2020 Mathematical Methods Trial Exam 2 Solutions

SECTION A – Multiple-choice questions

1	2	3	4	5	6	7	8	9	10
B	B	D	C	E	B	A	B	A	C

11	12	13	14	15	16	17	18	19	20
D	A	C	D	B	A	E	C	E	A

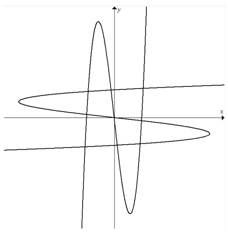
Q1 $y = ax^2 + \frac{b}{a}$, $ay = (ax)^2 + b$ B

Q2 $a_0 = 1$, $a_1 = -1$, $a_2 = 1$, $a_3 = -1$, …, $a_{n-1} = 1$
.: sum $= 1$ B

Q3 $\alpha = \beta$, .: $\alpha^{-1} = \beta^{-1}$ D

Q4 $5^{\log_a b} = 5^{\frac{\log_5 b}{\log_5 a}} = \left(5^{\log_5 b}\right)^{\frac{1}{\log_5 a}} = b^{\frac{1}{\log_5 a}} = b^{(\log_5 a)^{-1}}$ C

Q5 E



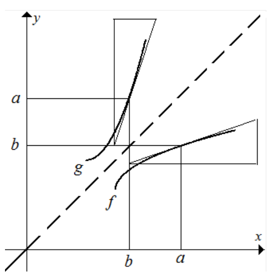
Q6 $\frac{2^n - 1}{1-0} = \frac{\int_0^1 (x+1)^n \, dx}{1-0}$, $n \approx 1.53$ B

Q7 $A = 4 \times \int_0^1 \left(x - \frac{1}{3}x\left(4x^2 - 1\right)\right) dx = \frac{4}{3}$ A

Q8 $y = \sin\left(\frac{x}{m}\right)$ is an odd function. The number of intersections it makes with $y = mx$ cannot be 2. B

Q9 $y = a\cos(nx)$ is an even function. When $b > 1$, $0 < \frac{1}{b} < 1$.
If p is a solution, then $-p$ is also a solution. .: Sum $= 0$ A

Q10 $g'(b) = \frac{1}{f'(a)} = \frac{1}{b} = b^{-1}$ C



Q11 $y = (1-x)\left(x^2 + bx + c\right)$ has three x-intercepts.
.: $\Delta = b^2 - 4c \geq 0$, $b \leq -2\sqrt{c}$ or $b \geq 2\sqrt{c}$. D

Q12 $f(t+10) = f(t)$ has a period of 10.
Given $f(5+a) = -f(5-a)$ for $0 < a < 5$, .: $f(6) = -f(4)$
Hence $f(26) = -f(34)$. A

Q13 If the common tangent touches $y = f(x)$ at (a, b), then it touches $y = f(x+h) + k$ at $(a-h, b+k)$.
Gradient of the common tangent $= \frac{(b+k) - b}{(a-h) - a} = -\frac{k}{h}$ C

Q14 Let $P(x, y)$ on the curve be the point closest to O.
Gradient $OP = \frac{y}{x}$, gradient of tangent to curve at $P = \frac{1}{x}$.
At P, $\frac{y}{x} \times \frac{1}{x} = -1$, $x^2 + y = 0$, .: $x^2 + \log_e x = 0$ D

Q15 $a \times 1 + \int_2^4 \left(\left(\frac{b-a}{2} \right) x + 2a - b \right) dx = 1$, .: $2a + b = 1$ B

Q16 $\Pr(B') = \Pr(A \cap B') + \Pr(A' \cap B')$
$\Pr(A \cap B') = \Pr(B') - \Pr(A' \cap B') = \frac{2}{3} - \frac{7}{12} = \frac{1}{12}$
$\Pr\left((A' \cup B)' \right) = \Pr(A \cap B') = \frac{1}{12}$ A

Q17 E

Q18 Sum of hidden numbers $= 63 - 58 = 5$
Possibilities: (113), (131), (311), (122), (212), (221)
$\Pr(58) = \frac{6}{6 \times 6 \times 6} = \frac{1}{36}$ C

Q19 $\mathrm{E}(\hat{P}) = 0.20$, $\mathrm{sd}(\hat{P}) = \sqrt{\frac{0.20 \times 0.80}{100}} = 0.04$
$\Pr(\hat{P} < 0.22) \approx 0.6915$
$\Pr(\text{at least one}) = 1 - \Pr(\text{none}) \approx 1 - 0.0028 = 0.9972$ E

Q20 $\Pr(X \leq 1) < 0.1$
$\Pr(X = 0) + \Pr(X = 1) < 0.1$
$(0.2)^n + n\left(0.2^{n-1}\right)(0.8) < 1$, .: $n \geq 4$ A

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SECTION B

Q1a $h = \frac{2}{9}\left(\frac{10}{t+1} - 1\right) = 0$, $t = 9$, time taken $= 9.000$ min.

Q1b $V(t) = \pi \times 1^2 \times \frac{2}{9}\left(\frac{10}{t+1} - 1\right)$, $\frac{dV}{dt} = -\frac{2\pi}{9} \times \frac{10}{(t+1)^2}$

$\left|\frac{dV}{dt}\right| = \frac{20\pi}{9(t+1)^2}$ m^3 min^{-1}

Q1c Max $= \frac{20\pi}{9}$ m^3 min^{-1}, min $= \frac{2\pi}{90}$ m^3 min^{-1}

Q1di At lowest level, $\frac{dV}{dt} = 1 - \frac{20\pi}{9(t+1)^2} = 0$, $(t+1)^2 = \frac{20\pi}{9}$

$t \approx 1.642$ min.

Q1dii $V(t) = 1 \times t + \frac{2\pi}{9}\left(\frac{10}{t+1} - 1\right)$,

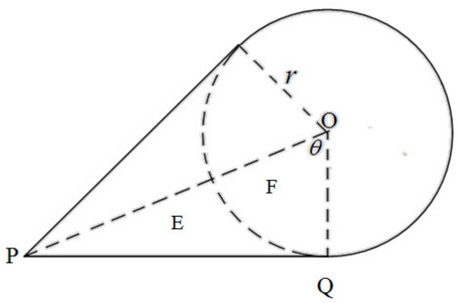
$V(1.642) \approx 1.642 + \frac{2\pi}{9}\left(\frac{10}{1.642+1} - 1\right) \approx 3.586\,\text{m}^3$

Q1diii $V(t) = 1 \times t + \frac{2\pi}{9}\left(\frac{10}{t+1} - 1\right) = 2\pi$, $t \approx 5.981$ min.

Q1div Let r m^3 min^{-1} be the constant rate,

$r \times 9 + \frac{2\pi}{9}\left(\frac{10}{9+1} - 1\right) = 2\pi$, $r = \frac{2\pi}{9} \approx 0.698$

Q2a



$PQ = \sqrt{1-r^2}$,

Area E $=$ area $\Delta POQ -$ sector F $= \frac{1}{2} r\sqrt{1-r^2} - \frac{1}{2} r^2 \theta$

$A =$ area of circle $+ 2 \times$ area E $= \pi r^2 + r\sqrt{1-r^2} - r^2 \theta$

Q2b $0 < r < 1$

Q2ci $\lim_{r \to 1} \theta = 0$

Q2cii $\lim_{r \to 1} A = \pi \times 1^2 + 1\sqrt{1-1^2} - 1^2 \times 0 = \pi$

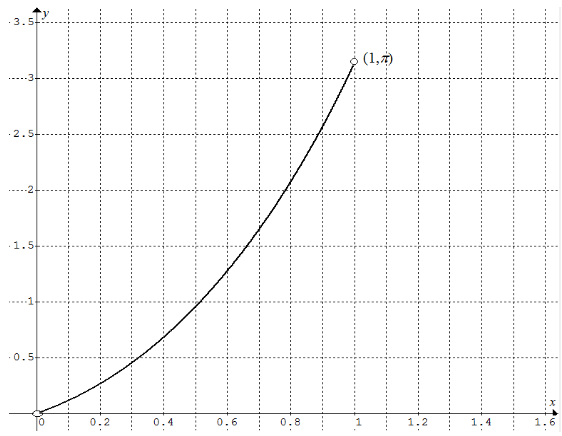
Q2di $\frac{dA}{dr} = 2\pi r + \sqrt{1-r^2} - 2r\cos^{-1} r$, $0 < r < 1$

Q2dii $0 < r < 1$, .: $0 < \theta < \frac{\pi}{2}$,

.: $\pi r + \sqrt{1-r^2} < \frac{dA}{dr} < 2\pi r + \sqrt{1-r^2}$, .: $\frac{dA}{dr} > 0$,

.: $A(r)$ is a strictly increasing function in $0 < r < 1$

Q2e



Q3a $\frac{2\pi}{m} = 12$, $m \approx 0.5236$

Q3b $T_P = 12 \times \frac{34}{24} = 17$, $\frac{2\pi}{n} = 17$, $n \approx 0.3696$

Q3c $h_P(t) = 17\sin(0.3696t) + 20 = 17\cos\left(\frac{\pi}{2} - 0.3696t\right) + 20$

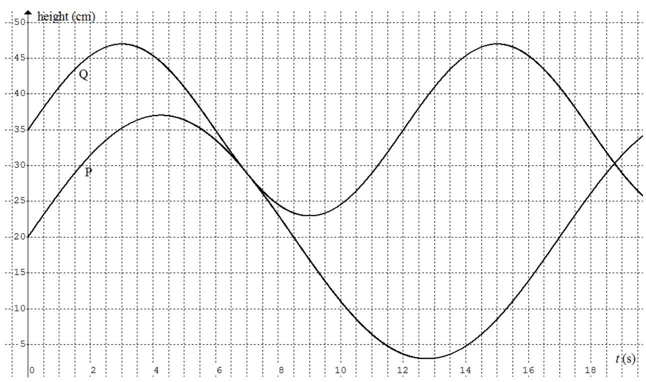
$= 17\cos(0.3696(4.25 - t)) + 20$

Q3d $h_Q(5) = 12\sin(0.5236 \times 5) + 34.95 \approx 40.95$

Q3e $h_P(t) = 17\sin(0.3696t) + 20 = 30$

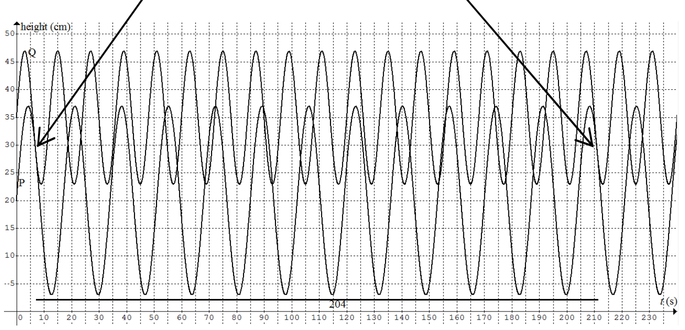
$t = \frac{1}{0.3696}\sin^{-1}\left(\frac{10}{17}\right) \approx 1.70$

Q3f $t \approx 7.03$ (± 0.02)



Q3g $LCM(T_P, T_Q) = LCM(12,17) = 204$ s or by graphs (see below)

First time Second time



Q4a $y = 5e^{\frac{x}{3}-2}+5$, $\frac{y}{5}-1 = e^{\frac{x}{3}-2}$

Sequence of transformations: Right 2, up 1, dilate from y-axis by 3, dilate from x-axis by 5.

$c=-2$, $d=-1$, $a=3$, $b=5$

Q4b Equation of curve B: $\frac{x}{5}-1 = e^{\frac{y}{3}-2}$, $\frac{y}{3}-2 = \log_e\left(\frac{x}{5}-1\right)$

$y = 3\log_e\left(\frac{x}{5}-1\right)+6$

Q4c Range of A: When $x=0$, $y=5e^{-2}+5$

Domain of B is the range of A, i.e. $\left[5e^{-2}+5, 40\right]$

Curve B: When $x=40$, $y = 3\log_e\left(\frac{40}{5}-1\right)+6 = 3\log_e 7+6$

Range of B is $\left[0, 3\log_e 7+6\right]$.

Q4d Curve A: $y=5e^{\frac{x}{3}-2}+5$. Let $\frac{dy}{dx}=\frac{5}{3}e^{\frac{x}{3}-2}=1$, $\frac{x}{3}-2=\log_e\frac{3}{5}$

$x = 3\log_e\frac{3}{5}+6$, $y=3+5=8$, $\left(3\log_e\frac{3}{5}+6,\ 8\right)$

Curve B: $\left(8,\ 3\log_e\frac{3}{5}+6\right)$

Shortest distance

$=\sqrt{\left(8-3\log_e\frac{3}{5}-6\right)^2+\left(3\log_e\frac{3}{5}+6-8\right)^2}=\sqrt{2}\left(2-3\log_e\frac{3}{5}\right)$

Q4e Area between curve A and curve B

$=40^2-2\int_{5e^{-2}+5}^{40}\left(3\log_e\left(\frac{x}{5}-1\right)+6\right)dx \approx 977.30\ \text{m}^2$

Q4f At (a,b), $\frac{dy}{dx}=\frac{5}{3}e^{\frac{x}{3}-2}=\frac{5}{3}e^{\frac{a}{3}-2}$, gradient from (p,p) to $(a,b)=\frac{p-b}{p-a}$, ∴ $\frac{5}{3}e^{\frac{a}{3}-2}\times\frac{p-b}{p-a}=-1$, $\frac{p-b}{p-a}=-\frac{3}{5}e^{2-\frac{a}{3}}$

Q4g $p=25.17$

Q5a $k\times$area under graph $=1$

$k\left(\frac{1}{2}(2\times0.5+3\times3)+\frac{1}{2}(0.5+3)3\right)=1$, $k=\frac{4}{41}$

Q5b $\Pr(\text{bus A late by} > 5)=\frac{4}{41}\left(\frac{1}{2}\times3\times3\right)\approx 0.4390$

Q5c $\Pr(\text{miss either bus})$

$=\Pr(\text{bus A late by} < 7)+\Pr(\text{bus B late by} < 2)$

$-\Pr(\text{bus A late by} < 7 \text{ and bus B late by} < 2)$

$=\left(1-\frac{4}{41}\left(\frac{1}{2}\times1\times1\right)\right)+\frac{4}{41}\left(\frac{1}{2}\times0.5\times2\right)$

$-\left(1-\frac{4}{41}\left(\frac{1}{2}\times1\times1\right)\right)\times\frac{4}{41}\left(\frac{1}{2}\times0.5\times2\right)\approx 0.9536$

Q5d Binomial: $p\approx0.4390244$, $n=5$

$\Pr(X\ge 2)\approx 0.7271$

Q5e

$$f(t)=\begin{cases}\frac{t}{4} & 0\le t\le 2\\ \frac{5t-7}{6} & 2<t\le 5\\ 8-t & 5<t\le 8\\ 0 & \text{elsewhere}\end{cases}$$

Mean

$=\int_0^2 t\times kf(t)\,dt+\int_2^5 t\times kf(t)\,dt+\int_5^8 t\times kf(t)\,dt$

$=\frac{8}{123}+\frac{81}{41}+\frac{108}{41}=\frac{575}{123}\approx 4.674797$

Q5f $\int_{4.674797}^{8} kf(t)\,dt\approx 0.5299$

Q5g Binomial: $n=20$, $p\approx 0.529907$

$\Pr(X=12)\approx 0.1473$

Q5h $\left(\frac{11}{20}-1.96\sqrt{\frac{\frac{11}{20}\times\frac{9}{20}}{20}},\ \frac{11}{20}+1.96\sqrt{\frac{\frac{11}{20}\times\frac{9}{20}}{20}}\right)$

$\approx(0.3320, 0.7680)$

Let p be the long term proportion of week days the 7:35 am bus of Company B is late for more than $\frac{575}{123}$ min.

If many similar surveys were carried out and the 95% confidence interval calculated in each survey, 95% of them would have p in the interval.

Please inform mathline@itute.com re conceptual and/or mathematical errors