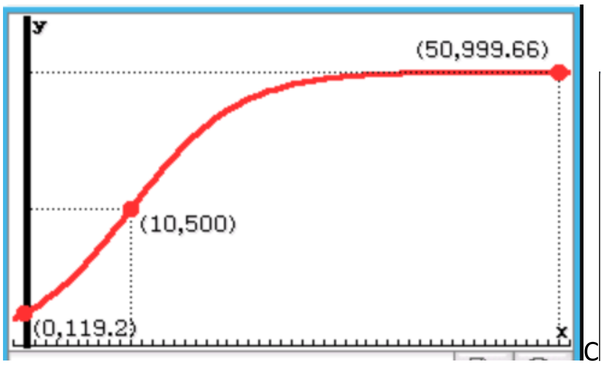
$$C'(t)=\frac{\left(200e^{\frac{1}{5}(t-10)}\right)\left(1+e^{\frac{1}{5}(t-10)}-e^{\frac{1}{5}(t-10)}\right)}{\left(1+e^{\frac{1}{5}(t-10)}\right)^2}$$

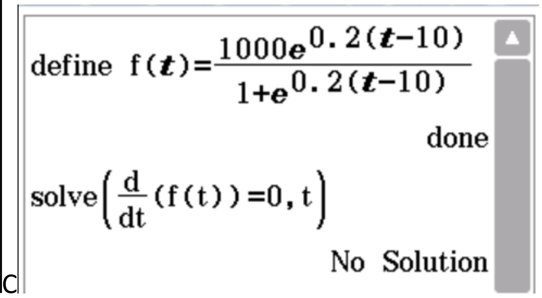
$$C'(t)=\frac{200e^{\frac{1}{5}(t-10)}}{\left(1+e^{\frac{1}{5}(t-10)}\right)^2}$$ **1M Show that**



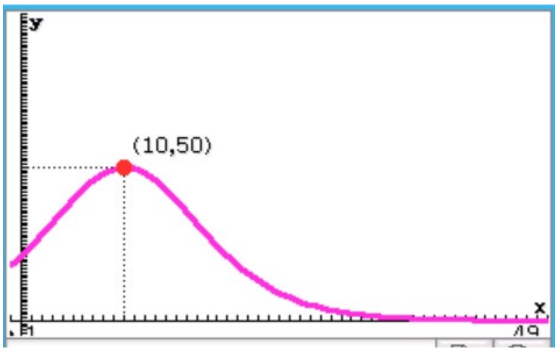
c. Let $C'(t)=0$ for stationary points.

Let $200e^{\frac{1}{5}(t-10)}=0$ gives no solutions.
So no stationary points. **1M Show that**





d. The gradient is at a maximum at $t = 10$. **1A**

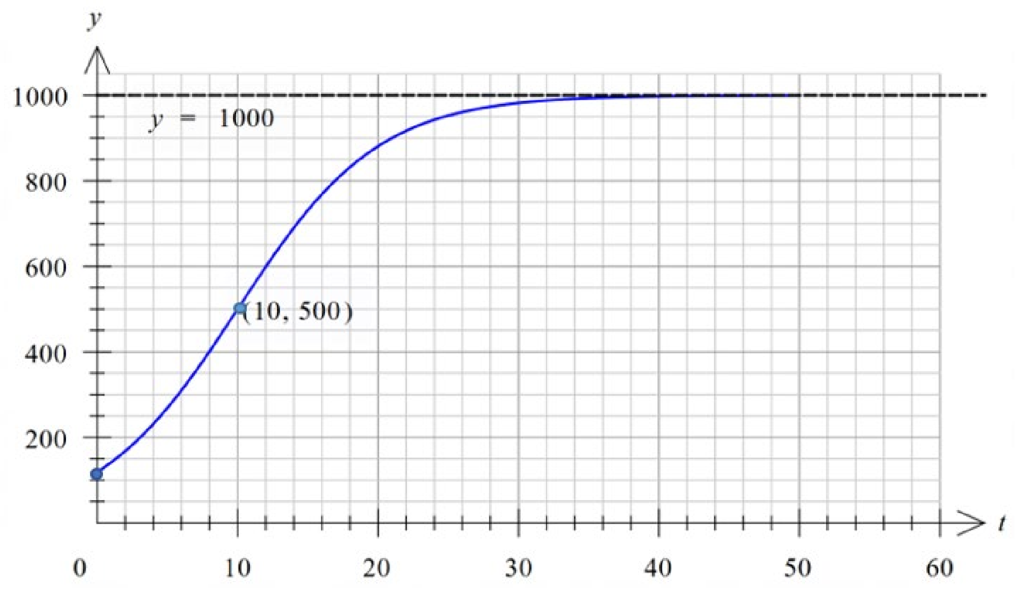


e. Gradient = 50 at t = 10. **1A**

f. Correct dilations **1A**, Correct translation **1A**

- Dilate from the t-axis by a factor of 1000
- Dilate from the y-axis by a factor of 5
- Translate in the positive t direction by 10 units

g. Correct shape and correct coordinate **1A**, Asymptote **1A**



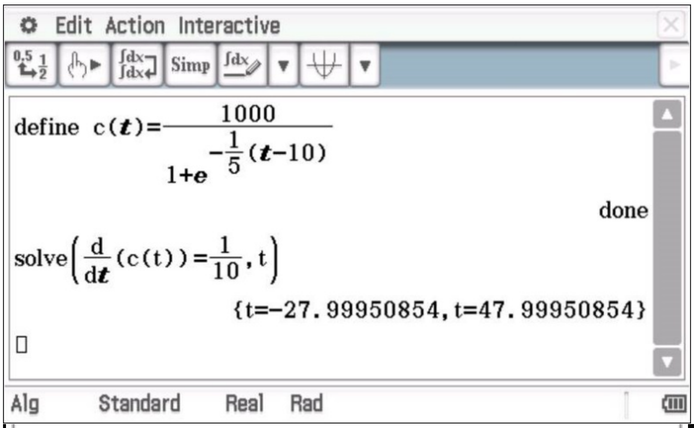
h. Will never reach 1000 confirmed cases as there is a horizontal asymptote at $C(t) = 1000$. **1A**

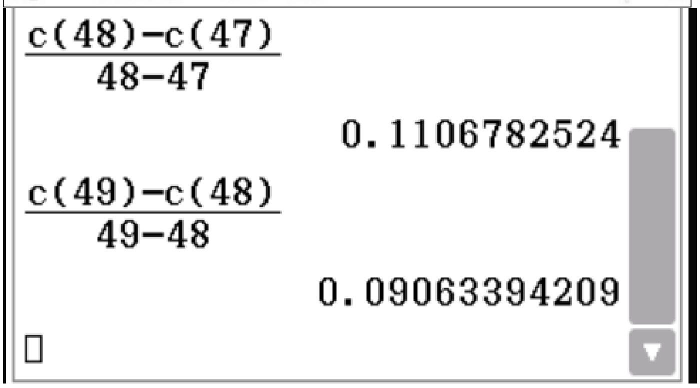
i. Solve $C'(t) = \frac{1}{10}$, $t = 48$ correct to the nearest integer

Average rate of change per day for the 48th day = $\frac{C(48) - C(47)}{48 - 47} = 0.11...$ **1M**

Average rate of change per day for the 49th day = $\frac{C(49) - C(48)}{49 - 48} = 0.09...$

Day 59 **1A**

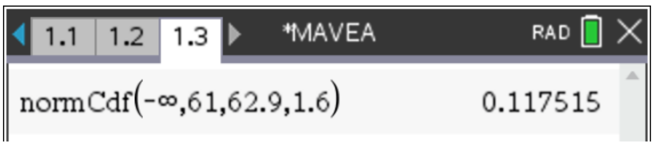




Question 3

a. $X \sim \mathrm{N}\left(62.9, 1.6^2\right)$

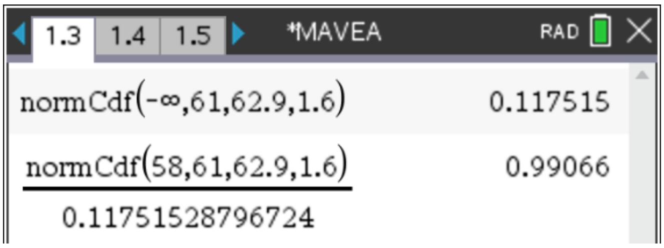
$\Pr\left(X < 61\right) = 0.1175$ correct to four decimal places **1A**



b. $\Pr\left(X > 58 \mid X < 61\right)$ **1M**

$= \frac{\Pr\left(58 < X < 61\right)}{\Pr\left(X < 61\right)}$

$= 0.991$ correct to three decimal places **1A**

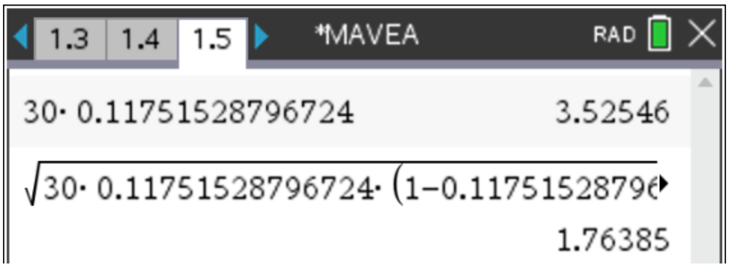


c. $Y \sim \text{Bi}(30, 0.1175...)$ **1A**

$\Pr(6 \le Y \le 30) = 0.1330$ correct to four decimal places **1A**

d. $E(Y) = np = 3.525$ correct to three decimal places **1A**

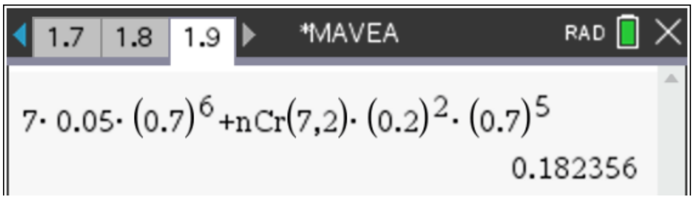
$\text{sd}(Y) = \sqrt{np(1-p)} = 1.764$ correct to three decimal places **1A**



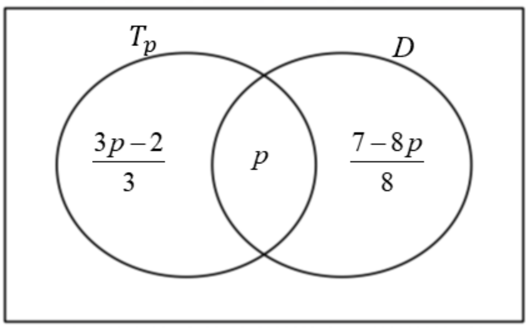
e. She could buy 2 packets on one day and none on the other 6 days or she could buy one packet on two days and none on the other days. **1M**

$\binom{7}{1}(0.05)(0.7)^6 + \binom{7}{2}(0.2)^2(0.7)^5$

$= 0.1824$ correct to four decimal places **1A**



f. Let T_p be the probability he plays with toilet paper and D the probability he eats his dinner.

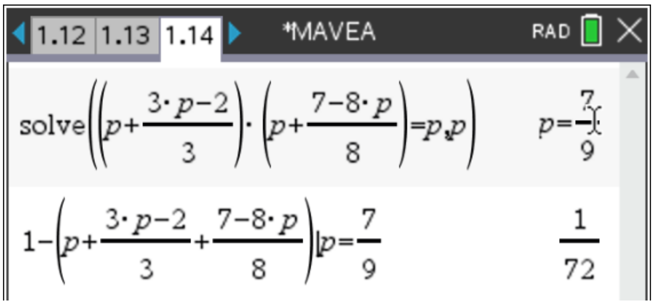


$\Pr(T_p) = \left(p + \frac{3p-2}{3}\right)$, $\Pr(D) = p + \frac{7-8p}{8}$

For independent events $\Pr\left(T_p \cap D\right) = \Pr\left(T_p\right) \times \Pr\left(D\right)$.

Solve $\left(p + \dfrac{3p-2}{3}\right)\left(p + \dfrac{7-8p}{8}\right) = p$ for p, $p = \dfrac{7}{9}$ **1M**

$\Pr\left(T_p' \cap D'\right) = 1 - \left(p + \dfrac{3p-2}{3} + \dfrac{7-8p}{8}\right) = \dfrac{1}{72}$ **1A**

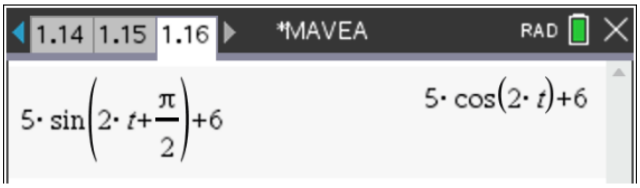


Question 4

a. $w_2 = 5\sin\left(2t + \dfrac{\pi}{2}\right) + 6$

Amplitude is 5, Period $\dfrac{2\pi}{2} = \pi$ **1A**

b. $w_2 = 5\cos\left(2t\right) + 6$ **1A**

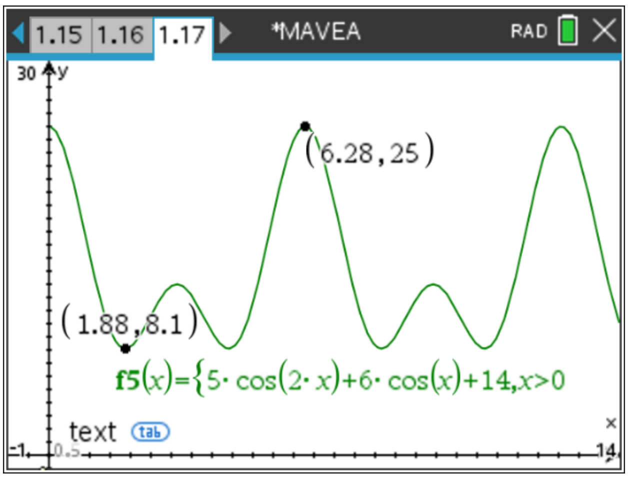


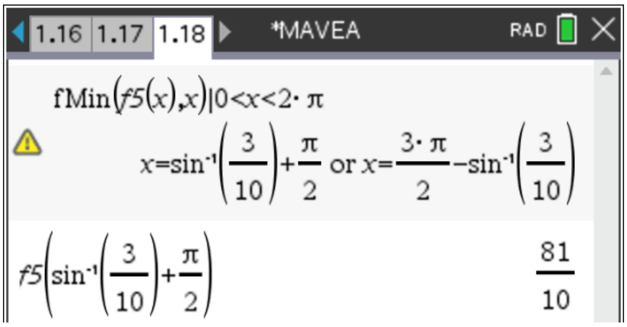
c. $w_r = w_1 + w_2 = 6\cos(t) + 8 + 5\cos(2t) + 6$

$w_r = 6\cos(t) + 5\cos(2t) + 14$ **1A**

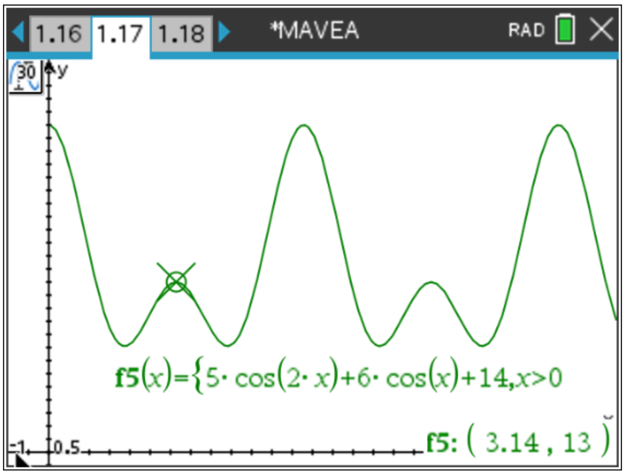
d. Period $= 2\pi$ **1A**

Range is $\left[\dfrac{81}{10},\ 25\right]$ **1A**





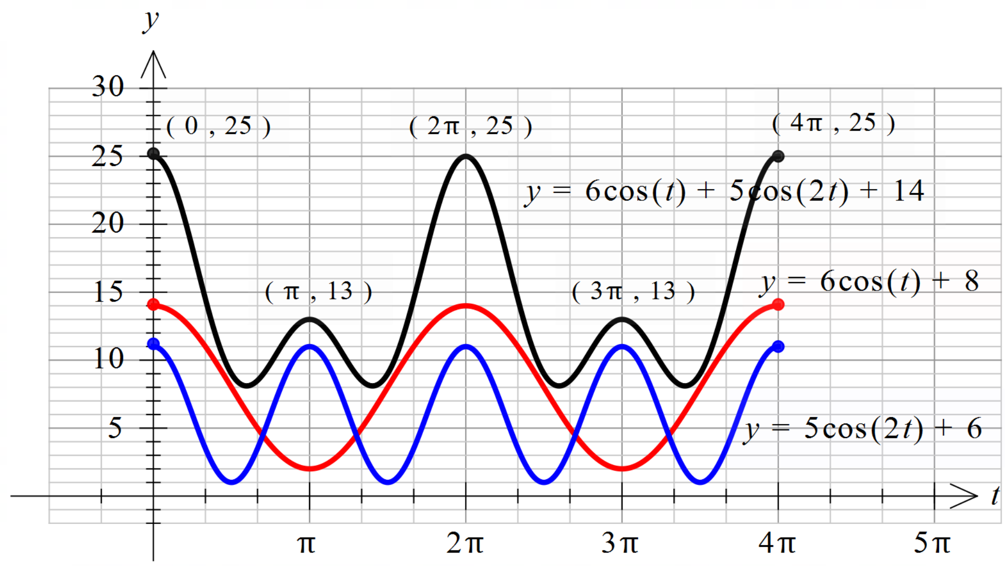
e. π and 3π **1A**



f. w_1 and w_2 labelled correctly **1A**

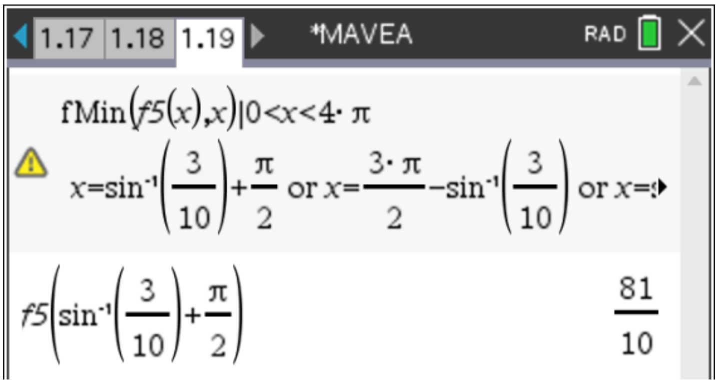
w_r drawn correctly **1A**

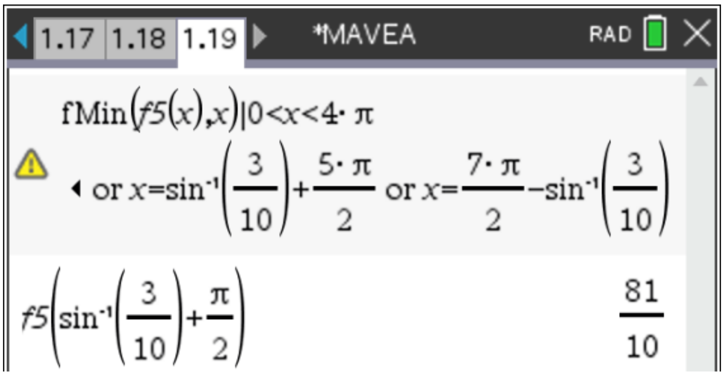
Correct coordinates **1A**



g. $\left(\sin^{-1}\left(\frac{3}{10}\right)+\frac{\pi}{2},\ \frac{81}{10}\right),\ \left(\frac{3\pi}{2}-\sin^{-1}\left(\frac{3}{10}\right),\ \frac{81}{10}\right)$

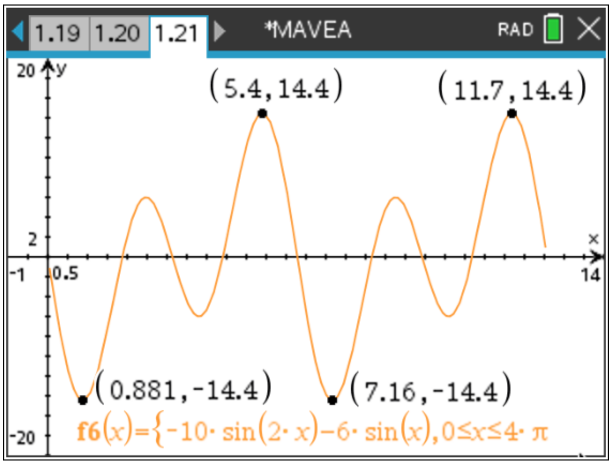
$\left(\sin^{-1}\left(\frac{3}{10}\right)+\frac{5\pi}{2},\ \frac{81}{10}\right),\ \left(\frac{7\pi}{2}-\sin^{-1}\left(\frac{3}{10}\right),\ \frac{81}{10}\right)$ **1A**

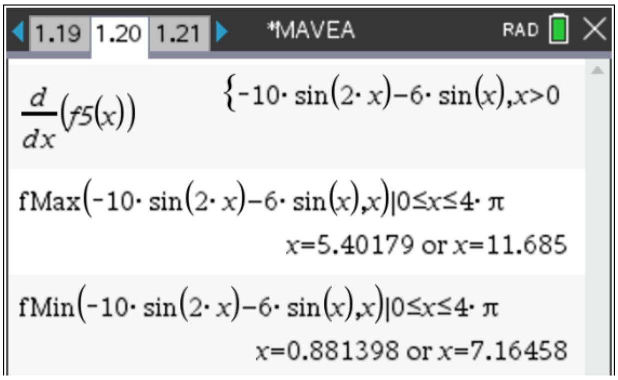




h. Sketch the graph of the derivative and find the coordinates of the maximum and minimum turning points or find the derivative and use the maximum and minimum functions. **1M**

$t=0.9,\ 5.4,\ 7.2,\ 11.7$ **1A**

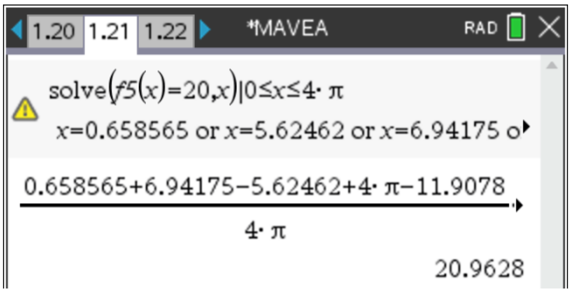




i. $w_r(t) = 20$

$$\frac{0.658... + (6.941... - 5.624...) + 4\pi - 11.907...}{4\pi} \times 100$$ **1M**

$= 20.96\%$ **1A**



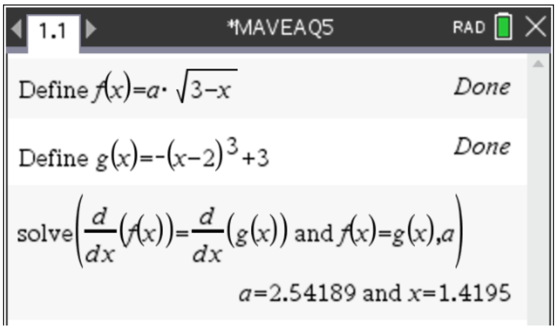
Question 5

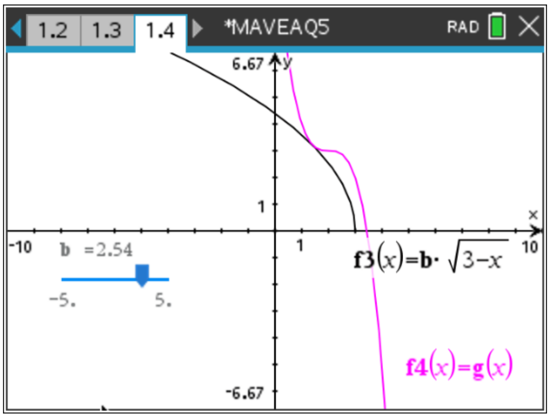
$f(x) = a\sqrt{3-x}$ and $g(x) = -(x-2)^3 + 3$, where $a \in R \setminus \{0\}$

a. Solve $f(x) = g(x)$ and $f'(x) = g'(x)$ for a. **1M**

There will be one solution if the curves touch and their gradients will also be equal at the point of intersection. This occurs when $x = 1.419\ldots$

$a = 2.542$ correct to three decimal places **1A**



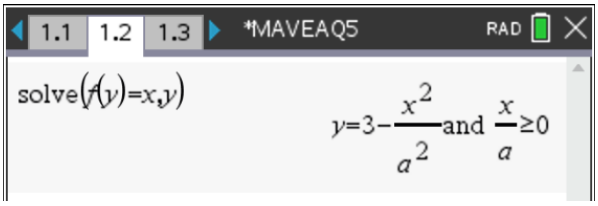


b. Let $y = a\sqrt{3-x}$

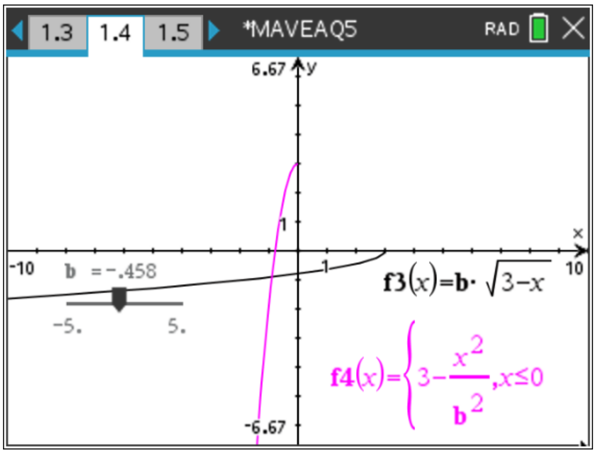
Inverse swap x and y and solve for y.

$x = a\sqrt{3-y}$

$f^{-1}(x) = 3 - \dfrac{x^2}{a^2}$ and $x \le 0$ **1A**

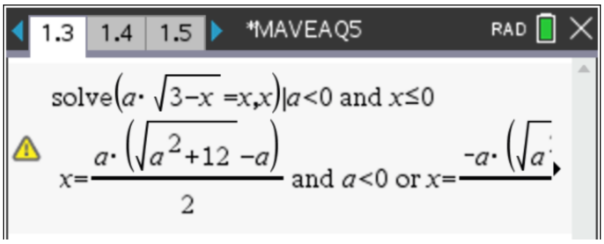


Note: use a different variable name in the graphing section.



c. Solve $f(x) = x$ for x.

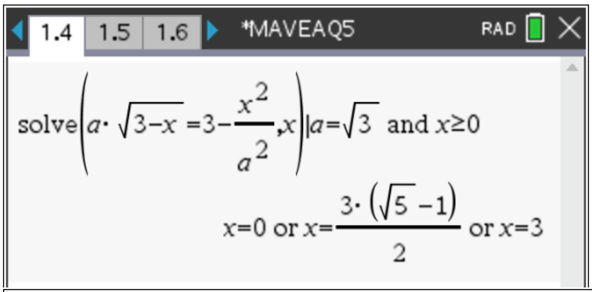
Intersection $\left(\dfrac{-a^2 + a\sqrt{a^2+12}}{2}, \dfrac{-a^2 + a\sqrt{a^2+12}}{2}\right)$ **1A**

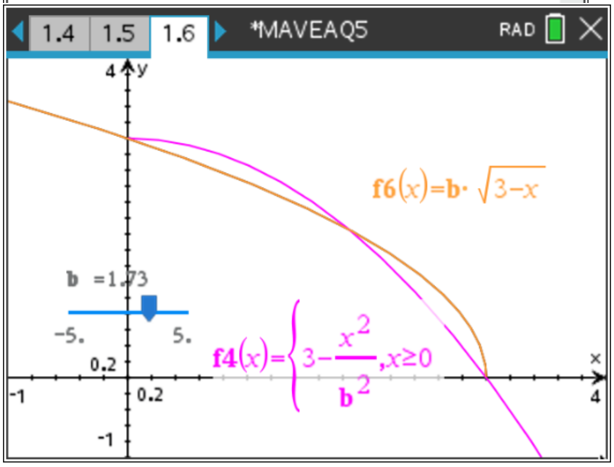


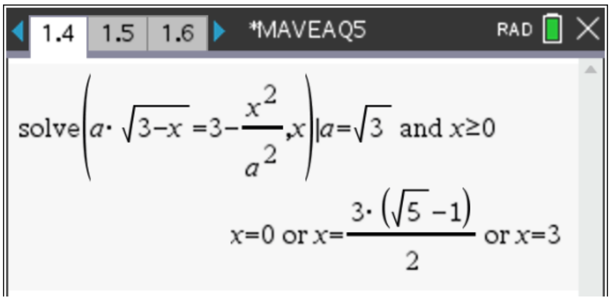
d. Substitute (0, 3) into $f(x) = a\sqrt{3-x}$ and solve for a.
Since there are three points of intersections, one point of intersection is on the line $y = x$ and the other two are at (0, 3) and (3, 0).

$3 = a\sqrt{3}$

$a = \sqrt{3}$ **1A**

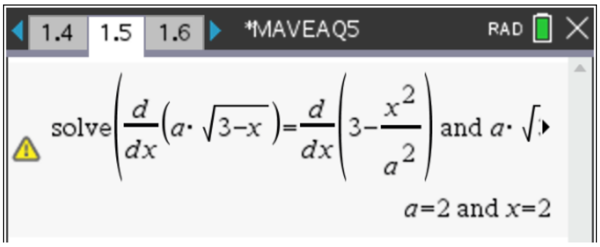


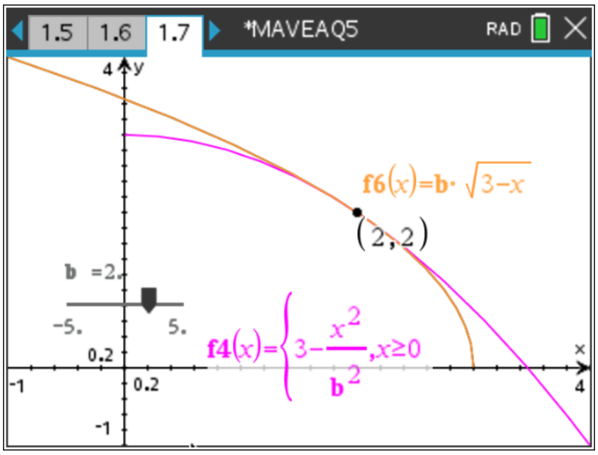




e. Solve $f(x) = f^{-1}(x)$ and $f'(x) = f^{-1\prime}(x)$ for a. **1M**
There will be one solution if the curves touch and their gradients will also be equal at the point of intersection. This occurs when $a = 2$ and $x = 2$.

$a \in \left[\sqrt{3}, 2\right)$ **1A**





END OF SOLUTIONS