UNIT 1 & 2 MATHEMATICAL METHODS

EXTENDED RESPONSE QUESTION

POLYNOMIAL FUNCTIONS QUESTION 1: CUBIC FUNCTION

QUESTION 1 (16 Marks)
A cubic function has the rule $f(x) = ax^3 + bx^2 + d$. The graph of the function cuts the y-axis at $y = -3$ and passes through the points $(1, 3)$ and $(7, -3)$.

a. Show that $a = -1,\ b = 7,\ d = -3$.

3 marks

b. Hence find:

(i) $f(0)$

1 mark

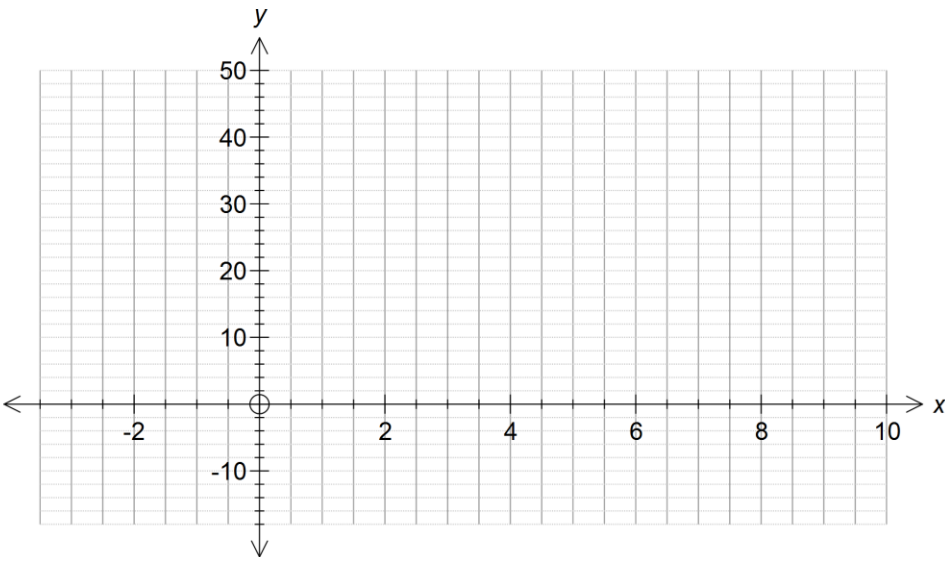
(ii) $\{x : f(x) = 3\}$

1 mark

(iii) $f'(1.5)$

1 mark

c. Sketch the graph of $f(x) = ax^3 + bx^2 + d$ for your values of a, b and d.



4 marks

d. Find $\{x : f(x) < 3\}$.

1 mark

e. Is $f(x)$ a one-to-one function? Explain your answer.

1 mark

f. Find m such that $f_1 : [m, \infty) \to R,\ f_1(x) = ax^3 + bx^2 + d$ is a one-to-one function.

1 mark

g. Sketch the graph of $f_1^{-1}(x)$ for this one-to-one function on the axes in **c**.

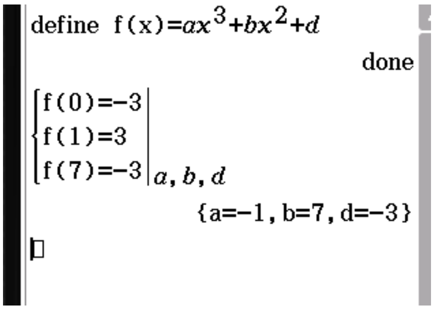
3 marks

SOLUTION

QUESTION 1

a. Set up the equations $f(0) = -3, f(1) = 3, f(7) = -3$ **2M**

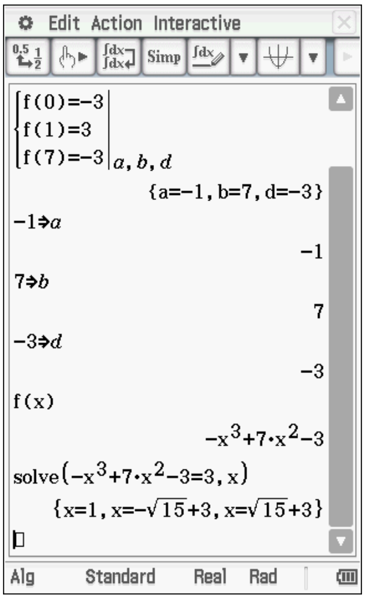
Solve simultaneously to show that $a = -1, b = 7, d = -3$ **1M**



b. Equation is $f(x) = -x^3 + 7x^2 - 3$

(i) $f(0) = -3$ **1A**

(ii) Find $\{x : f(x) = 3\}$

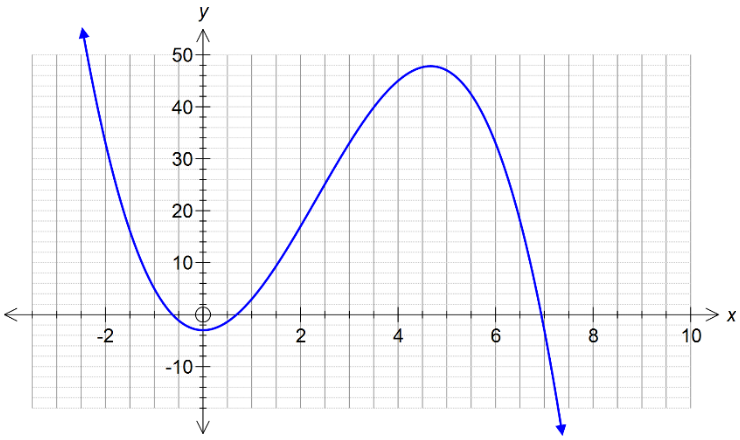


Gives $x = 1, 3 \pm \sqrt{15}$ **1A**

(iii) $f'(1.5)=\dfrac{57}{4}$ **1A**

```
define f(x)=-x^3+7x^2-3
                                done
diff(f(x),x,1,1.5)
                                57/4
```

c.



4A

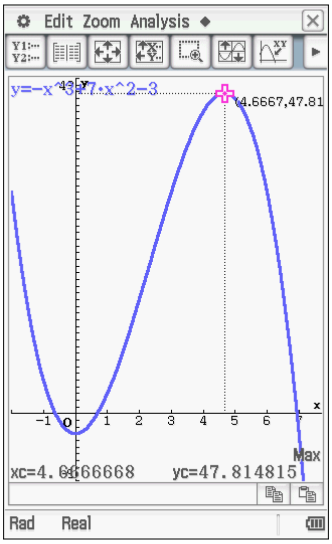
d.

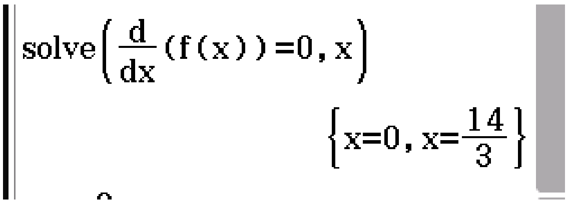
```
solve(f(x)<3,x)
        {-√15+3<x<1, √15+3<x}
```

$\{x: f(x)<3\}$ for $\{-\sqrt{15}+3<x<1\}\cup\{x>\sqrt{15}+3\}$ **1A**

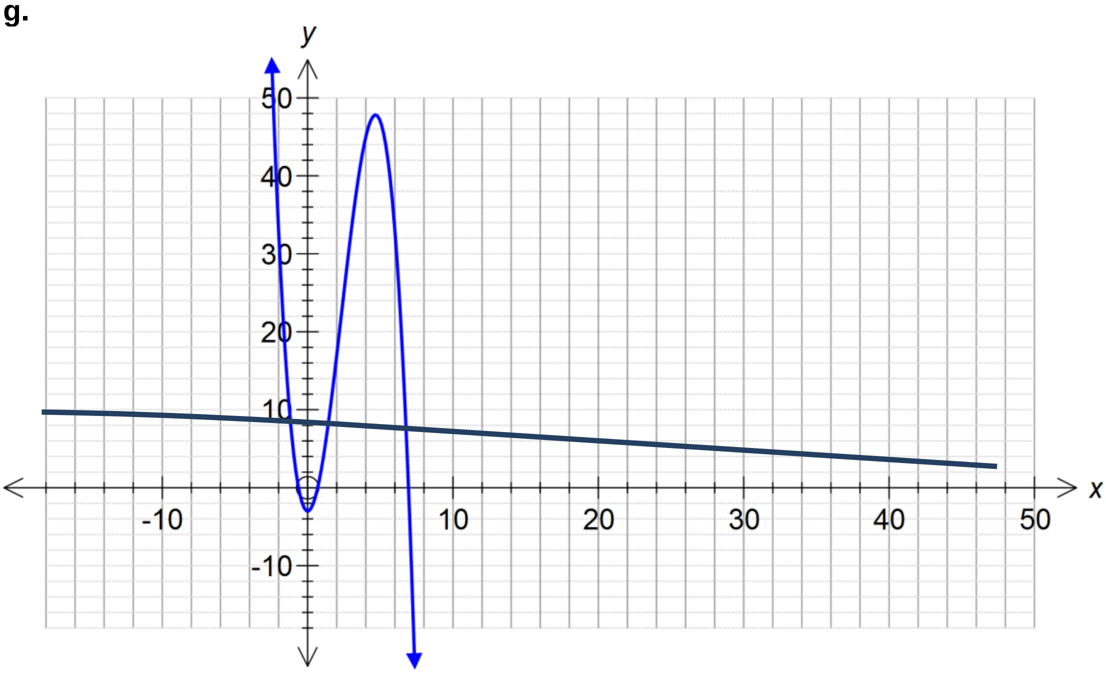
e. $f(x)$ is not a one-to-one function. It is a many-to-one function. **1A**

f. Restrict for domain $[m,\infty)$ where $m \approx 4.667$ or $m = \dfrac{14}{3}$. **1A**





g.



UNIT 1 & 2 MATHEMATICAL METHODS

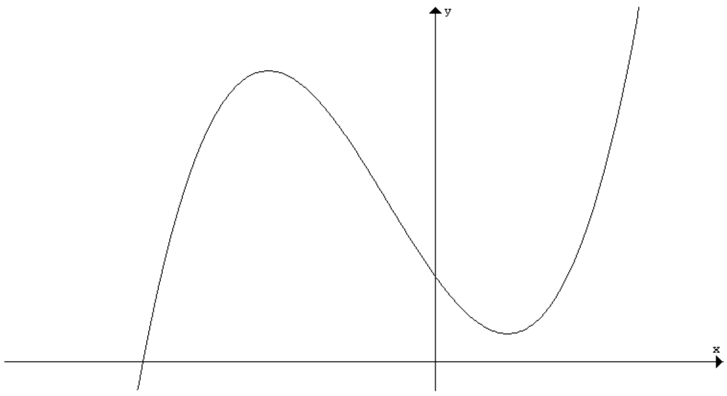
EXTENDED RESPONSE QUESTION

POLYNOMIAL & CALCULUS QUESTION 1: CUBIC FUNCTION

QUESTION 1 (12 Marks)

The graph of the function $f(x) = (x + a)(x - b)^2 + 2$ where a and b are real constants is shown below. It is known that $a > b$.

.



a. Given that $f'(1) = f'\left(-\frac{7}{3}\right) = 0$, use calculus to show that $a = 4$ and $b = 1$.

4 marks

b. Find the coordinates of the turning point of the graph of $y = f(x)$ at $x = 1$.

1 mark

c. If the coordinates of the other turning point of the graph of $y = f(x)$ is $\left(-\frac{7}{3}, c\right)$, find the value of c correct to 2 decimal places.

1 mark

d. Find the real values of m for which the equation $f(x) = m$ has three distinct solutions. (Non-integer values are to be given correct to 2 decimal places).

2 marks

The graph of $y = f(x)$ is dilated by a factor of k from the *y*-axis to form another function $g(x)$ so that the **horizontal** distance between the two turning points is 10 units.

e. Find this value of k.

2 marks

f. Hence, or otherwise, write down the coordinates of the turning points of the graph with equation $y = g(x)$. (Non-integer values are to be given correct to 2 decimal places).

2 marks

SOLUTION

QUESTION 1

a. $f'(x)=(x+a)\times 2(x-b)+1\times(x-b)^2$ (using the Product Rule) **M1**

$=(x-b)[2(x+a)+(x-b)]$

$=(x-b)(3x+2a-b)$ **A1**

$=0$ if $x=b$ or $x=\dfrac{b-2a}{3}$ **A1**

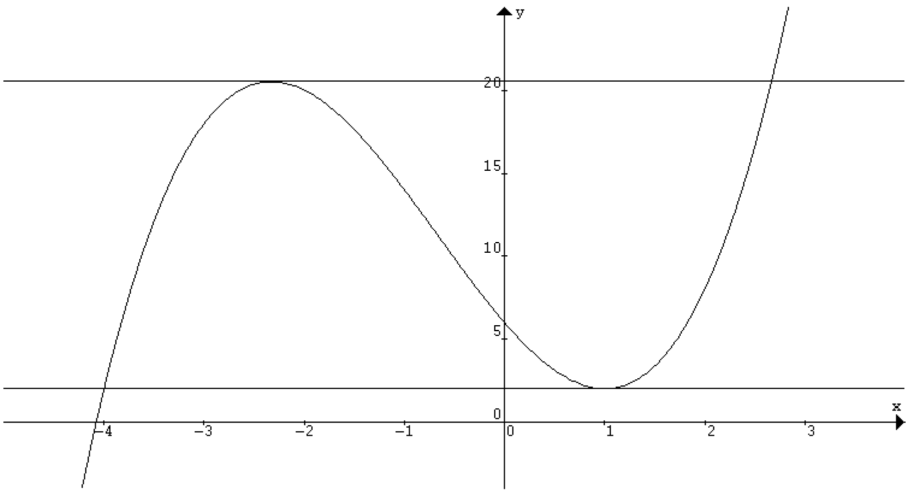
Since $f'(1)=0$ then b could be 1. If this is the case, see if $a=4$ satisfies the other stationary value. $-\dfrac{7}{3}=\dfrac{1-2a}{3}$ so $-7=1-2a$ which means that $a=4$, as req. **M1**

b. $f(x)=(x+4)(x-1)^2+2$

When $x=1$, $y=(1+4)(1-1)^2+2=2$ and so the turning point is at $(1,2)$. **A1**

c. When $x=-\dfrac{7}{3}$, $y=(-\dfrac{7}{3}+4)(-\dfrac{7}{3}-1)^2+2=20.52$ and so $c=20.52$ **A1**

d. The lines $y=2$ and $y=20.52$ have been drawn showing that each of them meets the graph at two points.



If $2<m<20.52$ then the equation $f(x)=m$ will have three distinct solutions.

A $4\times\dfrac{1}{2}\downarrow$ **(2 , < , <, 20.52)**

e. If the turning points of $f(x)$ are at $(1, 2)$ and $\left(-\frac{7}{3}, 20.52\right)$ then the horizontal distance between them is $1 - -\frac{7}{3} = \frac{10}{3}$ units. This would need to be multiplied by 3 to give the required result of being 10 units apart. Hence $k = 3$.

M1 (horizontal distance idea)
A1 (*k* = 3)

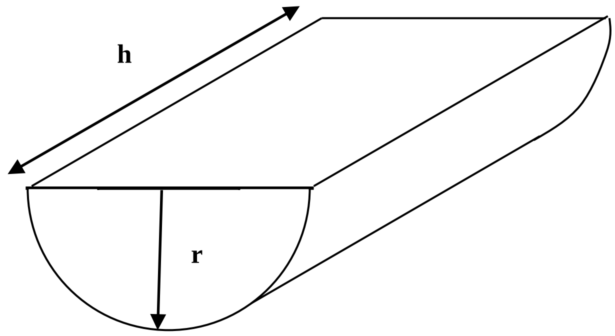
f. (-7, 20.52) and (3, 2) **A2** (1 for each pair)

UNIT 1 & 2 MATHEMATICAL METHODS

EXTENDED RESPONSE QUESTION

POLYNOMIAL & CALCULUS QUESTION 2: PRECIOUS METAL

QUESTION 2 (16 Marks)



A quantity of precious metal whose volume is $500\,cm^3$ is melted and moulded into a shape of length ***h*** centimetres and uniform semi-circular cross section of radius ***r*** centimetres, as shown in the diagram.

a. Show that the total surface area, A, is $\pi r^2 + 2rh + \pi rh$.

2 marks

b. Hence show that $A=\pi r^2+\dfrac{1000(2+\pi)}{\pi r}$.

2 marks

c. Using calculus, find the value of r, correct to 2 decimal places, for which this surface area is a minimum.

2 marks

d. Write down the minimum surface area, correct to 2 decimal places.

1 mark

To make this precious metal into a beautiful piece of jewellery, it is decided to cover the surface area with gold leaf. The flat surface area is easier to cover and costs $\$\boldsymbol{p}$ per cm^2 whereas the curved surface area is more difficult to coat with gold leaf and costs $\$\boldsymbol{q}$ per cm^2, where $\boldsymbol{p}$ and $\boldsymbol{q}$ are constants with $\boldsymbol{q} > \boldsymbol{p}$.

e. Show that an expression for the ***total cost*** (in dollars) of covering this piece of jewellery in terms of $\boldsymbol{r}$, $\boldsymbol{p}$ and $\boldsymbol{q}$ is given by $C = \boldsymbol{p}\left(\pi r^2 + \frac{2000}{\pi r}\right) + \frac{1000}{r}\boldsymbol{q}$.

2 marks

f. Find the value of r in terms of $\boldsymbol{p}$ and $\boldsymbol{q}$ for which this cost is a minimum.

3 marks

g. Hence, or otherwise, if $\boldsymbol{p} = 15$ and $\boldsymbol{q} = 35$ write down

i. the minimum cost (correct to the nearest dollar).

1 mark

ii. the radius (in centimetres, correct to 2 decimal places).

1 mark

h. For safety reasons, the height of the jewellery piece must **not** be less than 10 centimetres in length. Find the minimum cost to cover the surface area with gold leaf, if $\boldsymbol{p} = 15$ and $\boldsymbol{q} = 35$. Give your answer correct to the nearest dollar.

2 marks

SOLUTION

QUESTION 2

a. Total area = Two end semi-circles + flat surface + curved surface **M1**

$$A = 2\times(\frac{1}{2}\pi r^2) + 2r\times h + \frac{1}{2}\times 2\pi r h$$

$$A = \pi r^2 + 2rh + \pi r h$$ **A1**

b. Volume = $500 = \frac{1}{2}\pi r^2 h$ and so $h = \frac{1000}{\pi r^2}$ **A1**

$$A = \pi r^2 + 2rh + \pi r h$$

$$= \pi r^2 + (2r + \pi r)h$$

$$= \pi r^2 + (2r + \pi r)\times\frac{1000}{\pi r^2}$$ **M1**

$= \pi r^2 + (2 + \pi)r\times\frac{1000}{\pi r^2}$ which when cancelling the r gives

$A = \pi r^2 + \frac{1000(2+\pi)}{\pi r}$, as required.

c. $A = \pi r^2 + \frac{1000(2+\pi)}{\pi}\times r^{-1}$

$$\frac{dA}{dr} = 2\pi r - \frac{1000(2+\pi)}{\pi}\times r^{-2}$$ **H1**

$= 0$ for a minimum value

Therefore $2\pi^2 r^3 = 1000(2+\pi)$ and so $r = \left(\frac{1000(2+\pi)}{2\pi^2}\right)^{\frac{1}{3}}$ = 6.39 cm **A1**

d. 384.40 cm^2 (do not accept 384.4 cm^2) **A1**

e. $C = p(\pi r^2 + 2rh) + q\times\pi r\times\frac{1000}{\pi r^2} = p(\pi r^2 + 2r\times\frac{1000}{\pi r^2}) + q\times\pi r\times\frac{1000}{\pi r^2}$ **M2**

(Give a method mark for each part, curved and flat)

$$\therefore C = p\left(\pi r^2 + \frac{2000}{\pi r}\right) + \frac{1000}{r}q$$

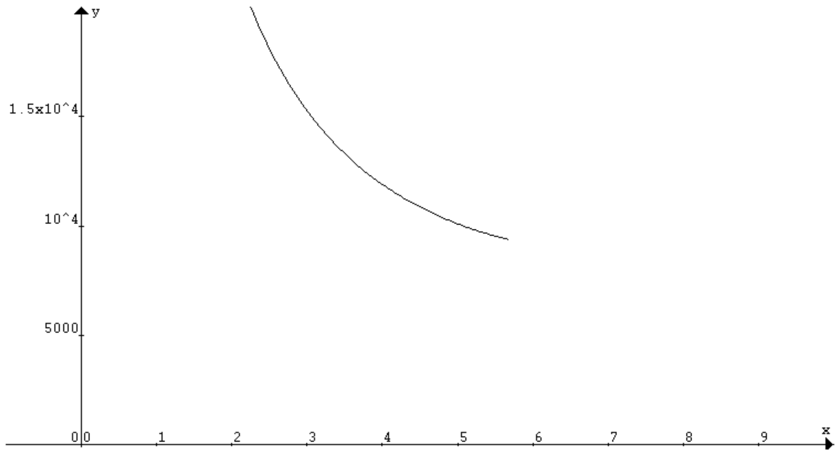
f. $\dfrac{dC}{dr} = p\left(2\pi r - \dfrac{2000}{\pi r^2}\right) - \dfrac{1000}{r^2}q$ **H1**

For minimum cost $\dfrac{dC}{dr} = 0$ and so $2\pi r\ p = \dfrac{2000p + 1000\pi q}{\pi r^2}$ **M1**

Hence $r = \left(\dfrac{1000(2p + q\pi)}{2\pi^2 p}\right)^{\frac{1}{3}}$ **A1**

g. **i.** \$8578
ii. 7.79 cm **A2**

h. Since $h = \dfrac{1000}{\pi r^2}$ then if $h = 10$, $r = \sqrt{\dfrac{100}{\pi}}$ (= 5.64 cm to 2 decimal places). **A1**



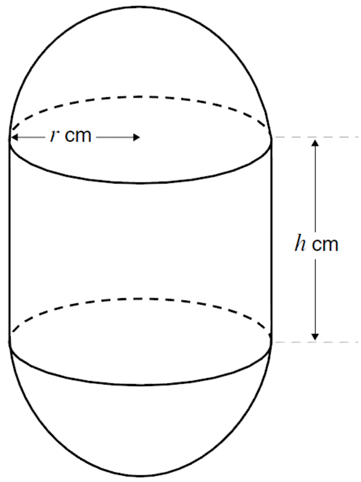
The minimum cost occurs when $r = 5.64\,cm$, and is approximately \$9396. **A1**

UNIT 1 & 2 MATHEMATICAL METHODS

EXTENDED RESPONSE QUESTION

POLYNOMIAL & CALCULUS QUESTION 3: TIME CAPSULE

QUESTION 3 (14 Marks) – Modified VCAA 2000
Victoria Jones wants to construct a time capsule in which to bury some of her treasures. The time capsule will be a right circular cylinder of height h cm, and radius r cm, with hemispherical caps of radius , r cm on each end, as shown in the diagram.



Let the total volume of the capsule be $V\ cm^3$.

a. Show that $V = \dfrac{4\pi r^3}{3} + \pi r^2 h$

1 mark

b. The total volume of the capsule will be $8000\ cm^3$.

(i) Show that $h = \dfrac{8000}{\pi r^2} - \dfrac{4r}{3}$.

2 marks

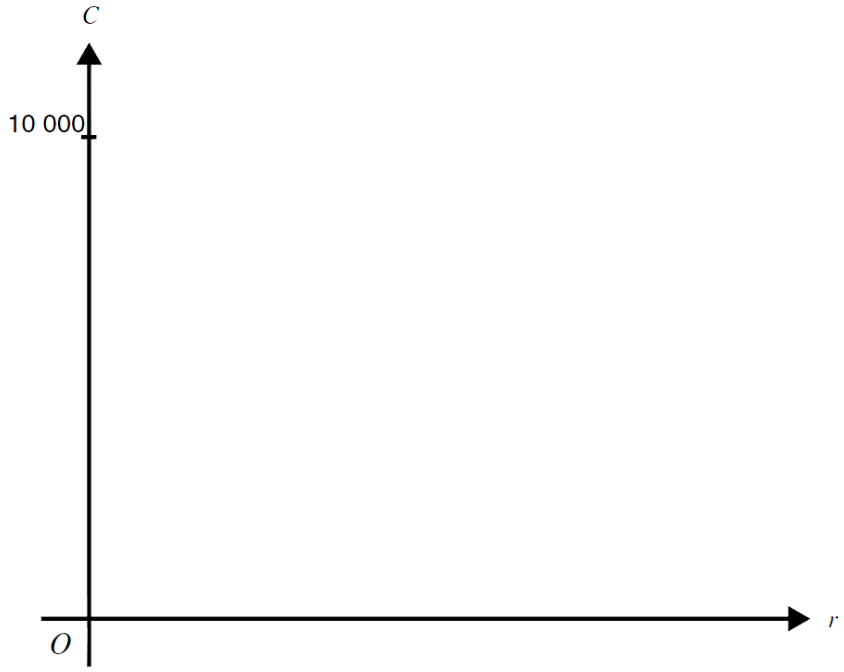
(ii) The values which r may take lie in an interval. Find the end points of this interval, correct to two decimal places.

2 marks

c. The material for the cylindrical part of the capsule costs 2 cents per cm^2 of surface. The material for the hemispherical caps costs 3 cents per cm^2 of surface. [The surface area of a sphere of radius r is $4\pi r^2$]. Find an expression for C cents, the total cost of the materials for the capsule, in terms or r.

2 marks

d. Sketch the graph of C over an appropriate domain on the axes below. Label any horizontal or vertical asymptote with its equation. You are not required to show the co-ordinates of any turning point.



3 marks

e. (i) Use calculus to find the value of r, correct to two decimal places, for which C is a minimum. [You do not need to justify that the value is a minimum].

3 marks

(ii) Hence calculate the minimum costs to construct the time capsule. State your answer correct to the nearest dollar.

1 mark

SOLUTION

QUESTION 3

a.

$$V = V_{sphere} + V_{cylinder}$$
$$= \frac{4}{3}\pi r^3 + \pi r^2 h$$

b. (i)

$$\frac{4}{3}\pi r^3 + \pi r^2 h = 8000$$
$$h = \frac{8000}{\pi r^2} - \frac{4r}{3}$$

(ii) $(0, 12.41]$

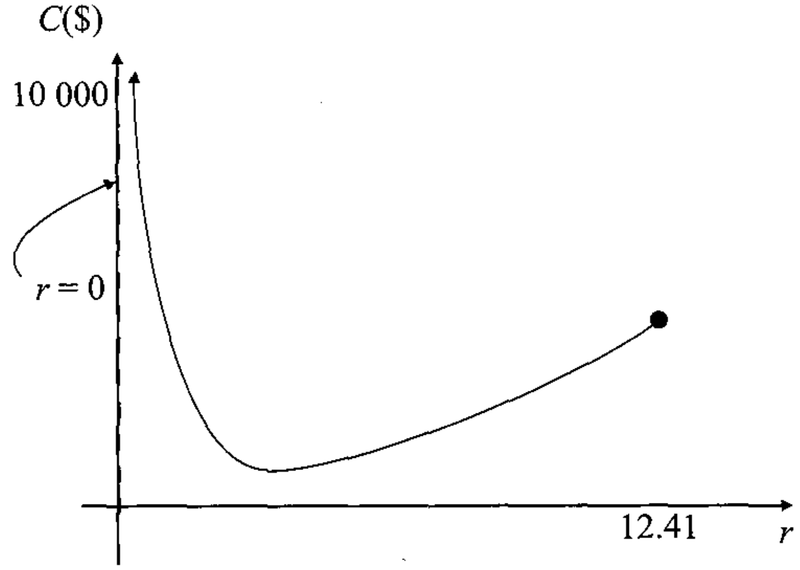
c.

$$Cost = Cost(cylinder) + cost(hemisphere)$$
$$= 2 \times SA_{cyl} + 3 \times SA_{hemi}$$
$$= (2 \times 2\pi r h) + 3 \times 4\pi r^2$$

Substitute $h = \dfrac{8000}{\pi r^2} - \dfrac{4r}{3}$

$$\therefore C = 2 \times 2\pi r\left(\frac{8000}{\pi r^2} - \frac{4r}{3}\right) + 3\left(4\pi r^2\right)$$
$$= \frac{32{,}000}{r} - r^2\left(\frac{16\pi}{3} - 12\pi\right)$$

d.



e. (i)

$$C = \frac{32\,000}{r} - r^2\left(\frac{16\pi}{3} - 12\pi\right)$$

$$= \quad 32\,000r^{-1} - r^2\left(\frac{16\pi}{3} - 12\pi\right)$$

$$\therefore \quad \frac{dC}{dr} = -32\,000r^{-2} - 2r\left(\frac{16\pi}{3} - 12\pi\right)$$

$= \quad$ 0 for a turning point.

$$\therefore \quad 32\,000r^{-2} = -2r\left(\frac{16\pi}{3} - 12\pi\right)$$

$$\therefore \quad 32\,000 = -2r^3\left(\frac{16\pi}{3} - 12\pi\right)$$

$$\therefore \quad r^3 = \frac{32\,000}{-2\left(\frac{16\pi}{3} - 12\pi\right)}$$

$$\therefore \quad r = \sqrt[3]{\frac{32\,000}{-2\left(\frac{16\pi}{3} - 12\pi\right)}}$$

$\approx \quad 9.141\,563$

$\approx \quad 9.14$ cm

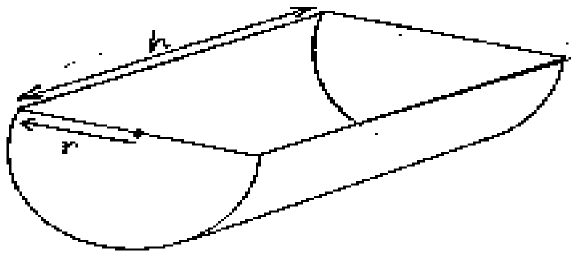
(ii) $1751

UNIT 1 & 2 MATHEMATICAL METHODS

EXTENDED RESPONSE QUESTION

POLYNOMIAL & CALCULUS QUESTION 4: THIRSTY HORSE

QUESTION 4 (16 Marks)
In outback Australia in the 1850s, drums were cut in half and used for water drinking tanks for horses.



a. Show that the surface area of the drinking tank is given by $S = \pi r^2 + \pi rh$.

2 marks

b. If a fully closed drum is made of 200 m^2 sheet of metal, show that $h = \dfrac{100 - \pi r^2}{\pi r}$.

2 marks

c. Show that the volume of the drinking tank is $V = 50r - \dfrac{\pi r^3}{2}$.

2 marks

d. Using calculus, find the radius and height of the tank that gives the maximum volume. Use an appropriate technique to prove that the volume is a maximum at this radius and height.

4 marks

e. How much water can the drum hold?

1 mark

f. If the tank is being filled at 0.5 m^3 / min, how long will it take to fill the tank? Express your answer in hours and minutes.

2 marks

g. A very thirsty horse comes to drink at the tank. He drinks at an average rate of $0.8\ m^3 / \min$. State the rate at which the volume of water in the tank is changing.

1 mark

h. If the horse drinks for 20 minutes, what volume of water remains in the tank when the horse finishes drinking? State your answer correct to 2 decimal places.

2 marks

SOLUTION

QUESTION 4

a. $S.A. = \pi.r^2 + \frac{1}{2}.2\pi.r \times h = \pi.r^2 + \pi.rh$

b. $100 = \pi.r^2 + \pi.rh$

$\pi.rh = 100 - \pi.r^2$

$h = \frac{100 - \pi.r^2}{\pi.r}$

c. $V = \frac{1}{2}\pi.r^2(\frac{100 - \pi.r^2}{\pi.r})$

$V = \frac{100r}{2} - \frac{\pi.r^3}{2}$

$V = 50r - \frac{\pi.}{2}r^3$

d. $\frac{dV}{dr} = 50 - \frac{3\pi}{2}r^2 = 0$

$r^2 = \frac{100}{3\pi}$

$r = \sqrt{10.61}$ (radius must be positive in a practical situation)
$r = 3.26m$

Show maximum at r = 3.26, using a sign diagram,
When r < 3.26, then $\frac{dV}{dr} > 0$
When r > 3.26, then $\frac{dV}{dr} < 0$,
Therefore maximum.

$h = \frac{100 - \pi(3.26)^2}{\pi(3.26)}$
$h = 6.504m$

e. $V = \frac{1}{2}\pi(3.26)^2(6.504) = 108.58m^3$

f. $time = \frac{108.58}{0.5} = 217.16 \min s = 3 hours\ \ 37 \min$

g. $-0.8m^3 / \min$

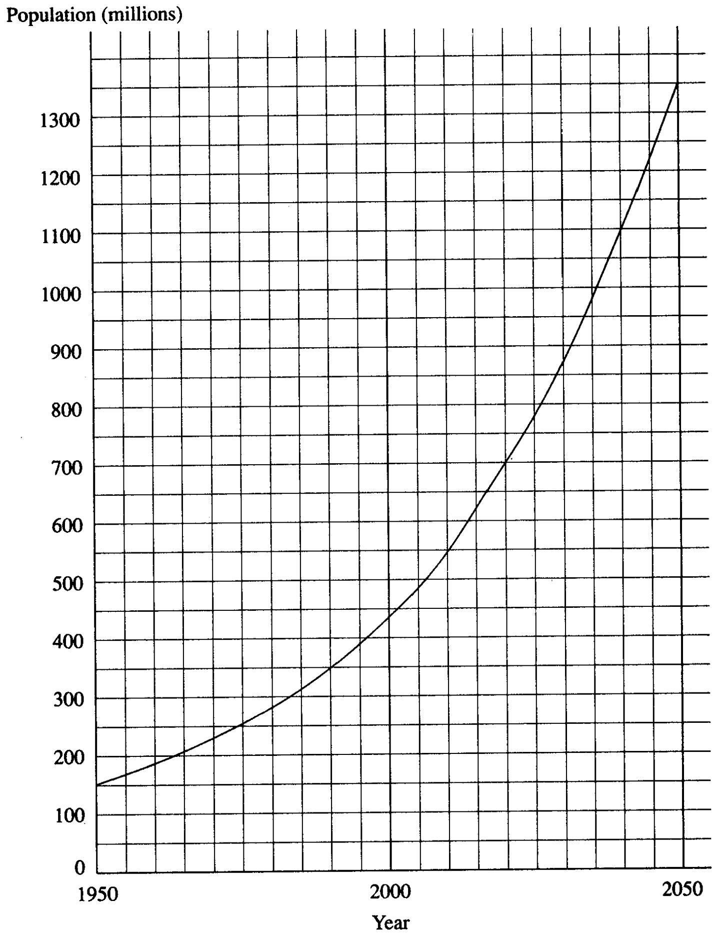
h. Volume decreases at $0.8m^3$ for 20 minutes = 0.8 x 20 = $16m^3$
Volume (after 20 mins) = 108.58 - 16 = $92.58m^3$

UNIT 1 & 2 MATHEMATICAL METHODS

EXTENDED RESPONSE QUESTION

POLYNOMIAL & CALCULUS QUESTION 5: EREHWEMOS

QUESTION 5 (17 Marks)
Erehwemos is a country whose population is growing rapidly. The below graph shows this country's population (in millions) from 1950 (when it was 150 million) to 1990 and shows what its population will be over the next 60 years if this growth continues unchecked.



a. (i) Use the graph to estimate the population of Erehwemos in 1990 and in 2020.

1 mark

(ii) Calculate the average rate of increase of population (in millions per year) from 1990 to 2020.

2 marks

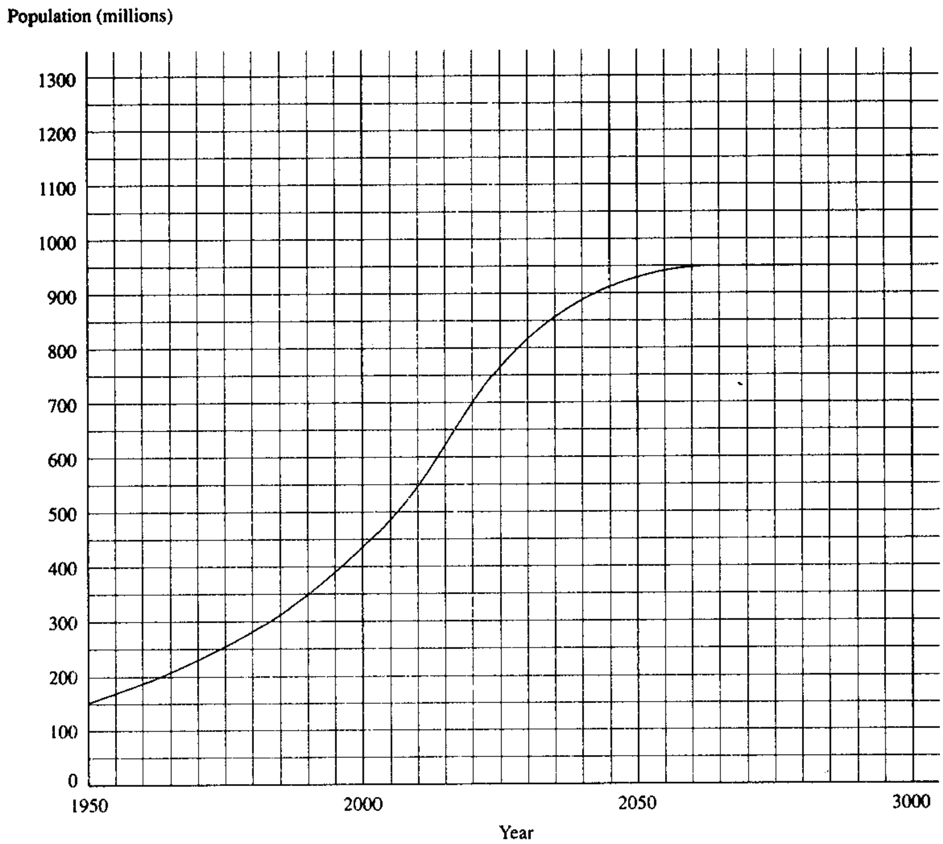
b. (i) Use the graph to estimate the rate at which the population is increasing in 1990. State your answer correct to 1 decimal place.

2 marks

(ii) Express the rate in 1990 as a percentage of the population in 1990, to give the rate, as a percentage, at which the population is increasing in 1990. State your answer correct to 3 decimal places.

1 mark

The government of Erehwemos is developing a population control policy, which is to be introduced in 1995; one that could achieve population equilibrium (zero growth rate) some years later. The situation is shown in the graph below, which is the same as the previous graph from 1950 to 1995, but thereafter gradually changes.



c. (i) Estimate when zero growth rate will be achieved.

1 mark

(ii) Estimate when the population growth will be greatest.

1 mark

(iii) Sketch a graph showing the rate of increase of population (in millions per year) from 1950 to 2100.

3 marks

d. It has been proposed that the population control policy can be modelled by the equation $P = at^3 + bt^2 + ct + d$ where t represents the number of years from 1950 and P represents the population in millions.

(i) State the value of d.

1 mark

(ii) Write 3 equations that can be used to find the values of a, b and c.

3 marks

(iii) Hence write a matrix equation that could be used to find the values of a, b and c.

1 mark

(iv) Hence solve for the values of a, b and c. State your answers correct to 3 decimal places.

1 mark

SOLUTION

QUESTION 5

a. (i) Read values off graph

In 1990 – Pop is 350 million

In 2020 – Pop is 700 million

(ii) Av rate = m line connecting the 2 points

$(1990, 350)$ x_1 y_1

$(2020, 700)$ x_2 y_2

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{700 - 350}{2020 - 1990} = 11\tfrac{2}{3} \text{ millions per yr}$$

b. (i) Draw a tangent to curve at $t = 1990$ and select 2 points on the tangent and calculate gradient

$(1950, 50)$ x_1 y_1

$(2050, 800)$ x_2 y_2

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{800 - 50}{2050 - 1950} = 7.5 \text{ millions per year}$$

Answer = 7.5 millions per year.

(ii) The population in 1990 is 350 million

$\therefore$ % increase $= \frac{7.5}{350} \times 100$ = 2.143%

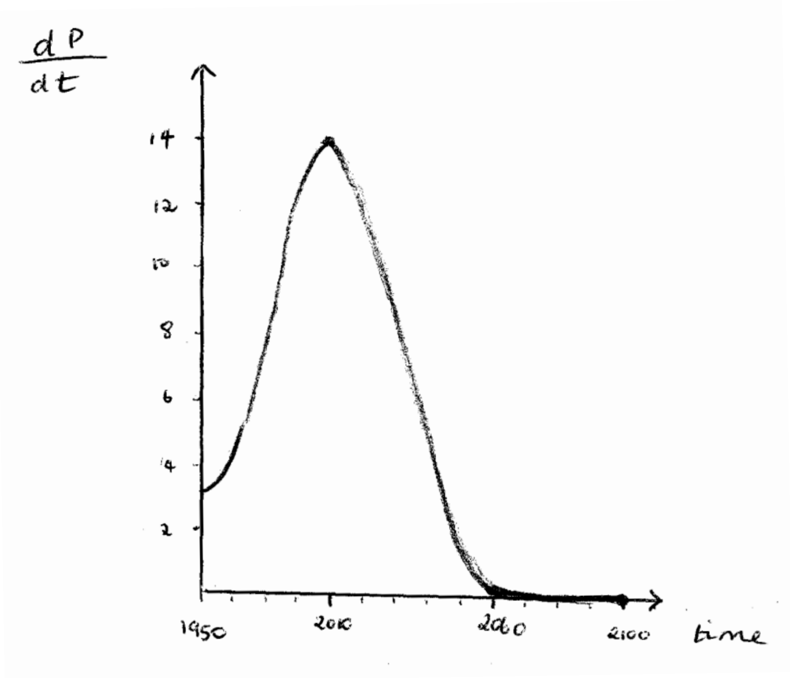
c. (i) Zero growth rate occurs when graph is flat.

i.e. In 2060.

(ii) i.e. Steepest gradient

The graph is most steep in 2010.

(iii) Draw tangents to the curve and plot.



d. (i) Substitute $(0,150)$ Answer gives $d = 150$

(ii) Substitute $(60, 550)$: $216,000a + 3,600b + 60c + 150 = 550$

$216,000a + 3,600b + 60c = 400$

Substitute $(85, 850)$: $614,125a + 7,225b + 85c + 150 = 850$

$614,125a + 7,225b + 85c = 700$

Substitute $(110, 950)$: $1,331,000a + 12,100b + 110c + 150 = 950$

$1,331,000a + 12,100b + 110c = 800$

(iii) $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 216,000 & 3,600 & 60 \\ 614,125 & 7,225 & 85 \\ 1,331,000 & 12,100 & 110 \end{bmatrix}^{-1} \begin{bmatrix} 400 \\ 700 \\ 800 \end{bmatrix}$

Accept $\begin{bmatrix} 216,000 & 3,600 & 60 \\ 614,125 & 7,225 & 85 \\ 1,331,000 & 12,100 & 110 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 400 \\ 700 \\ 800 \end{bmatrix}$

(iv) $a = -0.002, b = 0.356, c = -7.425$

UNIT 1 & 2 MATHEMATICAL METHODS

EXTENDED RESPONSE QUESTION

POLYNOMIAL & CALCULUS QUESTION 6: FUNCTIONS

QUESTION 6 (15 Marks)
Consider the graphs of the functions below:

$f(x) = x^2 + x + 2$
$g(x) = x + k$

a. (i) Without substitution, show that the solution for these equations is given by $x = \pm\sqrt{k-2}$.

2 marks

(ii) Hence find the coordinates of the point(s) of intersection of the curves when $k = 6$.

2 marks

b. (i) Using algebra, find the value(s) of k which will ensure that two points of intersection will be obtained.

2 marks

(ii) Hence find the equation of the normal at the point where curves $f(x)$ and $g(x)$ touch.

3 marks

Let A represent a point that lies on the graph of $f(x)$.

c. Use the results from **a.** and **b.** to find the equation of the tangent to the curve $f(x)$ at point A and which intersects with the line $g(x)$ at right angles.

3 marks

d. Consider the graphs of $f(x) = x^2 + x + 2$ and $h(x) = 4e^{x-1}$.

(i) Find the point(s) of intersection of the two curves.

1 mark

(ii) Although the two curves intersect, they cannot be joined smoothly. Show that a smooth join cannot be attained.

2 marks

SOLUTION

QUESTION 6

a. (i) Let $x^2+x+2 = x+k$

$x^2+x+2-x-k = 0$

$x^2+2-k = 0$

$x^2 = k-2$

$x = \pm\sqrt{k-2}$

(ii) Sub $k=6$

$x = \pm\sqrt{6-2} = \pm 2$

$\therefore (-2, 4)$ and $(2, 8)$

b. (i) 2 solutions when $b^2-4ac > 0$

$x^2 + 0x + (2-k) = 0$

$\Delta \Rightarrow 0^2 - 4(1)(2-k) > 0$

$-8+4k > 0$

$4k > 8$

$k > 2$

(ii) Curves touch when $b^2-4ac=0$

$\therefore k=2$

$g(x) = x+k = x+2 =$ tangent

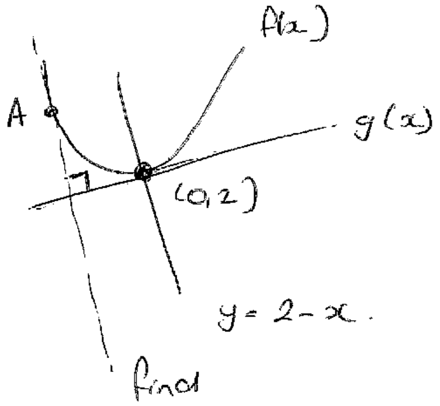
$m_{tangent} = 1 \quad \therefore m_{normal} = -1$

when $k=2$, $x = \pm\sqrt{2-2} = 0$

$\therefore y = 2$

$y-2 = -1(x-0) \quad \therefore y = 2-x$

c.



$m_{tangent} = m_{normal}$ of $y = 2 - x$

$\therefore m = -1$

$\therefore \frac{dy}{dx}$ at $A = -1$

$y = x^2 + x + 2$

$\frac{dy}{dx} = 2x + 1 = -1$

$2x = -2$

$x = -1$

$\therefore y = (-1)^2 + (-1) + 2 = 2 \qquad (-1, 2).$

$y - 2 = -1(x + 1)$

$y = 2 - x - 1$

$y = 1 - x$

d. (i) *Let f(x) = h(x)*

$x^2 + x + 2 = 4e^{x-1}$

Intersect at $(1, 4)$

(ii) To join smoothly ⇒ gradients at point of contact must be the same.

$\therefore$ show gradients at $x = 1$ are different.

On $y = x^2 + x + 2$

$\frac{dy}{dx} = 2x + 1$

At $x = 1$, $\frac{dy}{dx} = 3$

On $y = 4e^{x-1}$

Using technology:

$\frac{dy}{dx}$ at $x = 1$ is 4

As gradients are different, join is not smooth.

UNIT 1 & 2 MATHEMATICAL METHODS

EXTENDED RESPONSE QUESTION

RATIONAL FUNCTIONS & CALCULUS QUESTION 1: TITANIC 2

QUESTION 1 (10 Marks)
Leonardo Di Caprio has invested in a replica of the ship Titanic, called Titanic2. Leonardo is planning a 13,000 km trip from Los Angeles to Melbourne. The cost, ***C$ per hour***, at an average speed of ***v km/hr*** is given by the equation

$$C = 0.02v^3 + \frac{240}{v}$$

a. What is the ***cost per hour*** at a speed of 15km/hr?

1 mark

b. What would be the **cost** of the 13,000km journey at 15km/hr? State your answer correct to the nearest dollar.

2 marks

c. Show that the cost of a journey of 13,000km at *v km/hr* is $260v^2 + \frac{3.12 \times 10^6}{v^2}$.

2 marks

d. (i) Leonardo wants to keep costs to a minimum. Calculate the most economical speed for the journey. State your answer correct to 2 decimal places.

2 marks

(ii) What is the minimum cost of the journey? State your answer correct to the nearest dollar.

3 marks

SOLUTION

QUESTION 1

a. $C = 0.02(15)^3 + \dfrac{240}{15} = 67.5 + 16 = \$83.50\,per.hour$ **(1 mark)**

b. $Time = \dfrac{13000}{15} = 866.67$ **(1 mark)**

$Cost = 83.50 \times 866.67 = \$72{,}367$ **(1 mark)**

c. $Time = \dfrac{13000}{v}$

$Cost = \dfrac{13000}{v} \times (0.02v^3 + \dfrac{240}{v})$ **(1 mark)**

$= \dfrac{13000}{v} \times \dfrac{2v^4 + 24000}{100v} = \dfrac{26000v^4 + 3.12 \times 10^8}{100v^2}$

$= 260v^2 + \dfrac{3.12 \times 10^6}{v^2}$ **(1 mark)**

d. (i) $\dfrac{dC}{dv} = 520v - \dfrac{6.24 \times 10^6}{v^3} = 0$ **(1 mark)**

$\dfrac{6.24 \times 10^6}{v^3} = 520v$

$6.24 \times 10^6 = 520v^4$

$v^4 = 12000$

$v = 10.47 km/hr$ **(1 mark)**

(ii) $t = \dfrac{13000}{10.4664} = 1242.076 hrs$ **(1 mark)**

$C = 0.02(10.4664)^3 + \dfrac{240}{10.4664}$

$C = 22.9309 + 22.9305 = 45.861 \$/hr$ **(1 mark)**

Total Cost = 1242.076 x 45.861 = \$56,962.85=\$56,962 (nearest dollar) **(1 mark)**

UNIT 1 & 2 MATHEMATICAL METHODS

EXTENDED RESPONSE QUESTION

RATIONAL FUNCTIONS & CALCULUS QUESTION 2: HYPERBOLA

QUESTION 2 (24 Marks)

A function is defined by $f: R \setminus \{-2\} \to R$ where $f(x) = \dfrac{3x-4}{x+2}$.

a. State the domain and range of f.

2 marks

b. (i) Write $f(x)$ in the form $\dfrac{a}{x+h} + k$.

2 marks

(ii) Write f^{-1} in the form $\dfrac{a}{x+h} + k$ and state the value(s) of x for which f^{-1} is not defined.

3 marks

c. Write the equation $f(x) = x$ in the form $ax^2 + bx + c = 0$ and hence show that there are no real values of x which map themselves under the function f.

3 marks

The function g is defined by $g : R \setminus \{-2\} \to R$ where $g(x) = \frac{kx - 9}{x + 2}$.

d. Find the positive value of k for which there is only one value of x satisfying $g(x) = x$.

3 marks

Consider the function $f(x) = 2x^2 - 16x + 25$.

e. (i) Write $2x^2 - 16x + 25$ in the form $a[(x-b)^2 + c]$, where a, b, c are constants.

2 marks

(ii) The graph of $2x^2 - 16x + 25$ may be obtained from the graph of $y = x^2$ by means of appropriate translations and dilations. Describe suitable transformations in detail and state the order in which they would be applied.

3 marks

(iii) State the equations describing:

$y = f(x) + 2$

$y = f(x-3)$

$y = 2f(1-x)$

3 marks

f. A function f is defined for the domain $x \geq 4$ by $f(x) = 2x^2 - 16x + 25$. State the domain and range of f^{-1} and determine its equation.

3 marks

SOLUTION

QUESTION 2

a.

$$y = \frac{3x-4}{x+2},\ x \neq -2$$

dom $f = R\backslash\{-2\}$

To find the range, we solve for x:

$$(x+2)y = 3x-4$$
$$xy + 2y = 3x - 4$$
$$2y + 4 = 3x - xy$$
$$= x(3-y)$$
$$\therefore \quad x = \frac{2y+4}{3-y},\ y \neq 3$$

$\therefore$ ran $f = R\backslash\{3\}$.

b. (i)

$$\begin{array}{r|l} & 3 \\ x+2 & 3x-4 \\ - & 3x+6 \\ \hline & -10 \end{array}$$

$$y = 3 - \frac{10}{x+2}$$

b. (ii)

$$x = \frac{-10}{y+2} + 3$$

$$x - 3 = \frac{-10}{y+2}$$

$$y + 2 = \frac{-10}{x-3}$$

$$\therefore f^{-1}(x) = \frac{-10}{x-3} - 2$$

$f^{-1}(x)$ is not defined when $x = 3$

c.

If $f(x) = x$,

then $\frac{3x-4}{x+2} = x$

$$3x - 4 = x(x+2)$$
$$3x - 4 = x^2 + 2x$$
$$-x^2 + x - 4 = 0$$
$$x^2 - x + 4 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(4)}}{2 \times 1}$$
$$= \frac{1 \pm \sqrt{1-16}}{2}$$
$$= \frac{1 \pm \sqrt{-15}}{2} \notin R$$

$\therefore$ The equation has no real roots. Hence there are no real values of x which map onto themselves under the function f.

d.

If $g(x) = x$,

then $\frac{kx-9}{x+2} = x$

$$kx - 9 = x(x+2)$$
$$= x^2 + 2x$$
$$-x^2 + kx - 2x - 9 = 0$$
$$x^2 - kx + 2x + 9 = 0$$
$$x^2 + (2-k)x + 9 = 0$$

The condition for a root is $\Delta = 0$

i.e. $b^2 - 4ac = 0$

$$(2-k)^2 - 4 \times 1 \times 9 = 0$$
$$(2-k)^2 - 36 = 0$$
$$(2-k)^2 = 36$$
$$2 - k = \pm 6$$
$$-k = \pm 6 - 2$$
$$= 4 \text{ or } -8$$
$$\therefore \quad k = -4 \text{ or } 8$$

So, the positive value of k is 8.

e. (i)

$$
\begin{aligned}
2x^2 - 16x + 25 &= 2(x^2 - 8x + \tfrac{25}{2}) \\
&= 2(x^2 - 8x + 16 - 16 + \tfrac{25}{2}) \\
&= 2[(x-4)^2 - \tfrac{32}{2} + \tfrac{25}{2}] \\
&= 2[(x-4)^2 - \tfrac{7}{2}]
\end{aligned}
$$

(ii) Dilation of factor 2 from the X axis.
Translation of 4 units in the positive direction along the X axis.
Translation of 7 units in the negative direction along the Y axis.

(iii)

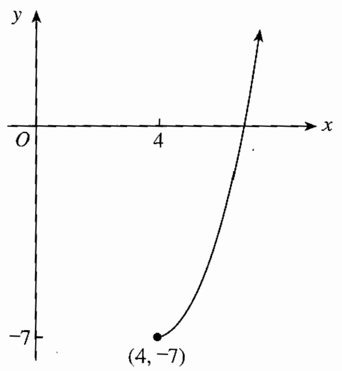
$$
\begin{aligned}
y = f(x)+2 &= 2x^2 - 16x + 25 + 2 \\
&= 2x^2 - 16x + 27
\end{aligned}
$$

$$
\begin{aligned}
y = f(x-3) &= 2(x-3)^2 - 16(x-3) + 25 \\
&= 2(x^2 - 6x + 9) - 16x + 48 + 25 \\
&= 2x^2 - 12x + 18 - 16x + 48 + 25 \\
&= 2x^2 - 28x + 91
\end{aligned}
$$

$$
\begin{aligned}
y = 2f(1-x) &= 2\left[2(1-x)^2 - 16(1-x) + 25\right] \\
&= 2\left[2(1 - 2x + x^2) - 16 + 16x + 25\right] \\
&= 2\left[2 - 4x + 2x^2 - 16 + 16x + 25\right] \\
&= 2\left(2x^2 + 12x + 11\right) \\
&= 4x^2 + 24x + 22
\end{aligned}
$$

f.

$$
\begin{aligned}
y &= 2x^2 - 16x + 25 \\
&= 2(x-4)^2 - 7,\ x \geqslant 4,\ y \geqslant -7
\end{aligned}
$$



The range is $\{y : y \geqslant -7\}$.

$\text{dom}\, f^{-1} = \{x : x \geqslant -7\}$
$\text{ran}\, f^{-1} = \{y : y \geqslant 4\}$

The inverse of
$y = 2(x-4)^2 - 7,\ x \geqslant 4,\ y \geqslant -7$
is $x = 2(y-4)^2 - 7,\ x \geqslant -7,\ y \geqslant 4$

$$
\begin{aligned}
\therefore \quad x + 7 &= 2(y-4)^2 \\
(y-4)^2 &= \tfrac{1}{2}(x+7) \\
y - 4 &= \pm\sqrt{\tfrac{1}{2}(x+7)} \\
y &= 4 \pm \sqrt{\tfrac{1}{2}(x+7)}, \\
&= 4 + \sqrt{\tfrac{1}{2}(x+7)}, \text{ as } y \geqslant 4
\end{aligned}
$$

$\therefore \quad f^{-1}(x) = 4 + \sqrt{\tfrac{1}{2}(x+7)},\ x \geqslant -7.$

UNIT 1 & 2 MATHEMATICAL METHODS

EXTENDED RESPONSE QUESTION

RATIONAL FUNCTIONS & CALCULUS QUESTION 3: HYPERBOLAE

QUESTION 3 (17 Marks)

The function $f(x)$ is described by the rule: $\{x : x > -2\} \to R,\ f(x) = \dfrac{x-2}{x+2}$.

a. Write $f(x)$ in the form $a + \dfrac{b}{x+2}$. Hence state the values of a and b.

2 marks

b. (i) Find the rule for the inverse function, $f^{-1}(x)$.

2 marks

(ii) State the domain of $f^{-1}(x)$.

1 mark

(iii) Explain why the inverse function $f^{-1}(x)$ exists.

1 mark

c. Sketch the graph of $f^{-1}(x)$ clearly labelling any intercepts with the X and Y axes as well as the equations of any asymptotes.

3 marks

d. Consider the function $h(x) = x - \dfrac{a}{x+b}$ where $a, b > 0$.

(i) Show that $h(x)$ has no stationary points for all values of x.

2 marks

(ii) Consider the specific case where $a = 3$ and $b = 2$. Find the value(s) of k for which the line $y = kx$ does not intersect with the graph of $y = h(x)$.

3 marks

e. Consider the function with rule $g(x) = \dfrac{ax+b}{cx+d}$ where $a, b, c, d \in R$.

(i) For what values of a does no stationary point exist on the graph of $g(x)$? State your answer in terms of b, c and/or d.

2 marks

(ii) What happens to the nature/properties of $g(x)$ when $ad - bc = 0$?

1 mark

SOLUTION

QUESTION 3

a.

$$\begin{array}{r} 1 \\ x+2 \overline{) x-2} \\ -\underline{x+2} \\ -4 \end{array}$$

$\therefore \ f(x) = 1 - \frac{4}{x+2}$

Dom $f(x)$: $x > -2$ $\qquad a = 1$

Ran $f(x)$: $y < 1$ $\qquad b = -4$

b. (i) $x = 1 - \frac{4}{y+2}$

$x - 1 = -\frac{4}{y+2}$

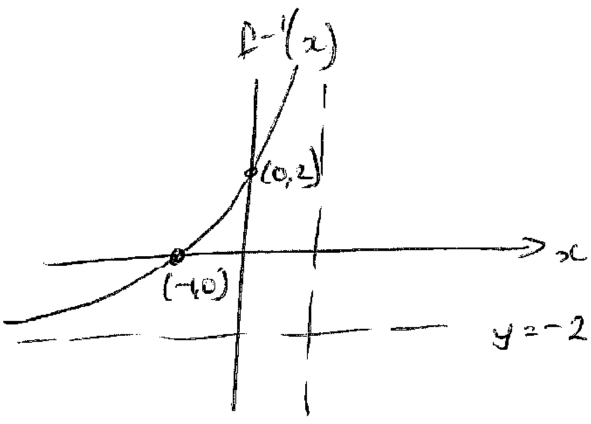
$\therefore \ y + 2 = -\frac{4}{x-1}$ $\qquad \therefore \ f^{-1}(x) = -2 - \frac{4}{x-1}$

(ii) Domain $f^{-1}(x)$ = Range $f(x)$

$\{x : x < 1\}$

(iii) Because $f(x)$ is a 1:1 function.

c.



$f^{-1}(x) = -2 - \frac{4}{x-1}$

$\underline{\text{X int, Let } f^{-1}(x) = 0}$

$-2 - \frac{4}{x-1} = 0$

$-\frac{4}{x-1} = 2$

$-2 = x - 1$

$x = -1$

$(-1, 0)$

$\underline{f^{-1}(x) \text{ int, Let } x = 0}$

$f^{-1}(x) = -2 - \frac{4}{-1} = -2 + 4 = 2 \qquad (0, 2)$

d. (i) $h(x) = x - \frac{a}{x+b} = x - a(x+b)^{-1}$

$h'(x) = 1 + a(x+b)^{-2} = 1 + \frac{a}{(x+b)^2}$

Let $h'(x) = 0$:

$1 + \frac{a}{(x+b)^2} = 0$

$\therefore \frac{a}{(x+b)^2} = -1$

As $(x+b)^2$ is always a positive number, equation can only equal -1 if a is negative. This is not possible as $a > 0$.

$\therefore$ No stationary points exist.

(ii)

$x - \frac{3}{x+2} = kx$

$\frac{x(x+2) - 3}{x+2} = kx$

$x^2 + 2x - 3 = kx(x+2)$

$x^2 + 2x - 3 = kx^2 + 2kx$

$x^2 - kx^2 + 2x - 2kx - 3 = 0$

$x^2(1-k) + x(2-2k) - 3 = 0$

For <u>no</u> intersection points: $\Delta < 0$

$\therefore b^2 - 4ac < 0$

$(2-2k)^2 - 4(1-k)(-3) < 0$

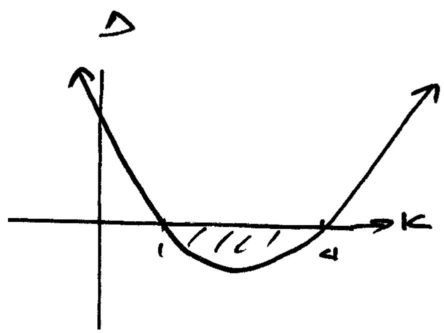
$4 - 8k + 4k^2 + 12 - 12k < 0$

$4k^2 - 20k + 16 < 0$

$k^2 - 5k + 4 < 0$

$(k-4)(k-1) < 0$

$1 < k < 4$



e. (i)

$$g(x) = \frac{ax+b}{cx+d}$$

$$g'(x) = \frac{(cx+d)a - c(ax+b)}{(cx+d)^2} = \frac{acx + ad - acx - ac}{(cx+d)^2}$$

Let $g'(x) = 0$

$$a\cancel{c}x + ad - a\cancel{c}x - bc = 0$$

$$ad - bc = 0$$

A stationary point does not exist when $ad - bc \neq 0$.

i.e. when $a \neq \frac{bc}{d}$

(ii) When $ad - bc = 0$ then $a = \frac{bc}{d}$. Substituting into $g(x)$ gives $\frac{ax+b}{ax+\frac{bc}{a}} = \frac{a}{c}$

The function $g(x)$ becomes a straight line with no stationary points.

$$a = \frac{bc}{d}$$

$$g(x) = \frac{\frac{bc}{d}x + b}{cx + d}$$

$$= \frac{bcx + bd}{d(cx+d)}$$

$$= \frac{b(cx+d)}{d(cx+d)}$$

$$= \frac{b}{d}$$

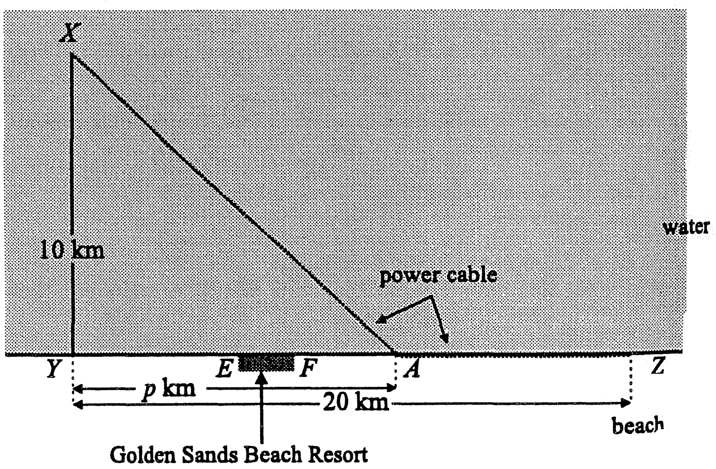
UNIT 1 & 2 MATHEMATICAL METHODS

EXTENDED RESPONSE QUESTION

RADICAL FUNCTIONS & CALCULUS QUESTION 1: ELECTRIC POWER

QUESTION 1 (14 Marks) **– ADVANCED QUESTION**
An island is located at X, 10 kilometres from the nearest point Y on a straight beach. Electric power is to be provided by laying a cable between X and a power generation plant located at Z, 20 kilometres along the beach from Y.

The cable contractor decides that the cable will go along the sea bottom from X to A, a point on the beach p kilometers from Y $(p \geq 0)$. It will then run alongside the beach to Z.

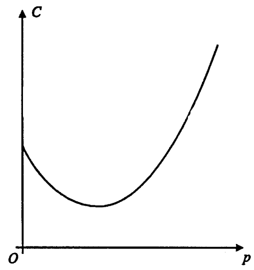


The cost of laying the cable is \$10 000 per kilometer along the beach and $w \times$ \$10 000 per kilometre along the sea bottom, where $w > 1$.

a. Let the total cost of laying the cable be $C \times \$10000$. Show that $C = 20 - p + w\sqrt{100 + p^2}$.

2 marks

The general shape of the graph of the function representing the cost of laying the cable is shown here. It shows that the function has a single local minimum.



b. Use calculus to show that the cost of laying the cable is minimum when $p^2 = \dfrac{100}{w^2 - 1}$.

3 marks

The Golden Sand Beach Resort is an expensive holiday destination located on the beach front between E and F as shown in the diagram. E and F are 9 and 10 kilometres from Y respectively.

c. If $w > \sqrt{2}$, show that if the cost of laying is to be minimum the cable will pass along the beach in front of a part or all of the resort.

2 marks

d. (i) If $w = \sqrt{5}$, find the position of A for which the total cost of laying the cable is a minimum.

2 marks

(ii) The local council has decided to impose a $20,000 penalty on the contractor if the cable passes along any part of the beach in front of the Golden Sands Beach Resort. Given this penalty, with $w = \sqrt{5}$, determine whether or not it will be cheaper for the contractor to lay the cable so that it does not pass in front of the resort.

3 marks

e. Find the range of values of w for which the cost of laying the cable would be a minimum if the cable was laid in a straight line from X to Z.

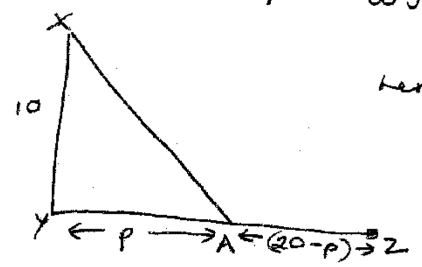
2 marks

SOLUTION

QUESTION 1

a.

Show
$C = 20 - p + w\sqrt{(100+p^2)}$



Length cable to be laid $= XA + AZ$
Using Pythagoras
$XA = \sqrt{10^2 + p^2}$
$= \sqrt{100 + p^2}$ units
$AZ = 20 - p$ units

$$\text{Cost} = w\,10{,}000 \times \sqrt{100+p^2} + 10{,}000(20-p)$$

$$C \times 10{,}000 = 10{,}000\, w\sqrt{100+p^2} + 10{,}000(20-p)$$

$$C = w\sqrt{(100+p^2)} + 20 - p$$

$$C = w(100+p^2)^{\frac{1}{2}} + 20 - p$$

b.

$$\frac{dC}{dp} = \frac{w}{2}(100+p^2)^{-\frac{1}{2}} \times 2p - 1$$

$$= wp(100+p^2)^{-\frac{1}{2}} - 1$$

$= 0$ for a turning point

$$\therefore \frac{wp}{\sqrt{100+p^2}} = 1$$

$$wp = \sqrt{100+p^2}$$

$$w^2p^2 = 100 + p^2$$

$$p^2w^2 - p^2 = 100$$

$$p^2(w^2-1) = 100$$

$p^2 = \dfrac{100}{w^2-1}$ As required

$$p = \sqrt{\frac{100}{w^2-1}}$$

c.

If $w = \sqrt{2}$

$p^2 = \frac{100}{(\sqrt{2})^2 - 1}$

$p^2 = \frac{100}{1}$

$p = \sqrt{100} = 10$

If $w > \sqrt{2}$, p will be less than 10, due to w being in the denominator

i.e test $w = 2$ $p = \sqrt{\frac{100}{3}} \simeq 5.77$

The cable is to the left of F

d. (i)

$w = \sqrt{5}$

$p^2 = \frac{100}{(\sqrt{5})^2 - 1} = \frac{100}{4} = 25$

$\therefore p = \sqrt{25} = 5$ km

i.e A is 5 km to the right of Y

(ii) If $p = 5$ and $w = \sqrt{5}$

$C = 20 - 5 + \sqrt{5}\sqrt{100 + (5)^2} = 40$ cost units

When $p = 10$ (i.e cable not in front of resort)

$C = 20 - 10 + \sqrt{5}\sqrt{100 + 10^2} = 41.623$ cost units

Difference of \$16,230 $\therefore$ Better to build past the resort to avoid the \$20,000 fine

e.

From (b) $p^2 = \frac{100}{w^2 - 1}$

$\therefore w^2 = 1 + \frac{100}{p^2}$

$p = 20$

$\therefore w^2 = 1 + \frac{100}{20^2}$

$= 1.25$

$\therefore w = \sqrt{1.25}$

As $w > 1$

then

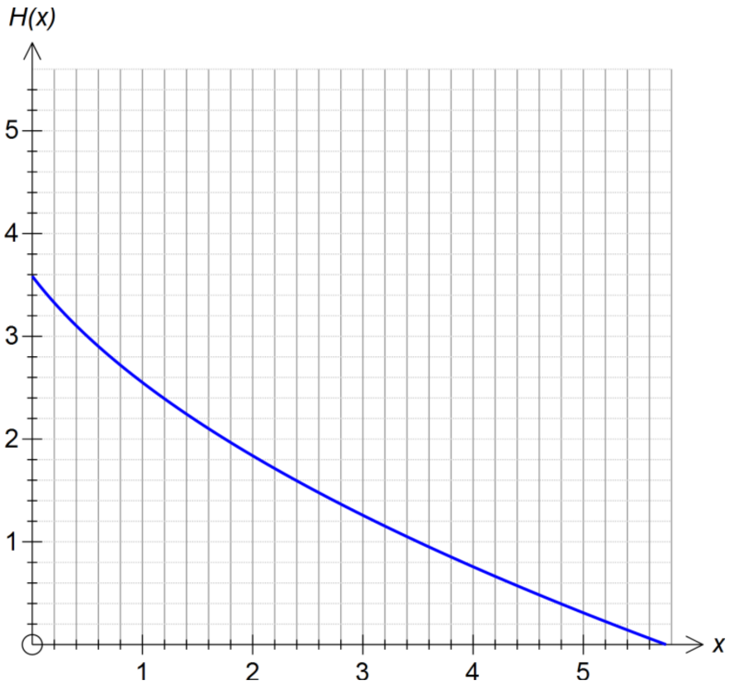
$1 < w < \sqrt{1.25}$

UNIT 1 & 2 MATHEMATICAL METHODS

EXTENDED RESPONSE QUESTION

RADICAL FUNCTIONS & CALCULUS QUESTION 2: SLIDE

QUESTION 2 (16 Marks)
The height, in metres, of a slide can be modelled by the function $H(x) = -\sqrt{4x+2}+5$ where x is the horizontal distance, in metres, from the base of the slide.



a. Find where $H(x)=0$.

____

1 mark

b. If the slide ends where it touches the ground, state an appropriate domain and range for $H(x)$.

____

____

____

2 marks

c. Find the average rate of change in the height of the slide from $x = 0$ to $x = 2$.

2 marks

d. State the instantaneous rate of change of the slide at $x = 1$.

1 mark

e. Explain if your answer for **d**. is steeper/equal/shallower than the average rate of change in the height of the slide from $x = 0$ to $x = 2$.

2 marks

f. If the height of the slide has to be halved for safety concerns and then moved to the right by 5 metres to avoid some wet ground, state the new function, $H_T(x)$ in terms of x.

2 marks

g. Find the rule of inverse function H^{-1}, also stating the domain and range of H^{-1}.

4 marks

h. Sketch the inverse function H^{-1}, on the given graph.

2 marks

SOLUTION

QUESTION 2

a. Solve $H(x) = -\sqrt{4x+2}+5=0$ gives $x=\dfrac{23}{4}$. **1A**

```
define f(x)=-√(4x+2)+5
                                done
solve(f(x)=0,x)
                            {x=23/4}
```

b. Domain $\left[0, \dfrac{23}{4}\right]$, Range $\left[0, 5-\sqrt{2}\right]$ **2A**

c. Average rate of change

$$= \frac{f(2)-f(0)}{2-0}$$

$$= \frac{\sqrt{2}-\sqrt{10}}{2}$$

1M 1A

```
(f(2)-f(0))/(2-0)
                    -(√10-√2)/2
```

d. $f'(1) = \dfrac{-\sqrt{6}}{3}$ **1A**

```
diff(f(x),x,1,1)
                          -√6/3
```

e. Answer for **d**. is $f^{/}(1) \approx -0.816$
Answer for **c**. is ≈ -0.874 **1M**

Instantaneous rate of change at x = 1 is shallower than the average rate of change from $x = 0$ to $x = 2$.

1A

```
f(2)-f(0)
---------
   2-0
                 -0.8740320489
diff(f(x),x,1,1)
                 -0.8164965809
```

f. Halved $y = \frac{1}{2}\left(-\sqrt{4x+2}+5\right) = -\frac{1}{2}\sqrt{4x+2}+\frac{5}{2}$ **1A**

Then moved to the right gives $y = -\frac{1}{2}\sqrt{4(x-5)+2}+\frac{5}{2}$

Giving $H_T(x) = -\frac{1}{2}\sqrt{4x-18}+\frac{5}{2}$ **1A**

g. $H(x) = -\sqrt{4x+2}+5$

Swap:

$x = -\sqrt{4y+2}+5$ and solve

Giving:

$$5 - x = \sqrt{4y+2}$$
$$\therefore (5-x)^2 = 4y+2$$
$$\therefore 4y = (5-x)^2 - 2$$
$$\therefore H^{-1}(x) = \frac{1}{4}(5-x)^2 - \frac{1}{2}$$

1M

Alternatively:

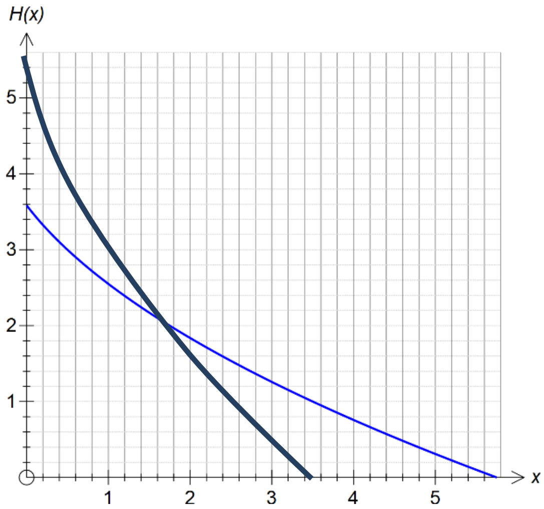
```
solve(f(y)=x,y)
```

$$\left\{y = \frac{x^2}{4} - \frac{5\cdot x}{2} + \frac{23}{4}\right\}$$

$$H^{-1}(x) = \frac{x^2}{4} - \frac{5x}{2} + \frac{23}{4} \quad \text{or} \quad H^{-1}(x) = \frac{1}{4}(5-x)^2 - \frac{1}{2}$$ **1A**

Domain $\left[0, 5-\sqrt{2}\right]$, Range $\left[0, \frac{23}{4}\right]$ **2A**

h.



2A