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2019

Mathematical Methods

Trial Examination 2
(2 hours)

SECTION A Multiple-choice questions

Instructions for Section A

Answer **all** questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Unless otherwise indicated, the diagrams in this examination are **not** drawn to scale.

Question 1 Which one of the following is the correct solution set when $\log_a(b-x)^2 = c$ is solved for x given $a, b, c \in R$?

A. $\left\{b - a^{\frac{c}{2}}\right\}$

B. $\left\{-b + a^{\frac{c}{2}}, b + a^{-\frac{c}{2}}\right\}$

C. $\left\{b - a^{\frac{c}{2}}, b + a^{\frac{c}{2}}\right\}$

D. $\left\{b - a^{\sqrt{c}}, b - a^{-\sqrt{c}}\right\}$

E. $\left\{-b + a^{\sqrt{c}}, b + a^{-\sqrt{c}}\right\}$

Question 2 Which one of the following is a correct solution when $\left(\cos\left(x + \frac{\pi}{4}\right)\right)\left(\tan\left(x - \frac{\pi}{4}\right)\right) = 0$ is solved for x?

A. $-\frac{9\pi}{4}$

B. $-\frac{5\pi}{4}$

C. $-\frac{3\pi}{4}$

D. $-\frac{\pi}{4}$

E. $\frac{7\pi}{4}$

Question 3 The domain of the inverse of $\{(3,-3),(-1,-2),(1,-1),(-2,2),(3,-1),(-2,-3)\}$ is D. Which one of the following statements is true?

A. D is $\{x: -2 \le x \le 3\}$

B. D is $\{x: -3 \le x \le 2\}$

C. A subset of D is $\{x: -2 \le x \le 2\}$

D. $3 \notin D$

E. There are 3 positive integers in D

Question 4 Given $a \in R \setminus \{0\}$, the **inverse** of $y = a\left(1 \pm \frac{1}{x-a}\right)$ is

A. $y = a\left(1 \pm \frac{1}{x-a}\right)$

B. $y = a\left(1 \pm \frac{1}{x+a}\right)$

C. $y = a\left(1 - \frac{1}{x \pm a}\right)$

D. $y = a\left(1 + \frac{1}{x \pm a}\right)$

E. $y = \pm a\left(1 - \frac{1}{x-a}\right)$

Question 5 The area of the region(s) bounded by the graphs of $y = x^a$ and $y = x^{\frac{1}{a}}$ for $0 < a < 1$ is A. The value of A is in the interval

A. $(0, 0.5)$

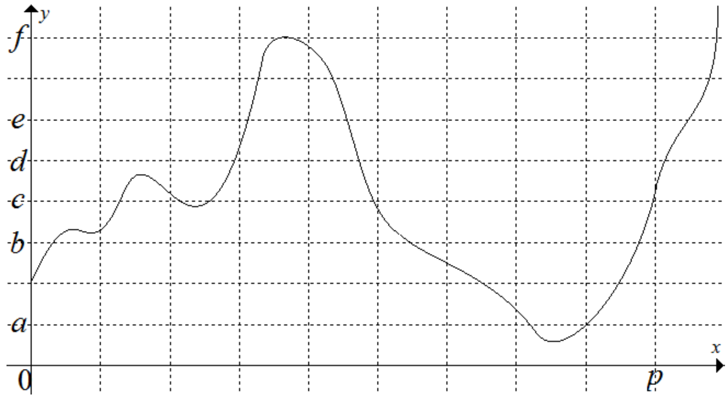
B. $(0, 1)$

C. $(1, 1.5)$

D. $(1, 2)$

E. $(0, 2)$

Question 6 The graph of $y = h(x)$ is shown below. The average value of $h(x)$ in the interval $[0, p]$ is H.



The value of H is in the interval

A. (a, b)

B. (b, c)

C. (c, d)

D. (d, e)

E. (e, f)

Question 7 Consider $h(x)$ in **Question 6**.
The average rate of change of $h(x)$ with respect to x in the interval $[0, p]$ is closest to

A. $\dfrac{c-a+b}{p}$

B. $\dfrac{b-a+c}{p}$

C. $\dfrac{2c-a-b}{2p}$

D. $\dfrac{2c-a+b}{2p}$

E. $\dfrac{0.5c-a+b}{p}$

Question 8 If $f(x-a)+f(a-x)=0$ and $a \in R$, $f(x)$ **cannot** be

A. $\tan x$

B. $2x^3$

C. $\sin 3x$

D. $4x^{-3}$

E. $\cos 5x$

Question 9

The number of solutions to simultaneous equations $y=\cos(nx)$ and $y=\cos(3nx)$ in the interval $\left[-\frac{\pi}{n}, \frac{\pi}{n}\right]$ for $n>0$ is

A. 1

B. 2

C. 3

D. 4

E. 5

Question 10 $f(x)$ is a cubic polynomial. The graph $y=f(x)$ cuts the *x*-axis at $x=a, b$ and c from left to right in that order. (m, n) and (p, q) are the local maximum and minimum points respectively. Using two triangles to approximate the regions bounded by $y=f(x)$ and the *x*-axis, the approximate area is closest to

A. $\frac{1}{2}(nc-(n+q)b+qa)$

B. $\frac{1}{2}(nc+(n+q)b+qa)$

C. $\frac{1}{2}((n-q)b-na-qc)$

D. $\frac{1}{2}((n-q)b+na-qc)$

E. $\frac{1}{2}((n-q)b-na+qc)$

Question 11 A fair die is rolled 18 times. The probability of getting 5 or 6 appearing on the uppermost face of the die 5 times or 6 times out of the 18 rolls of the die is closest to

A. 0.25

B. 0.33

C. 0.38

D. 0.43

E. 0.51

Question 12 A bag contains 3 green, 4 blue and 5 red marbles. A random sample of three marbles are taken out of the bag. From the following statements choose one which could be a random variable.

A. All three marbles in the sample are green.

B. The number of blue marbles is greater than the number of red marbles in the sample.

C. There are two blue marbles and one red marble in the sample.

D. The colour of each marble in the sample.

E. The number of red marbles in the sample.

Question 13 A large number of random samples of size 256 people are taken from a large city population. The distribution of $\hat{P}$, proportion of people wearing glasses in a sample, has a standard deviation of 0.025. The proportion of people wearing glasses in the population is closest to

A. 0.55

B. 0.45

C. 0.40

D. 0.30

E. 0.20

Question 14 A large number of random samples of size 300 people are taken from another large city. Many samples have $\hat{p} = 0.25$. A statistician decides to use 0.25 as an estimation of p. Using this value the proportion of random samples with sample proportion greater than 0.2 is closest to

A. 0.98

B. 0.78

C. 0.68

D. 0.58

E. 0.57

Question 15 The graph of $y = f(x)$ undergoes the following sequence of transformations. Firstly the graph is translated in the negative y-direction by b units, then the resulting graph is reflected in the x-axis, and lastly dilated from the x-axis by factor a.

The equation of the graph after the sequence of transformations is

A. $y = ab - af(x)$

B. $y = b - af(x)$

C. $y = -b - af(x)$

D. $y = -b + af(x)$

E. $y = -ab - af(x)$

Question 16 The probability distribution of random variable X is given by the table below.

X	1	2	3	4
$\Pr(X = x)$	0.50	a^2	0.35	$0.2a$

The standard deviation of X is closest to

A. 1.0436

B. 1.0522

C. 1.0713

D. 1.0715

E. 1.9007

Question 17 The probability density function of random variable X is given by

$$f(x) = \begin{cases} \dfrac{1}{1+a-x} & \text{for } 0 \le x \le a \\ 0 & \text{elsewhere} \end{cases}$$

The value of a is

A. e

B. e^{-1}

C. $e - 1$

D. $1 - e^{-1}$

E. $1 + e^{-1}$

Question 18 The shortest distance from the origin O to the curve $y = \frac{(x-2)^3}{3}$ is closest to

A. 1.01

B. 0.99

C. 0.97

D. 0.96

E. 0.95

Question 19
The graph of $y = 3x^3 + ax^2 + b^2x + c$ for $b, c \in R^+$ has no stationary points.
Which of the following statements is true?

A. $a < 3b$

B. $a > 3b$

C. $a = 3b$

D. $-b < 2a < 3b$

E. $a < 3b + c$

Question 20 Given $f(x) = \frac{a}{x-a} + b$ where $a, b \in R$ and $a \neq 0$, the number of intersections of the graphs of $y = f(x)$ and $y = f^{-1}(x)$ **cannot** be

A. 0

B. 1

C. 2

D. 3

E. infinitely many

SECTION B

Instructions for Section B

Answer **all** questions in the spaces provided.
In all questions where a numerical answer is required, an exact value must be given unless otherwise stated.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this examination are **not** drawn to scale.

Question 1 A parabola has its vertex at $(2, 0)$ and passes through point $(3, 1)$.

a. The equation of the parabola is $y = x^2 - 4x + 4$. Show working in finding the equation. 1 mark

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b. The inverse of the parabola in part a is given by $y = f(x)$ and $y = g(x)$ together with point $(0, 2)$. Given $f(x) > g(x)$, write down the rule for each of functions $f(x)$ and $g(x)$.

1 mark

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c. Show that the parabola and its inverse intersect at $(1, 1)$ and $(4, 4)$. 1 mark

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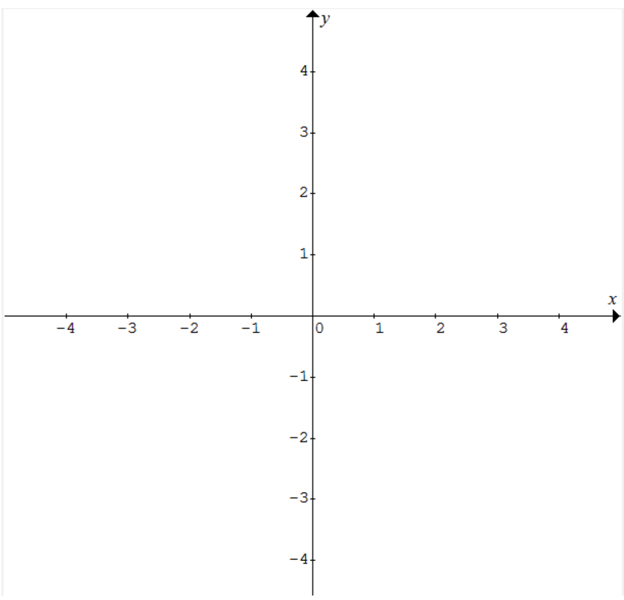
di. Sketch the parabola and its inverse with $(1, 1)$ and $(4, 4)$ as the endpoints of each curve. 2 marks



dii. Show that the area of the region bounded by the parabola and its inverse is 9. 2 marks

Horizontal line $y = d$ divides the bounded region into two regions of equal area.

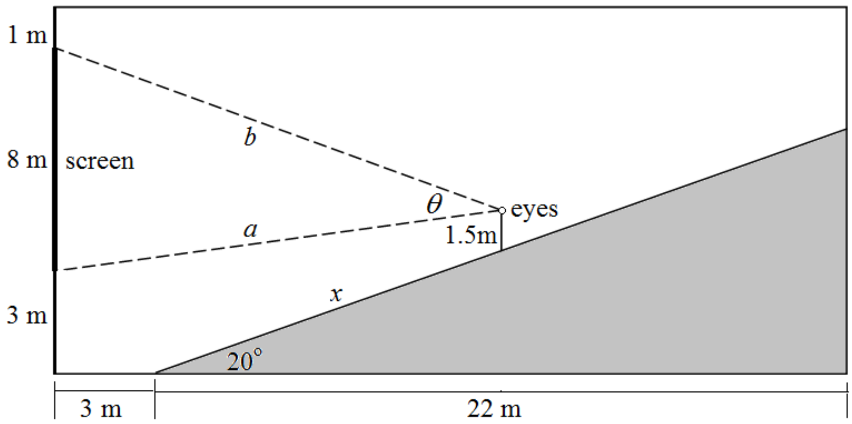
e. Determine the value of d. Give the value correct to two decimal places. 2 marks

A line segment of equation $y = -x + c$ is drawn inside the bounded region.

f. Find the maximum length of such a line segment inside the region and the corresponding value of c. Give exact values for your answers.

3 marks

Question 2 The following diagram shows the side elevation of a cinema.



There are 23 rows of seats. The first row is 3 m horizontally from the screen. The last row is at the top of the sloping floor inclined at $20°$ to the horizontal. The rows are 1 m apart horizontally. The diagram shows a viewer sitting in a row x m from the first row. The eyes of the viewer are 1.5 m above the sloping floor. The bottom and the top of the screen are 3 m and 11 m respectively above the ground.
The viewing angle is θ. Lengths are measured in metres and angles in degrees.

a. Determine the possible exact values of x. 1 mark

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b. Show that $a^2 = x^2 + (6\cos 20° - 3\sin 20°)x + 11.25$ 2 marks

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c. Show that $b^2 = x^2 + (6\cos 20° - 19\sin 20°)x + 99.25$ 1 mark

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di. Using $\cos\theta = \dfrac{a^2 + b^2 - c^2}{2ab}$ where c is the height of the screen, show that

$$\cos\theta = \frac{x^2 + (6\cos 20° - 11\sin 20°)x + 23.25}{\sqrt{(x^2 + (6\cos 20° - 3\sin 20°)x + 11.25)(x^2 + (6\cos 20° - 19\sin 20°)x + 99.25)}}.$$ 1 mark

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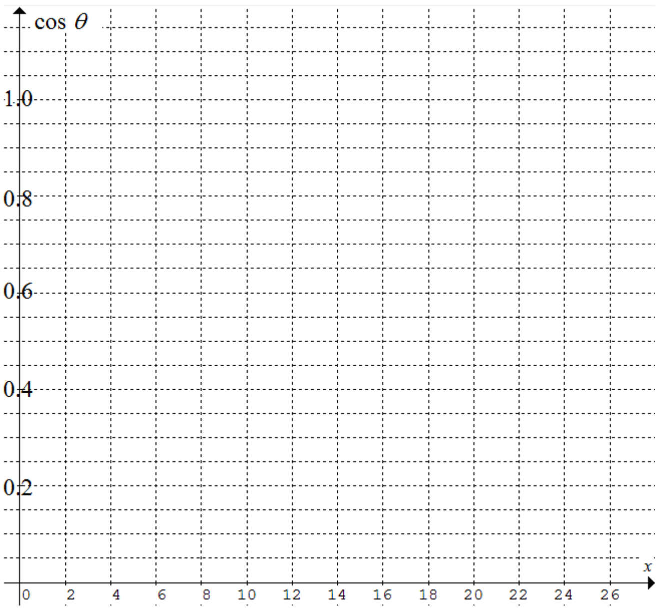
dii. Given $\cos\theta \approx \dfrac{x^2 + 1.88x + 23.25}{\sqrt{x^4 + bx^3 + 106.53x^2 + 448.07x + 1116.56}}$, show that $b \approx 3.75$. 1 mark

e. Calculate the viewing angle θ (correct to the nearest degree) if the viewer is seated in the tenth row.

2 marks

f. Use the expression in dii to sketch the graph of $\cos\theta$ versus x. 2 marks



g. Use the graph to determine the greatest viewing angle θ (correct to the nearest degree) and the value of x (correct to the nearest whole number) for maximum θ.

2 marks

h. State which row gives the viewer the greatest viewing angle. 1 mark

Question 3 Consider $f(x) = e^x - mx$ where $m > 0$.

a. If $f(x) = 0$ has exactly one solution, find the solution and the exact value of m. 2 marks

b. Write down the values of m such that $f(x) = 0$ has two distinct solutions. 1 mark

c. If the graph of $y = f(x)$ has x-intercepts at x_1 and x_2, where $x_2 > x_1$, write a definite integral for the area of the region bounded by the graph of $y = f(x)$ and the x-axis.

1 mark

d. Evaluate the definite integral in part c in terms of m, x_1 and x_2. 2 marks

e. Hence show that the sum of the two solutions to $f(x) = 0$ is always greater than 2. 3 marks

Now consider $g(x)=\log_e x-nx^2$ where $n>0$.

f. If $g(x)=0$ has exactly one solution $x=a$, find the exact value of a and the exact value of n.

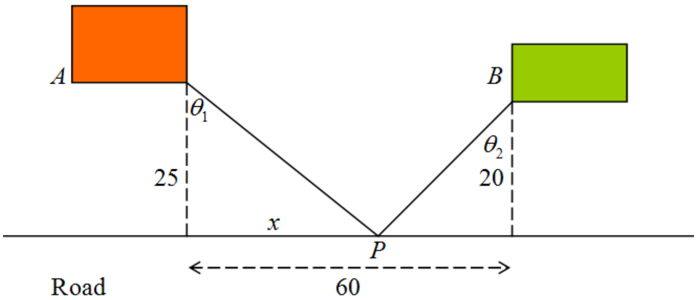
3 marks

g. Write down the n values such that the equation $g(x)=0$ has two distinct solutions. 1 mark

h. If $g(x)=\log_e x-nx^2$ where $n<0$, and $x=b$ is the solution to $g(x)=0$, explain why $0<b<1$.

1 mark

Question 4 Two farm houses, A and B, are 60 m apart. Farm House A is 25 m from the road and Farm House B is 20 m from the road.
A telephone pole marked as P is located at the edge of the road x m from A along the edge of the road.
The telephone lines make angles θ_1 and θ_2 as shown in the following diagram.
Assume that the telephone lines are straight and horizontal.



Aerial view

a. Given $x = 30$ and $\tan(\theta_1 + \theta_2) = \dfrac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2}$, find the exact value of $\theta_1 + \theta_2$. 2 marks

b. Find the value(s) of x such that $\theta_1 + \theta_2 = 90°$. 2marks

c. Show that the total length of the two telephone lines is a minimum when $\theta_1 = \theta_2$. 2 marks

d. Find the exact value of θ_1 when the total length of the two telephone lines is a minimum. 1 mark

e. Find the minimum total length of the two telephone lines. 1 mark

Question 5 The following piecewise function is a differentiable probability density function for continuous random variable X, where k is a real constant.

$$f(x)=\begin{cases} ke^{-2}x, & 0\le x<1 \\ ke^{-\frac{(x-5)^2}{8}}, & 1\le x\le 9 \\ -ke^{-2}(x-10), & 9<x\le 10 \\ 0 & \text{elsewhere} \end{cases}$$

a. Show that $k\approx 0.203232$. 2 marks

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Let random variable X be the length (in cm) of an earth worm in Worm Farm A.
The probability distribution of X is given by $f(x)$ in part a.

b. Evaluate $\Pr(X<6 \mid X>1)$, correct to 4 decimal places. 1 mark

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Let p be the proportion of earth worms in the farm with $X>6$.
Random samples of 100 worms are taken from the farm.

c. Determine the value of p, correct to 4 decimal places. 1 mark

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d. Determine the mean and standard deviation of sample proportion of earth worms with $X>6$.
Correct your answers to 4 decimal places.

2 marks

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e. Find the probability, correct to 4 decimal places, that the proportion of earth worms in a sample with $X>6$ is less than 0.4

1 mark

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For parts f and g, let $p = 0.3$ be the proportion of earth worms in the farm with $X > c$.

f. Find the value of c, correct to 2 decimal places. 1 mark

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g. Find how large a sample is required so that $\text{sd}(\hat{P}) \le 0.01$. 1 mark

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Let random variable Y be the length (in cm) of an earth worm in Worm Farm *B*.
Random variable Y has a normal distribution.
A random sample of 400 earth worms is taken from Worm Farm *B*.
The sample contains 100 worms of length $Y > 8$.
Correct your answers to parts h, i and j to 2 decimal places.

h. For $Y < 8$ cm, write down a suitable approximate value of p in the expression for estimating the standard deviation of p in Worm Farm *B*.

1 mark

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i. Calculate the estimated standard deviation of p in Worm Farm *B*. 1 mark

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j. Find an approximate 70% confidence interval for p in Worm Farm *B*. 2 marks

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End of Examination 2

