TRIUMPH
PUBLICATIONS

**Fortify Sample Exam 2A**

# MATHEMATICAL METHODS

## Written examination 2

**Reading time: 15 minutes**
**Writing time: 2 hours**

## QUESTION AND ANSWER BOOK

**Structure of book**

| *Section* | *Number of questions* | *Number of questions to be answered* | *Number of marks* |
|---|---|---|---|
| A | 20 | 20 | 20 |
| B | 5 | 5 | 60 |
| | | | Total 80 |

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set squares, aids for curve sketching, one bound reference, one approved technology (calculator or software) and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared. For approved computer-based CAS, full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or correction fluid/tape.

**Materials supplied**
- Question and answer booklet of 25 pages.
- Formula sheet.
- Working space is provided throughout the book.

**Instructions**
- Write your **student number** in the space provided above on this page.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

**At the end of the examination**
- You may keep the formula sheet.

**Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.**

## SECTION A - Multiple-choice questions

**Instructions for Section A**

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1; an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

### Question 1

The function with rule $g(x) = -4\tan(3\pi x)$ has period

**A.** $\frac{1}{3}$

**B.** $3$

**C.** $\frac{3}{\pi}$

**D.** $\frac{1}{6}$

**E.** $3\pi$

### Question 2

The point $P\,(2,\ -1)$ lies on the graph of a function $f$. The graph of $f$ is translated three units up, four units left, then reflected in the $x$-axis. The coordinates of the final image of $P$ are

**A.** $(2,\ -2)$

**B.** $(5,\ 5)$

**C.** $(2,\ 2)$

**D.** $(5,\ -5)$

**E.** $(-2,\ -2)$

**Question 3**

Let $f$ be a function with domain $R$ such that $f'(-1) = 0$ and $f'(x) > 0$ for $x \in R\backslash\{-1\}$. At $x = -1$, the graph of $f$ has a

**A.** Gradient of $1$

**B.** Gradient of $-1$

**C.** Stationary point of inflection

**D.** Local minimum

**E.** Local maximum

**Question 4**

If $f(x) = 3e^{2x+1}$ and $g(x) = \log_e\left(\frac{x}{3}\right) - 1$ then $g(f(x))$ is equal to

**A.** $2x - \log_e(3)$

**B.** $2x$

**C.** $\frac{1}{3}x^2 + \log_e(x)$

**D.** $\log_e(3x) - 1$

**E.** $\frac{x^2}{3e}$

**Question 5**

The continuous random variable $X$ has a normal distribution with mean 3.4 and standard deviation 0.5. The continuous random variable $Z$ follows the standard normal distribution.
$\Pr(-1 < Z < 3)$ is equal to

**A.** $\Pr(1.9 < X < 3.9)$

**B.** $\Pr(2.9 < X < 4.4)$

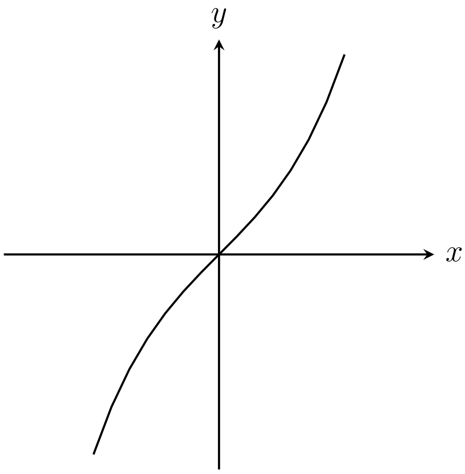
**C.** $\Pr(3.15 < X < 3.65)$

**D.** $\Pr(2.4 < X < 4.4)$

**E.** $\Pr(1.9 < X < 2.9)$

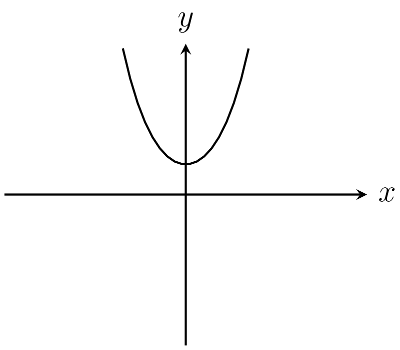
**Question 6**

The graph of a function $f$ is shown below.

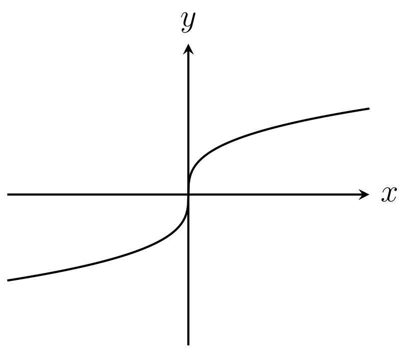



The graph of an antiderivative of $f$ could be

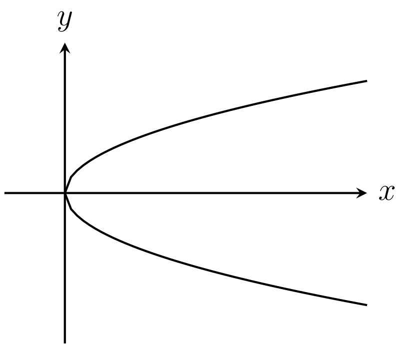
**A.**



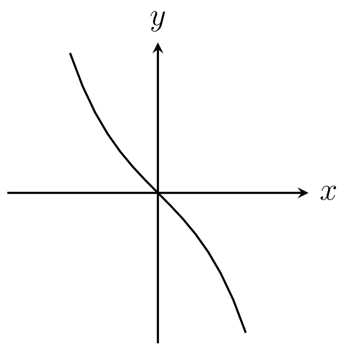

**B.**



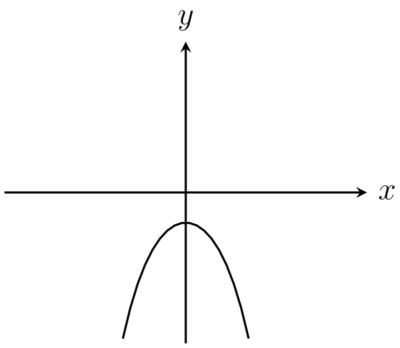

**C.**




**D.**




**E.**



**Question 7**

The function $g$ has rule $g(x) = 3\log_e\left(\frac{x}{2}\right)$. If $g(4x) = \log_e(a)$, then $a$ is equal to

**A.** $\frac{x^3}{8}$

**B.** $64x^3$

**C.** $12x$

**D.** $8x^3$

**E.** $4x$

**Question 8**

The graph of $y = 2ax - 3$ intersects the graph of $y = 2x^2 - 4x$ at two distinct points for

**A.** $a > 16$

**B.** $a = -2 - \sqrt{6}$

**C.** $-2 - \sqrt{6} \leq a \leq -2 + \sqrt{6}$

**D.** $a < -2 - \sqrt{6}$ and $a > -2 + \sqrt{6}$

**E.** $-2 \leq a \leq \sqrt{6}$

**Question 9**

The normal to the graph of $y = ae^{x^2}$ has a gradient of $-\frac{1}{2}$ at the the point where $x = 1$. The value of $a$ is

**A.** $-\frac{1}{2\text{e}}$

**B.** $\frac{2}{\text{e}}$

**C.** e

**D.** $-\frac{1}{4\text{e}}$

**E.** $\frac{1}{\text{e}}$

**Question 10**

A graph with a rule $f(x) = 2x^3 + 6x^2 + d$, where $d$ is a real number, has three distinct $x$-intercepts. The set of all possible values of $d$ is

**A.** $R^+$

**B.** $(-\infty,\ -8)$

**C.** $(0,\ 8)$

**D.** $R$

**E.** $(-8,\ 0)$

**Question 11**

$A$ and $B$ are events of a sample space. It is known that:

- $\Pr(A \cup B) = \dfrac{8}{9}$
- $\Pr(A \cap B) = \dfrac{1}{5}$
- $\Pr(A \cap B') = \Pr(A' \cap B)$

$\Pr(A \mid B)$ is equal to

**A.** $\dfrac{9}{40}$

**B.** $\dfrac{8}{9}$

**C.** $\dfrac{8}{45}$

**D.** $\dfrac{18}{49}$

**E.** $\dfrac{49}{90}$

**Question 12**

For the cosine function $f(x)$, its period is $\frac{5\pi}{3}$, its range is $[-4,\ 3]$ and its horizontal translation is $\frac{\pi}{3}$ units in the positive direction of the $x$-axis.

The equation for $f(x)$ in the form $f(x) = a\cos(b(x+c)) + d$ is

**A.** $f(x) = 7\cos\left(\frac{4}{3}\left(x - \frac{\pi}{3}\right)\right) + 3$

**B.** $f(x) = \frac{6}{5}\cos\left(\frac{7}{2}\left(x - \frac{\pi}{3}\right)\right) - \frac{1}{2}$

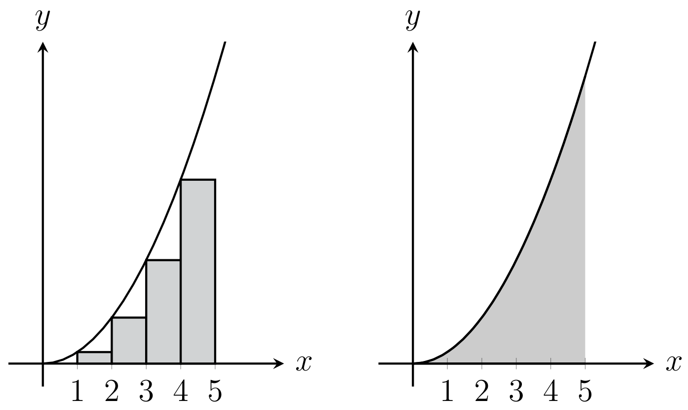
**C.** $f(x) = \frac{7}{2}\cos\left(\frac{6}{5}\left(x + \frac{\pi}{3}\right)\right) + \frac{1}{2}$

**D.** $f(x) = \frac{7}{2}\cos\left(\frac{6}{5}\left(x - \frac{\pi}{3}\right)\right) - \frac{1}{2}$

**E.** $f(x) = \frac{6}{5}\cos\left(\frac{7}{2}\left(x + \frac{\pi}{3}\right)\right) - \frac{1}{2}$

**Question 13**

Part of the graph of $g : R \to R,\ g(x) = \frac{x^2}{2}$ is shown below. Matt finds the approximate area under this curve by drawing rectangles as shown in the left diagram. He then finds the exact area of the shaded region using his calculator as shown in the right diagram.




Matt's approximation is $k\%$ less than the exact value of the area.

The value of $k$ is closest to

**A.** 15

**B.** 20

**C.** 25

**D.** 30

**E.** 35

**Question 14**

An electronics company, Robot Inc., is 95% certain that $45\% - 60\%$ of consumers prefer to use their products over any other.
How many people did they need to survey to achieve this level of confidence?

**A.** $160$

**B.** $165$

**C.** $170$

**D.** $175$

**E.** $180$

**Question 15**

The graph of the function $f : D \to R,\ f(x) = \dfrac{2x-1}{3-x}$, where $D$ is the maximal domain has asymptotes

**A.** $x = -3,\ \ y = 2$

**B.** $x = 3,\ \ y = \dfrac{1}{2}$

**C.** $x = -2,\ \ y = 3$

**D.** $x = 3,\ \ y = -2$

**E.** $x = 1,\ \ y = 2$

**Question 16**

The function $f : [b,\ \infty) \to R,\ f(x) = \dfrac{2x^3}{3} + \dfrac{5x^2}{2} - 3x + 2$ will have an inverse function provided

**A.** $b \le \dfrac{1}{2}$

**B.** $b \ge 0$

**C.** $b \le -3$

**D.** $b \ge \dfrac{1}{2}$

**E.** $b \le 0$

**Question 17**

A continuous random variable $X$ has a probability density function given by

$$f(x) = \begin{cases} ax^2, & 0 \leq x \leq b \\ 0, & \text{elsewhere} \end{cases}$$

Given that the mean of $X$ is $\frac{1}{2}$, the values of $a$ and $b$ are

**A.** $a = 1,\ b = 2$

**B.** $a = 64,\ b = \frac{\sqrt[3]{3}}{4}$

**C.** $a = \frac{81}{8},\ b = \frac{2}{3}$

**D.** $a = 1,\ b = \sqrt[3]{3}$

**E.** $a = 2,\ b = 1$

**Question 18**

There are 3 black coins and $k$ white coins in a bag. Two coins are taken from the bag without replacement. The probability that both coins are black is $\frac{1}{7}$. The value of $k$ is

**A.** $2$

**B.** $3$

**C.** $4$

**D.** $5$

**E.** $6$

**Question 19**

Let $y = \mathrm{e}^{2u}$ where $u$ is a function of $x$ such that $\frac{du}{dx} = \log_{\mathrm{e}}(x) + 1$, and the graph of $u$ passes through the point $(3,\ \log_{\mathrm{e}}(27))$.

The value of $\frac{dy}{dx}$ when $u = \log_{\mathrm{e}}(4)$ is

**A.** $32\log_{\mathrm{e}}(2) + 32$

**B.** $16\log_{\mathrm{e}}(2) + 16$

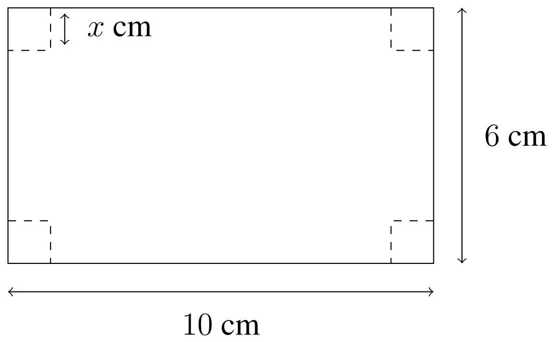
**C.** $\log_{\mathrm{e}}(27)$

**D.** $\mathrm{e}^8$

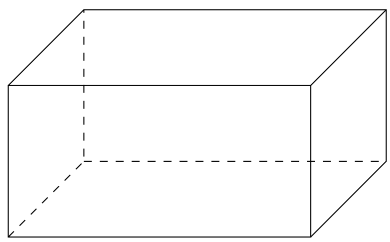
**E.** $\mathrm{e}^{16}$

**Question 20**

Kristian has a rectangular piece of cardboard that is 10cm long and 6cm wide. Kristian cuts squares of side length $x$ centimetres from each of the corners of the cardboard, as shown in the diagram below.




Kristian turns up the sides to form an open box.



The volume of this box is a maximum. The value of $x$ in this case is closest to

**A.** $4.1$

**B.** $1.2$

**C.** $3$

**D.** $0.8$

**E.** $5$

**END OF SECTION A**

## SECTION B

**Instructions for Section B**

Answer **all** questions in the spaces provided.

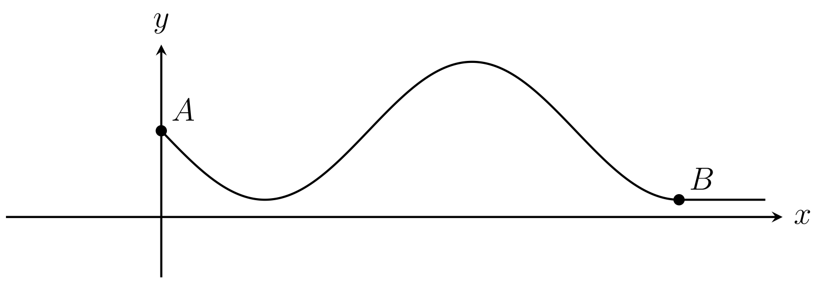
In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

**Question 1** (9 marks)

A section of a rollercoaster ride is shown below.




This rollercoaster track between points $A$ and $B$ is modelled by the function $h(x) = a + b\sin(cx)$, where $h$ is the coaster's height above the ground in metres and $x$ is the horizontal distance in metres from point $A$.

The following features are known about this rollercoaster:

- The maximum height the coaster reaches is 18 metres
- The horizontal distance between point $A$ and point $B$ is 30 metres
- The track beyond $B$ is modelled by the equation $g(x) = 2$
- Point $B$ is a smooth track; that is, $h(x)$ and $g(x)$ have the same gradient at point $B$

**a.** Find the values of $a$, $b$ and $c$.

3 marks

**b.** What is the horizontal distance of the coaster from point $A$ when it reaches its maximum height?

1 mark

**c.** State the period of the function $h(x)$.

1 mark

**d.** Find the first value of $x$ for which $h = 15$, giving your answer correct to two decimal places.

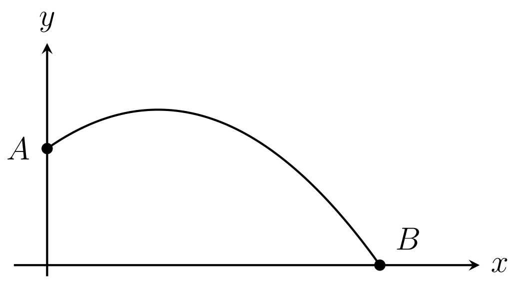
2 marks

**e.** Between points $A$ and $B$, how much horizontal distance does the coaster cover while lower than a height of 15 metres?

2 marks

**Question 2** (11 marks)

Lachie and Emily are conducting an experiment to see how far Lachie can throw a basketball off the roof of a building. Lachie stands on the roof at point $A$ and Emily stands at point $B$. When Lachie is at his strongest, the basketball's trajectory follows the curve $h(x) = a(x-b)^2 + c$ where $h$ is the height of the ball above the ground and $x$ is the horizontal distance of the ball from the building, where $a$, $b$ and $c$ are real numbers.




**a.** Emily is standing 30 metres from the building. If the ball starts falling when it reaches a height of 14 metres and a horizontal distance of 10 metres, what is the equation for $h(x)$?

2 marks

**b.** How tall is the building that Lachie is standing on?

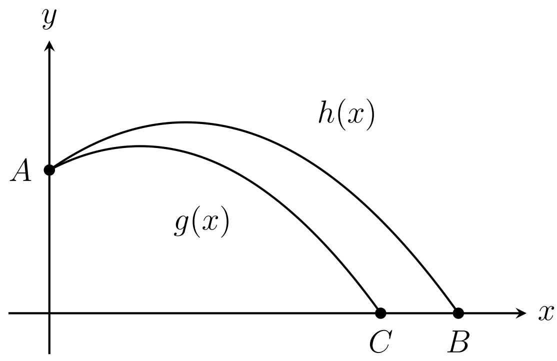
1 mark

**c.** What is the angle of depression from Lachie to Emily, correct to two decimal places?

2 marks

**d.** Shortly after they conduct this experiment, Lachie goes to the United States for a holiday and stops going to the gym. As he starts eating more junk food and putting on more weight, his upper body strength declines rapidly.

Lachie eventually comes back to Australia, and they decide to conduct the experiment again to test if his holiday has had an effect on his ability to throw a basketball.




Lachie stands on the same building and throws the ball as far as he can. They discover that his strength has diminished such that the ball now follows the curve $g(x)$, where $h(x)$ is applied the following transformations to give $g(x)$:

- a dilation by a factor of $\frac{2}{3}$ in the $y$-axis
- a dilation by a factor of $\frac{1}{2}$ in the $x$-axis
- a translation of $\frac{21}{4}$ in the positive direction of the $y$-axis

**i.** Find the rule for the function $g(x)$.

3 marks

**ii.** Find the coordinates of point C. Hence, by how many metres has Lachie's throw's maximal horizontal distance decreased? Give your answer correct to two decimal places.

2 marks

**iii.** At what height does the ball start falling in the second experiment?

1 mark

**Question 3** (15 marks)

Troy goes to the gym every day in the winter in an attempt to get huge. The amount of time he spends training each day is a continuous random variable, $X$ hours, with probability density function given by

$$f(x) = \begin{cases} \frac{1}{9}\left(\frac{x^3}{3} - \frac{5x^2}{2} + 6x\right), & 1 \le x \le 3 \\ 0, & \text{elsewhere} \end{cases}$$

**a.** On any given day in winter, how much time is Troy expected to spend at the gym? Give your answer correct to two decimal places.

1 mark

**b.** On any given day in winter, Troy has a 43% chance to spend more than $n$ hours at the gym. Find the value of $n$, correct to two decimal places.

2 marks

**c.** During the summer, Troy does not go to the gym every day. If he goes one day, the probability that he goes the following day is $\frac{1}{2}$. If he doesn't go one day, the probability that he doesn't go the next day is $\frac{1}{4}$.

He goes to the gym on a particular Wednesday. What is the probability that:

**i.** he will go to the gym on all of the following three days?

1 mark

**ii.** he doesn't go to the gym on the following Friday?

2 marks

**iii.** he goes to the gym on Thursday, given that he doesn't go on Friday?

2 marks

**d.** Troy starts working on his power lifting, and is using very heavy weights. On any given attempt for a power lift, the probability that he successfully lifts the weight is $\frac{7}{20}$. Assume that the success of any lift is completely independent of any other lift.
In a set of 12 attempts, what is the probability that he:

**i.** completes no successful lifts?

1 mark

**ii.** completes more than 5 successful lifts?

2 marks

**e.** Troy enters a strength competition in which he must complete 200 attempts at a power lift over the course of a day. Troy does his first 20 attempts and uses the results from this sample set to estimate what his overall results will be. For this sample set, $\hat{p}$ is the random variable of the distribution of sample proportions of successful lift attempts.

Find:

**i.** $\Pr\left(\hat{p} \leq \dfrac{3}{10} \mid \hat{p} \geq \dfrac{1}{10}\right)$. Give your answer to four decimal places. Do not use a normal approximation.

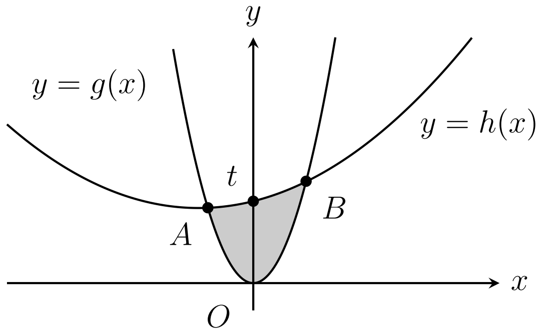
3 marks

**ii.** the 95% confidence interval for Troy's estimate of the proportion of interest, correct to three decimal places, given that he completes 6 successful lifts in his first sample of 20.

1 mark

**Question 4** (9 marks)

The graphs of $g(x) = x^2$ and $h(x) = \dfrac{x^2}{2} + 2x + t$ are shown below.




The graph of $h$ is sliding upwards at a rate of 1 unit per second, such that $t$ is equal to both $h(0)$ and the time, in seconds, since the graph started sliding upwards.

The points $A$ and $B$ are the points of intersection between $y = g(x)$ and $y = h(x)$.

**a.** Find:

**i.** the coordinates of points $A$ and $B$ in terms of $t$.

3 marks

**ii.** the area, $A$ units$^2$, of the shaded region above bound by the graphs of $y = g(x)$ and $y = h(x)$ in terms of $t$.

2 marks

Two seconds after the graph of $h$ begins its upward slide, the shaded region above is equal to $k$ units$^2$.
Find:

**b.** the exact value of $k$.

1 mark

**c.** $\dfrac{dA}{dt}$, the rate of increase of the area of the shaded region over time.

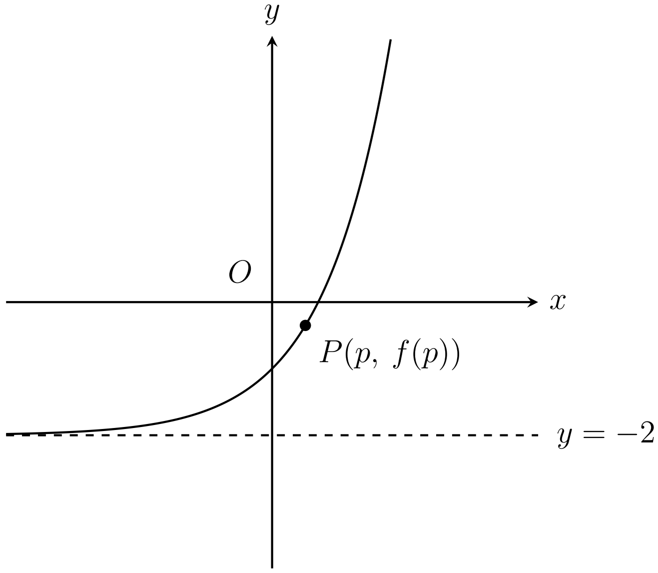
2 marks

**d.** the value of $t$ for which the rate of increase of the area of the shaded region is $\log_e(2)$ units$^2$/second.

1 mark

**Question 5** (16 marks)

The graph of $f : R \to R,\ f(x) = k\mathrm{e}^x - 2$ is shown below, where $k$ is a positive real number.




**a.** Find the set of values of $k$ for which $f(0) < 0$.

1 mark

**b.** Let $k = 1$.

**i.** Show that $L = \sqrt{p^2 + \mathrm{e}^{2p} - 4\mathrm{e}^p + 4}$, where $L$ is the length of the line segment $OP$.

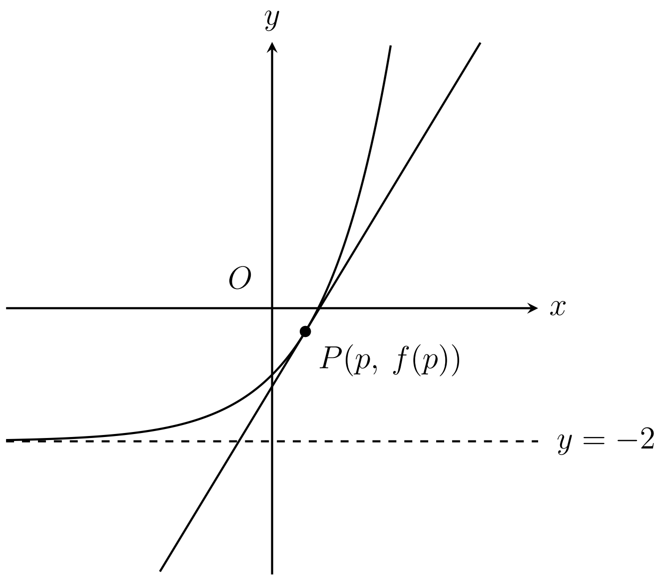
2 marks

**ii.** The length of $OP$ is at a minimum. Find the coordinates of point $P$, correct to 2 decimal places.

3 marks

**iii.** Find the minimum possible length of $OP$, correct to 3 decimal places.

1 mark




**c.** A tangent to the graph of $f(x)$ is drawn at point $P$ as shown above.

Find, correct to two decimal places:

**i.** the equation, $g(x)$, of the tangent to the graph of $f(x)$ at point $P$.

2 marks

**ii.** the area bound by the graphs of $y = f(x)$, $y = g(x)$, the $y$-axis and the line $x = \log_e(2)$.

2 marks

Suppose that $k = a$. Point $Q$ now represents a point on the graph of $f(x)$ such that $OQ$, the distance from origin $O$ to point $Q$, is at a minimum. The coordinates of point $Q$ can be obtained by applying the following transformations to point $P$ from parts **b - c**:

- A dilation by a factor of 0.43 from the $y$-axis
- A translation of 0.19 units in the positive direction of the $y$-axis

**d.** Find, correct to two decimal places:

**i.** the coordinates of point $Q$.

2 marks

**ii.** the value of $a$, given that $a > 1$.

3 marks

**END OF QUESTION AND ANSWER BOOK**

# MATHEMATICAL METHODS

## Written examination 2

### FORMULA SHEET

### Instructions

This formula sheet is provided for your reference.
A questions and answer book is provided with this formula sheet.

**Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.**

# Formula Sheet

## Mensuration

| | | | |
|---|---|---|---|
| area of a trapezium | $\frac{1}{2}(a+b)h$ | volume of a pyramid | $\frac{1}{3}Ah$ |
| curved surface area of a cylinder | $2\pi rh$ | volume of a sphere | $\frac{4}{3}\pi r^3$ |
| volume of a cylinder | $\pi r^2 h$ | area of a triangle | $\frac{1}{2}bc\sin(A)$ |
| volume of a cone | $\frac{1}{3}\pi r^2 h$ | | |

## Calculus

| | |
|---|---|
| $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$ | $\int x^n \, dx = \frac{1}{n+1}x^{n+1} + c,\ n \neq -1$ |
| $\frac{d}{dx}\left((ax+b)^n\right) = an(ax+b)^{n-1}$ | $\int (ax+b)^n \, dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c,\ n \neq -1$ |
| $\frac{d}{dx}\left(\mathrm{e}^{ax}\right) = a\mathrm{e}^{ax}$ | $\int \mathrm{e}^{ax} \, dx = \frac{1}{a}\mathrm{e}^{ax} + c$ |
| $\frac{d}{dx}(\log_{\mathrm{e}}(x)) = \frac{1}{x}$ | $\int \frac{1}{x} \, dx = \log_{\mathrm{e}}(x) + c,\ x > 0$ |
| $\frac{d}{dx}(\sin(ax)) = a\cos(ax)$ | $\int \sin(ax) \, dx = -\frac{1}{a}\cos(ax) + c$ |
| $\frac{d}{dx}(\cos(ax)) = -a\sin(ax)$ | $\int \cos(ax) \, dx = \frac{1}{a}\sin(ax) + c$ |
| $\frac{d}{dx}(\tan(ax)) = \frac{a}{\cos^2(ax)} = a\sec^2(ax)$ | |

| | | | |
|---|---|---|---|
| product rule | $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$ | quotient rule | $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ |
| chain rule | $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ | | |

# Formula Sheet

## Probability

| | | | |
|---|---|---|---|
| $\Pr(A) = 1 - \Pr(A')$ | | $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$ | |
| $\Pr(A \mid B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$ | | | |
| mean | $\mu = \mathrm{E}(X)$ | variance | $\mathrm{var}(X) = \sigma^2 = \mathrm{E}((X-\mu)^2) = \mathrm{E}(X^2) - \mu^2$ |

| **Probability distribution** | | **Mean** | **Variance** |
|---|---|---|---|
| discrete | $\Pr(X = x) = p(x)$ | $\mu = \Sigma\, x\, p(x)$ | $\sigma^2 = \Sigma\,(x-\mu)^2\, p(x)$ |
| continuous | $\Pr(a < X < b) = \int_a^b f(x)\, dx$ | $\mu = \int_{-\infty}^{\infty} x\, f(x)\, dx$ | $\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$ |

## Sample proportions

| | | | |
|---|---|---|---|
| $\hat{P} = \dfrac{X}{n}$ | | mean | $\mathrm{E}(\hat{P}) = p$ |
| standard deviation | $\mathrm{sd}(\hat{P}) = \sqrt{\dfrac{p(1-p)}{n}}$ | approximate confidence interval | $\left(\hat{p} - z\sqrt{\dfrac{p(1-p)}{n}},\ \hat{p} + z\sqrt{\dfrac{p(1-p)}{n}}\right)$ |

**END OF FORMULA SHEET**