**The Mathematical Association of Victoria**

# Trial Examination 2020

# MATHEMATICAL METHODS

## Trial Written Examination 1 - SOLUTIONS

**Question 1**

$h(x) = \dfrac{x^2}{\tan(2x)}$ Use the quotient rule.

$h'(x) = \dfrac{2x\tan(2x) - 2x^2\sec^2(2x)}{\tan^2(2x)}$ **1A**

Other forms
**OR**

$h'(x) = \dfrac{2x\left(\sin(2x)\cos(2x) - x\right)}{\sin^2(2x)}$ **1A**

**Question 2**

$f(x) = \log_e\left(\sin(2x)\right)$ Use the chain rule.

$f'(x) = \dfrac{2\cos(2x)}{\sin(2x)}$ **1A**

$f'\left(\dfrac{\pi}{3}\right) = \dfrac{2\cos\left(\dfrac{2\pi}{3}\right)}{\sin\left(\dfrac{2\pi}{3}\right)}$

$= \dfrac{2 \times -\dfrac{1}{2}}{\dfrac{\sqrt{3}}{2}}$

$= \dfrac{-2}{\sqrt{3}}$ **1A**

**OR**

$f'\left(\dfrac{\pi}{3}\right)$

$= \dfrac{2}{\tan\left(\dfrac{2\pi}{3}\right)}$

$= \dfrac{-2}{\sqrt{3}}$ **1A**

**Question 3**

$$\int_0^a \left(\frac{x}{x^2+4}\right)dx = 3.$$

$$\frac{1}{2}\int_0^a \left(\frac{2x}{x^2+4}\right)dx = 3$$

$$\frac{1}{2}\left[\log_e\left(x^2+4\right)\right]_0^a = 3$$ **1M**

$$\frac{1}{2}\left(\log_e\left(a^2+4\right)-\log_e(4)\right) = 3$$

$$\log_e\left(\frac{a^2+4}{4}\right) = 6$$ **1M**

$$\frac{a^2+4}{4} = e^6$$

$$a = \pm\sqrt{4e^6-4} = \pm 2\sqrt{e^6-1}$$ **1A**

**Question 4**

$$f(x) = \frac{1}{9}(3x-1)e^{3x}$$

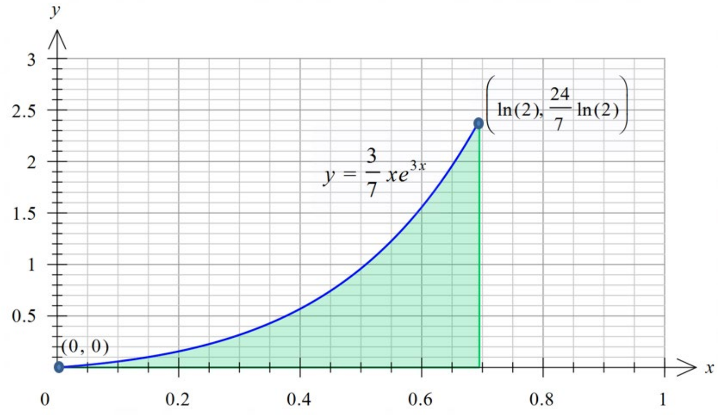
**a.** $f'(x) = \frac{3}{9}e^{3x} + \frac{3}{9}(3x-1)e^{3x}$ product rule

$= \frac{1}{3}e^{3x} + xe^{3x} - \frac{1}{3}e^{3x}$

$= xe^{3x}$ as required **1M Show that**

**b.** $h:[0,\log_e(2)] \to R, h(x) = \frac{3}{7}e^{3x}$

$y = xh(x)$



$\frac{3}{7}\int_{0}^{\log_e(2)}\left(xe^{3x}\right)dx$ **1M**

$=\frac{3}{7}\times\frac{1}{9}\left[(3x-1)e^{3x}\right]_{0}^{\log_e(2)}$ **1M**

$=\frac{1}{21}\left(\left(3\log_e(2)-1\right)e^{3\log_e(2)}+1\right)$

$e^{3\log_e(2)}=e^{\log_e\left(2^3\right)}=e^{\log_e(8)}=8$

$=\frac{1}{21}\left(24\log_e(2)-8+1\right)$

$=\frac{8}{7}\log_e(2)-\frac{1}{3}$ **1A**

**Question 5**

$f:[0,\infty)\to R, f(x)=e^{2x}$ and $g(x)=x^2-1$

**a.** Test to see if $g\left(f(x)\right)$ exists.

If $g(f(x))$ exists then $r_f\subset d_g$.

Range $f=[1,\infty)$ Domain $g=R$

$[1,\infty)\subset R$

Therefore $g\left(f(x)\right)$ exists. **1A**

$g_1:D\to R, g_1(x)=x^2-1$, where $D$ is the maximal domain of $g_1$ such that $h(x)=f\left(g_1(x)\right)$ exists.

**b.** Test to see if $f\left(g_1(x)\right)$ exists.

If $f(g(x))$ exists then $r_g\subset d_f$.

Range $g=[-1,\infty)$, Domain $f=[0,\infty)$

$[-1,\infty)\not\subset[0,\infty)$

Restrict Range $g=[-1,\infty)$ to Range $g_1=[0,\infty)$ **1M**

Giving Domain $g_1=(-\infty,-1]\cup[1,\infty)$

$D=(-\infty,-1]\cup[1,\infty)$ OR $R\setminus(-1,1)$ **1A**

**c.** Rule $h(x)=e^{2(x^2-1)}$ **1A**

$h'(x)=4xe^{2(x^2-1)}$

$h'(2)=8e^6$ **1A**

**Question 6**

$f:[-1,\infty)\to R, f(x)=a\sqrt{x+b}+c$

**a.** $f(x)=a\sqrt{x+1}+3$ **1A**

Substitue (0, 5) into $f(x)$.

$a\sqrt{0+1}+3=5,\ a=2$

$f(x)=2\sqrt{x+1}+3$ **1A**

**b.** $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}m&0\\0&n\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}p\\q\end{bmatrix}$

$f(x)=2\sqrt{x+1}+3$, $g(x)=-\sqrt{2x-1}-1=-\sqrt{2\left(x-\frac{1}{2}\right)}-1$

1. Dilate by a factor of $\frac{1}{2}$ from the $x$-axis. $f_1(x)=\sqrt{x+1}+\frac{3}{2}$

2. Reflect in the $x$-axis. $f_2(x)=-\sqrt{x+1}-\frac{3}{2}$

3. Dilate by a factor of $\frac{1}{2}$ from the $y$-axis. $f_3(x)=-\sqrt{2x+1}-\frac{3}{2}=-\sqrt{2\left(x+\frac{1}{2}\right)}-\frac{3}{2}$

4. Translate 1 unit to the right and a $\frac{1}{2}$ unit up. $g(x)=-\sqrt{2\left(x-\frac{1}{2}\right)}-1$

$n=-\frac{1}{2}$ (from 1 and 2), $m=\frac{1}{2}$ (from 3), $p=1$ and $q=\frac{1}{2}$ (from 4)

A correct method **1M,** 2 correct **2H,** all correct **3A**

**OR**

$x'=mx+p,\ x=\frac{x'-p}{m}$

$y'=ny+q,\ y=\frac{y'-q}{n}$

$y'=2n\sqrt{\frac{x'-p}{m}+1}+3n+q$

$y'=2n\sqrt{\frac{x'}{m}+1-\frac{p}{m}}+3n+q$

$=-\sqrt{2x-1}-1$

$2n=-1,\ n=-\frac{1}{2},\ \frac{1}{m}=2,\ m=\frac{1}{2},\ 1-2p=-1,\ p=1,\ -\frac{3}{2}+q=-1,\ q=\frac{1}{2}$

A correct method **1M,** 2 correct **2H,** all correct **3A**

**Question 7**

$X \sim \text{Bi}\left(10, \frac{1}{10}\right)$

$\Pr(X \le 2) = \Pr(X=0) + \Pr(X=1) + \Pr(X=2)$ **1M**

$= \binom{10}{0}\left(\frac{1}{10}\right)^0\left(\frac{9}{10}\right)^{10} + \binom{10}{1}\left(\frac{1}{10}\right)^1\left(\frac{9}{10}\right)^9 + \binom{10}{2}\left(\frac{1}{10}\right)^2\left(\frac{9}{10}\right)^8$ **1M**

$= \left(\frac{9}{10}\right)^8\left(\left(\frac{9}{10}\right)^2 + 10\left(\frac{1}{10}\right)\left(\frac{9}{10}\right) + 45\left(\frac{1}{10}\right)^2\right)$

$= \left(\frac{9}{10}\right)^8\left(\frac{81}{100} + \frac{9}{10} + \frac{45}{100}\right)$

$= \left(\frac{216}{100}\right)\left(\frac{9}{10}\right)^8$

$= \left(\frac{54}{25}\right)\left(\frac{9}{10}\right)^8$ **1A**

**OR**

$\Pr(X \le 2) = \Pr(X=0) + \Pr(X=1) + \Pr(X=2)$ **1M**

$= \binom{10}{0}\left(\frac{1}{10}\right)^0\left(\frac{9}{10}\right)^{10} + \binom{10}{1}\left(\frac{1}{10}\right)^1\left(\frac{9}{10}\right)^9 + \binom{10}{2}\left(\frac{1}{10}\right)^2\left(\frac{9}{10}\right)^8$ **1M**

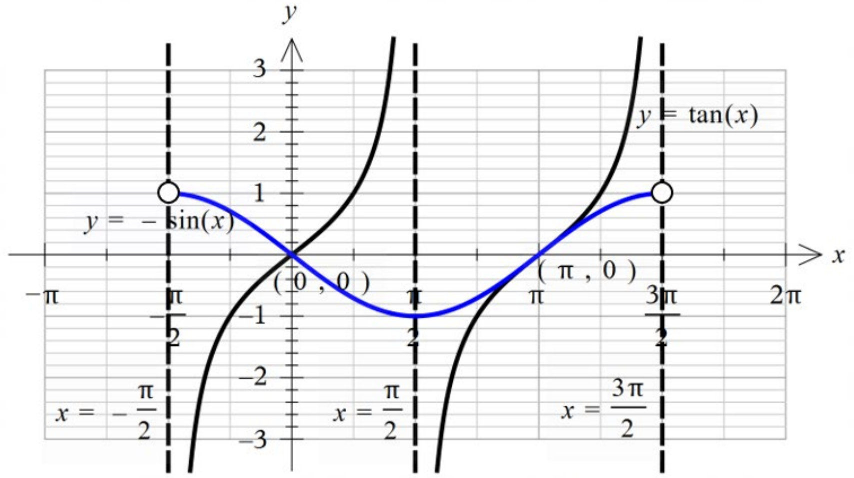
$= \left(\frac{9}{10}\right)^8\left(\left(\frac{9}{10}\right)^2 + 10\left(\frac{1}{10}\right)\left(\frac{9}{10}\right) + 45\left(\frac{1}{10}\right)^2\right)$

$= \left(\frac{9}{10}\right)^{10}\left(1 + \frac{10}{9} + \frac{5}{9}\right)$

$= \left(\frac{8}{3}\right)\left(\frac{9}{10}\right)^{10}$ **1A**

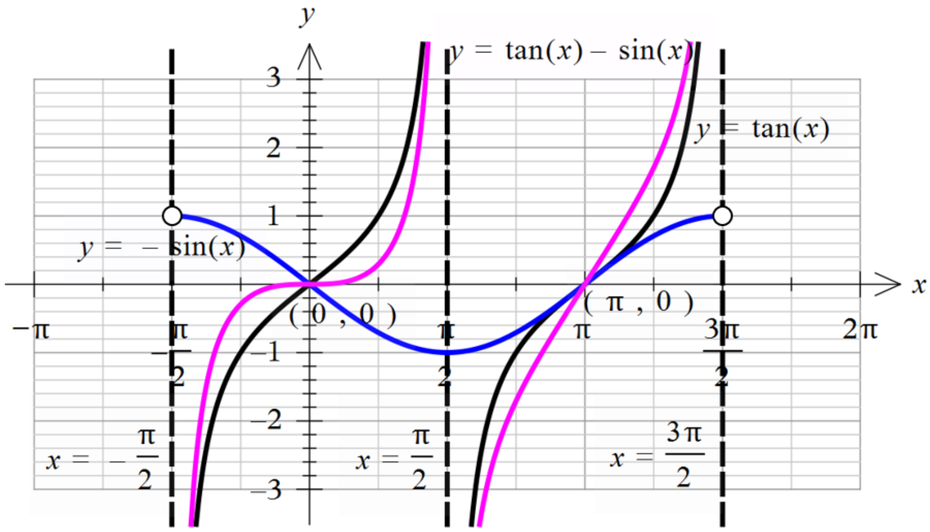
**Question 8**

**a.** $y=\tan(x)$ for $x\in\left(-\frac{\pi}{2},\frac{3\pi}{2}\right)\setminus\left\{\frac{\pi}{2}\right\}$




Shape **1A**
Asymptotes and intercepts labelled **1A**

**b.**




Evidence of addition of ordinates either graphically or a table of values. **1M**

| $x$ | $-\frac{\pi}{2}$ | $-\frac{\pi}{4}$ | 0 | $\frac{\pi}{4}$ | $\frac{\pi}{2}$ | $\frac{3\pi}{4}$ | $\pi$ | $\frac{5\pi}{4}$ | $\frac{3\pi}{2}$ |
|---|---|---|---|---|---|---|---|---|---|
| $\tan(x)$ | undefined | $-1$ | 0 | 1 | undefined | $-1$ | 0 | 1 | undefined |
| $-\sin(x)$ | undefined | $\frac{\sqrt{2}}{2}$ | 0 | $-\frac{\sqrt{2}}{2}$ | $-1$ | $-\frac{\sqrt{2}}{2}$ | 0 | $\frac{\sqrt{2}}{2}$ | undefined |
| $\tan(x)-\sin(x)$ | undefined | $-1+\frac{\sqrt{2}}{2}$ | 0 | $1-\frac{\sqrt{2}}{2}$ | undefined | $-1-\frac{\sqrt{2}}{2}$ | 0 | $1+\frac{\sqrt{2}}{2}$ | undefined |

Correct graph **1A**

**c.** $x=0,\ x=\pi$ **1A**

**Question 9**

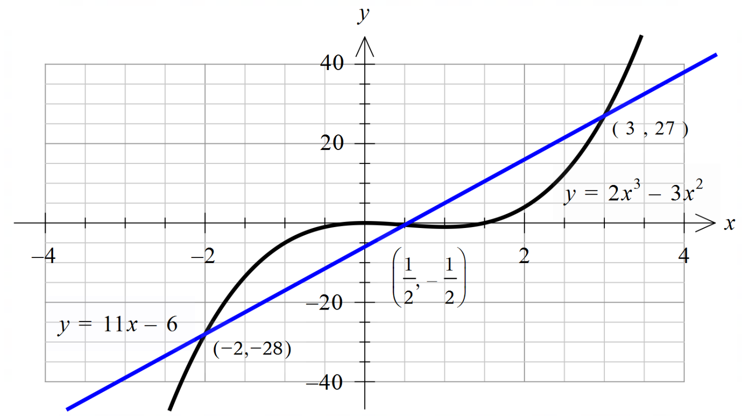
**a.** $x^2(2x-3)=11x-6$

$2x^3-3x^2-11x+6=0$ **1M**

$(x+2)\left(x-\frac{1}{2}\right)(2x-6)=0$

$x=3$ **1A**

**b.** Area enclosed by the graphs of $y=2x^3-3x^2$ and $y=11x-6$




The bounded areas are equal.

Area = $2\int_{-2}^{\frac{1}{2}}\left(\left(2x^3-3x^2\right)-\left(11x-6\right)\right)dx$ **1M**

$=2\left[\frac{1}{2}x^4-x^3-\frac{11}{2}x^2+6x\right]_{-2}^{\frac{1}{2}}$

$=2\left(\left(\frac{1}{32}-\frac{1}{8}-\frac{11}{8}+3\right)-\left(8+8-22-12\right)\right)$ **1M**

$=2\left(\left(-\frac{47}{32}+3\right)+18\right)$

$=2\left(-\frac{47}{32}+\frac{672}{32}\right)$

$=\frac{625}{16}=39.0625=39\frac{1}{16}$ (any correct form) **1A**

**Question 10**

$f:(-\infty,1)\to R, f(x)=\dfrac{2}{(x-1)^2}-\dfrac{20}{9}$

**a.** Let $y=\dfrac{2}{(x-1)^2}-\dfrac{20}{9}$.

Inverse swap $x$ and $y$.

$$x=\frac{2}{(y-1)^2}-\frac{20}{9}$$ **1M**

$$x+\frac{20}{9}=\frac{2}{(y-1)^2}$$

$$\left(x+\frac{20}{9}\right)(y-1)^2=2$$

$$(y-1)^2=\frac{2}{\left(x+\frac{20}{9}\right)}$$

$$y=\pm\sqrt{\frac{2}{\left(x+\frac{20}{9}\right)}}+1$$

$y=-\sqrt{\dfrac{2}{\left(x+\frac{20}{9}\right)}}+1$ as range is $(-\infty,1)$.

$$f^{-1}(x)=-\sqrt{\frac{2}{\left(x+\frac{20}{9}\right)}}+1=-\sqrt{\frac{18}{9x+20}}+1$$ **1A**

Domain $\left(-\dfrac{20}{9},\infty\right)$ **1A**

**b.** Solve $\frac{2}{(x-1)^2}-\frac{20}{9}=x$ for $x$. **1M**

$$\frac{2}{(x-1)^2}=x+\frac{20}{9}$$

$$2=\left(x+\frac{20}{9}\right)(x-1)^2$$

$$x^3+\frac{20}{9}x^2-2x^2-\frac{40}{9}x+x+\frac{20}{9}-2=0$$

$$x^3+\frac{20}{9}x^2-2x^2-\frac{40}{9}x+x+\frac{20}{9}-2=0$$

$$x^3+\frac{2}{9}x^2-\frac{31}{9}x+\frac{2}{9}=0$$ **1M**

$$(x+2)\left(x^2-\frac{16}{9}x+\frac{1}{9}\right)=0$$

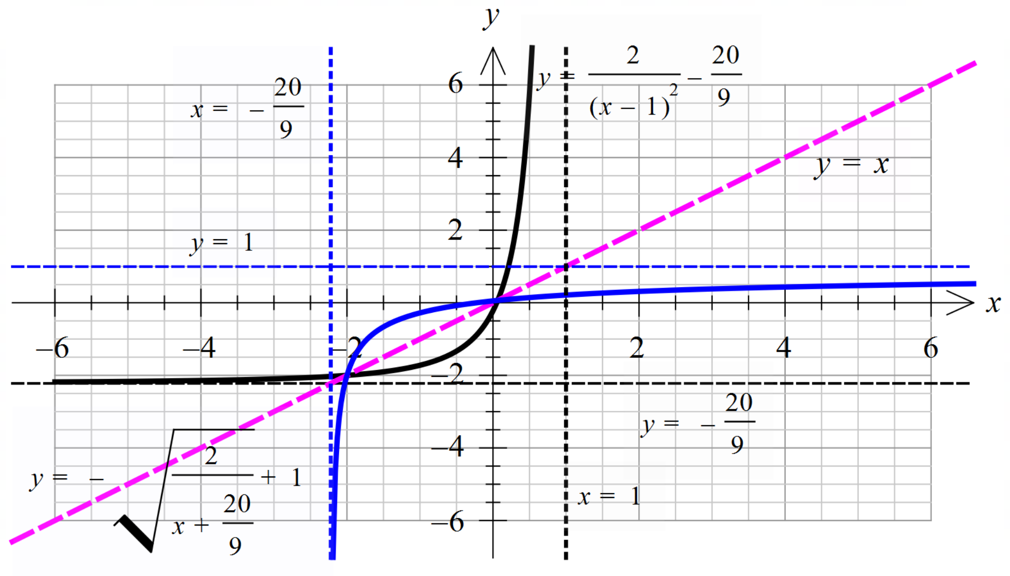
$$(x+2)(9x^2-16x+1)=0$$

$x=-2$, $(-2,-2)$ **1A**

$$x=\frac{16\pm\sqrt{256-36}}{18}=\frac{16\pm\sqrt{220}}{18}=\frac{8\pm\sqrt{55}}{9}$$

Select the negative branch as the range is $(-\infty,1)$.

$x=\frac{8-\sqrt{55}}{9}$, $\left(\frac{8-\sqrt{55}}{9},\frac{8-\sqrt{55}}{9}\right)$ **1A**




**END OF SOLUTIONS**